# Quantum spectral curve for twisted ABJM model

Andrei Onishchenko
in collaboration with Roman Lee

BLTP JINR, MIPT, SINP MSU

**Supersymmetries and quantum symmetries, 2017**

# Outline

ABJM is $\mathcal{N}=6$ superconformal Chern-Simons-matter theory with gauge group $U(N)\times U(N)$ on $\mathbb{R}^{1,2}$ and Chern-Simons levels $k$ and $-k$. The field content is given by gauge fields $A_\mu$ and $\hat{A}_\mu$, four complex scalars $Y^A$ and four Weyl spinors $\psi_A$.

In planar limit $k,N\to\infty,\quad \lambda\equiv\frac{N}{k}=\text{fixed}$ it is dual to superstring theory on $AdS_4\times\mathbb{CP}^3$

We will be interested in anomalous dimensions of operators

$$\mathrm{tr}\left[D_+^S(Y^1Y_4^\dagger)^L\right]$$

with Dynkin labels $[L+S,S;L,0,L]$ under $\mathrm{OSp}(6|4)$

Aharony, Bergman, Jafferis, Maldacena 2008

## The $\gamma$ deformation of ABJM theory: Moyal-like $*$-product

$$f * g = e^{i\pi Q_f \times Q_g} fg = e^{i\pi Q_f^i C_{ij} Q_g^j} fg.$$

where

$$\mathbf{C} = \begin{pmatrix} 0 & -\gamma_3 & +\gamma_2 \\ +\gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & +\gamma_1 & 0 \end{pmatrix}$$

and charges under $SU(4)_R$ are given by

| Fields | $Y^1$ | $Y^2$ | $Y^3$ | $Y^4$ | $\psi^{\dagger 1}$ | $\psi^{\dagger 2}$ | $\psi^{\dagger 3}$ | $\psi^{\dagger 4}$ |
|---|---|---|---|---|---|---|---|---|
| $Q_f^1$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 |
| $Q_f^2$ | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| $Q_f^3$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |

Emeroni 2008

He, Wu 2013

Bai, Chen, Ding, Li, Wu 2016

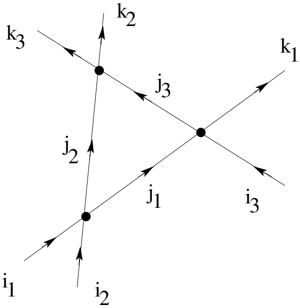
## Graded Yang-Baxter equation ($\mathbf{q}_X \times \mathbf{q}_Y = \mathbf{q}_X^{\mathrm{T}} \mathbf{C} \mathbf{q}_Y = C_{ab} q_X^a q_Y^b$)

$$(-)^{[j_2]([j_1]+[k_1])} \mathcal{R}_{i_1 i_2}^{j_1 j_2}(u-v) \mathcal{R}_{j_1 i_3}^{k_1 j_3}(u) \mathcal{R}_{j_2 j_3}^{k_2 k_3}(v) = (-)^{[j_2]([i_1]+[j_1])} \mathcal{R}_{i_2 i_3}^{j_2 j_3}(v) \mathcal{R}_{i_1 j_3}^{j_1 k_3}(u) \mathcal{R}_{j_1 j_2}^{k_1 k_2}(u-v)$$

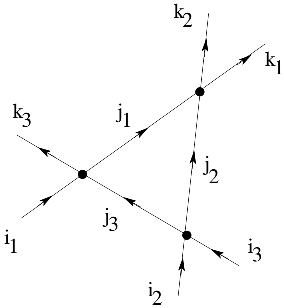
## Deformed $R$-matrix

$$\tilde{\mathcal{R}}_{ij}^{lk} = e^{i(\mathbf{q}_k \times \mathbf{q}_l - \mathbf{q}_i \times \mathbf{q}_j)/2} \mathcal{R}_{ij}^{lk}$$

$$\sum_{j_1, j_2, j_3}$$




$$= \sum_{j_1, j_2, j_3}$$




Beisert, Roiban, 2005

## $OSp(6|4)$ spin chain excitations ($\eta = -1$ $sl(2)$-favored grading)

$$
\begin{aligned}
\mathcal{B}_1 &: Y^3 \to \psi_{4-}, Y^4 \to \psi_{3-}, Y_1^\dagger \to \psi^{\dagger 2-}, Y_2^\dagger \to \psi^{\dagger 1-}, \\
\mathcal{B}_2 &: Y^2 \to Y^3, \psi_{3\pm} \to \psi_{2\pm}, \\
\mathcal{B}_3 &: \psi^{\dagger 1+} \to Y_3^\dagger, \psi_{4+} \to Y^2, \psi^{\dagger 3+} \to Y_1^\dagger, \psi_{2+} \to Y^4, \\
\mathcal{B}_4 &: Y^1 \to \psi_{4+}, Y^4 \to \psi_{1+}, Y_2^\dagger \to \psi^{\dagger 3+}, Y_3^\dagger \to \psi^{\dagger 2+}, \\
\mathcal{B}_{\bar{4}} &: Y^2 \to \psi_{3+}, Y^3 \to \psi_{2+}, Y_1^\dagger \to \psi^{\dagger 4+}, Y_4^\dagger \to \psi^{\dagger 1+}.
\end{aligned}
$$

## The twist matrix is

$$
\mathbf{A} = \frac{1}{2}\left(\begin{array}{c|ccccc}
0 & \gamma_2-\gamma_1 & 2(\gamma_1-\gamma_2) & \gamma_2-\gamma_1 & \gamma_1+\gamma_2-\gamma_3 & -\gamma_1-\gamma_2+\gamma_3 \\
\hline
\gamma_1-\gamma_2 & 0 & \gamma_1-\gamma_2 & \gamma_2-\gamma_1 & \gamma_1+\gamma_2 & -\gamma_1-\gamma_2 \\
2(\gamma_2-\gamma_1) & \gamma_2-\gamma_1 & 0 & \gamma_1-\gamma_2 & -\gamma_1-\gamma_2-\gamma_3 & \gamma_1+\gamma_2+\gamma_3 \\
\gamma_1-\gamma_2 & \gamma_1-\gamma_2 & \gamma_2-\gamma_1 & 0 & \gamma_3 & -\gamma_3 \\
-\gamma_1-\gamma_2+\gamma_3 & -\gamma_1-\gamma_2 & \gamma_1+\gamma_2+\gamma_3 & -\gamma_3 & 0 & 0 \\
\gamma_1+\gamma_2-\gamma_3 & \gamma_1+\gamma_2 & -\gamma_1-\gamma_2-\gamma_3 & \gamma_3 & 0 & 0
\end{array}\right)
$$

Chen, Liu, Wu, 2016
Caetano, Gurdogan, Kazakov, 2016

## Asymptotic Bethe ansatz in $\eta = -1$ grading

$$e^{-2\pi i(\mathbf{AK})_1} = e^{i\mathcal{Q}_1} \left.\frac{Q_2^- B^{(+)}}{Q_2^+ B^{(-)}}\right|_{u_{1,k}}, \quad x \equiv x(u) = \frac{u + \sqrt{u^2 - 4h^2}}{2h}$$

$$e^{-2\pi i(\mathbf{AK})_2} = -\left.\frac{Q_2^{++} Q_1^- Q_3^-}{Q_2^{--} Q_1^+ Q_3^+}\right|_{u_{2,k}},$$

$$e^{-2\pi i(\mathbf{AK})_3} = \left.\frac{Q_2^- R^{(+)}}{Q_2^+ R^{(-)}}\right|_{u_{3,k}},$$

$$e^{-2\pi i(\mathbf{AK})_4} \left(\frac{x_{4,k}^+}{x_{4,k}^-}\right)^L = \frac{B_1^+ R_3^+ B_4^{(+)+} R_{\bar{4}}^{(-)-}}{B_1^- R_3^- B_4^{(-)-} R_{\bar{4}}^{(+)+}} \left(\prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-}\right) S\Big|_{u_{4,k}},$$

$$e^{-2\pi i(\mathbf{AK})_{\bar{4}}} \left(\frac{x_{\bar{4},k}^+}{x_{\bar{4},k}^-}\right)^L = \frac{B_1^+ R_3^+ B_{\bar{4}}^{(+)+} R_4^{(-)-}}{B_1^- R_3^- B_{\bar{4}}^{(-)-} R_4^{(+)+}} \left(\prod_{j=1}^{K_{\bar{4}}} \frac{x_{\bar{4},j}^+}{x_{\bar{4},j}^-}\right) S\Big|_{u_{\bar{4},k}},$$

**where**

$\mathbf{K} = (L|K_1, K_2, K_3, K_4, K_{\bar{4}})$, $\quad f^{[\pm a]} \equiv f(u \pm ia/2), f^{\pm} \equiv f(u \pm i/2)$.

Gromov, Vieira, 2008; Chen, Liu, Wu, 2016 and the various functions are:

$$R_l^{(\pm)} = \prod_{j=1}^{K_l} \left(x(u) - x_{l,j}^{\mp}\right) \, , R_l = \prod_{j=1}^{K_l} \left(x(u) - x_{l,j}\right) \, ,$$

$$B_l^{(\pm)} = \prod_{j=1}^{K_l} \left(\frac{1}{x(u)} - x_{l,j}^{\mp}\right) \, , B_l = \prod_{j=1}^{K_l} \left(\frac{1}{x(u)} - x_{l,j}\right) \, ,$$

$$Q_l = \prod_{j=1}^{K_l} (u - u_{l,j}) \, , S_l = \prod_{j=1}^{K_l} \sigma_{BES} \left(x(u), x_{l,j}\right) \, ,$$

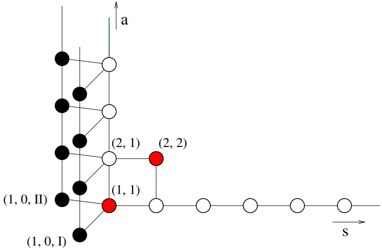
$$R = R_4 R_{\bar{4}}, B = B_4 B_{\bar{4}}, S = S_4 S_{\bar{4}}$$

momentum condition

$$\prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-} \prod_{j=1}^{K_{\bar{4}}} \frac{x_{\bar{4},j}^+}{x_{\bar{4},j}^-} = e^{-2\pi i (\mathbf{AK})_0},$$

anomalous dimension of single trace operator

$$E = h(\lambda)\left(\sum_{j=1}^{K_4}\left(\frac{i}{x_{4,j}^+} - \frac{i}{x_{4,j}^-}\right) + \sum_{j=1}^{K_{\bar{4}}}\left(\frac{i}{x_{\bar{4},j}^+} - \frac{i}{x_{\bar{4},j}^-}\right)\right).$$




**ABJM Y-system**

$$Y_{a,s}(u+i/h)Y_{a,s}(u-i/h) = \frac{(1+Y_{a,s+1}(u))(1+Y_{a,s-1}(u))}{(1+1/Y_{a+1,s}(u))(1+1/Y_{a-1,s}(u))}, \quad s>1, (a,s)\neq(2,2)$$

$$Y_{a,1}(u+i/h)Y_{a,1}(u-i/h) = \frac{(1+Y_{a,2}(u))(1+Y^{I}_{a,0}(u))(1+Y^{II}_{a,0}(u))}{(1+1/Y_{a+1,1}(u))(1+1/Y_{a-1,1}(u))},$$

$$Y^{\alpha}_{a,0}(u+i/h)Y^{\beta}_{a,0}(u-i/h) = \frac{(1+Y_{a,1}(u))}{(1+1/Y^{\alpha}_{a+1,0}(u))(1+1/Y^{\beta}_{a-1,0}(u))}, \quad \alpha,\beta\in\{I,II\}$$

Cavaglia, Fioravanti, Tateo, 2013
Gromov, Levkovich-Maslyuk 2013

## ABJM T-system

$$Y_{a,s}(u) = \frac{T_{a,s+1}(u)\,T_{a,s-1}(u)}{T_{a+1,s}(u)\,T_{a-1,s}(u)}, \qquad \text{for } s \geq 2, a \geq 1,$$

$$Y_{a,1}(u) = \frac{T_{a,2}(u)\,T^{I}_{a,0}(u)\,T^{II}_{a,0}(u)}{T_{a+1,1}(u)\,T_{a-1,1}(u)}, \qquad \text{for } a \geq 1,$$

$$Y^{\alpha}_{a,0}(u) = \frac{T_{a,1}(u)\,T^{\beta}_{a,-1}(u)}{T^{\alpha}_{a+1,0}(u)\,T^{\beta}_{a-1,0}(u)}, \qquad \text{for } a \geq 1, \quad \alpha, \beta \in \{I, II\}, \beta \neq \alpha.$$

## Discrete Hirota equation for T-functions:

$$T^{[+1]}_{a,s}\,T^{[-1]}_{a,s} = \prod_{(a'\sim a)_{\updownarrow}} T_{a',s} + \prod_{(s'\sim s)_{\leftrightarrow}} T_{a,s'},$$

**where the products are over horizontal ($\leftrightarrow$) and vertical ($\updownarrow$) neighbouring nodes**

**T-functions are gauge dependent!**

There are two special gauges **T** and $\mathbb{T}$, so that

$$\begin{aligned}\mathbb{T}_{1,s} &= \mathbf{P}_1^{[+s]}\mathbf{P}_2^{[-s]} - \mathbf{P}_2^{[+s]}\mathbf{P}_1^{[-s]}, \qquad \mathbb{T}_{0,s} = 1,\\ \mathbb{T}_{2,s} &= \mathbb{T}_{1,1}^{[+s]}\mathbb{T}_{1,1}^{[-s]}, \qquad \mathbb{T}_{3,2}/\mathbb{T}_{2,3} = \mu_{12}, \quad s \geq a\end{aligned}$$

and

$$\begin{aligned}\mathbf{T}_{n,s} &= (-1)^{n(s+1)}\mathbb{T}_{n,s}\left({\mu_{12}}^{[n+s-1]}\right)^{2-n}, \qquad s \geq 1\\ \mathbf{T}_{n,0}^{\alpha} &= (-1)^{n}\mathbb{T}_{n,0}^{\alpha}\left(\sqrt{{\mu_{12}}^{[n-1]}}\right)^{2-n},\\ \mathbf{T}_{n,-1}^{\alpha} &= \mathbb{T}_{n,-1}^{\alpha} = 1, \qquad \alpha = I, II,\end{aligned}$$

The $\mathbf{T}_{n,s}$ functions are required to satisfy:

$$\begin{aligned}\mathbf{T}_{n,0}^{\alpha} &\in \mathcal{A}_{n+1}, \qquad \alpha = I, II, \quad n \geq 0\\ \mathbf{T}_{n,1} &\in \mathcal{A}_{n}, \quad n \geq 1,\end{aligned}$$

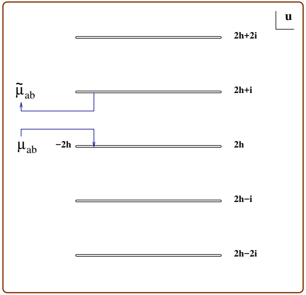
$\mathcal{A}_n$ is the class of functions free of branch cuts for $|\mathrm{Im}(u)| < \frac{n}{2}$.

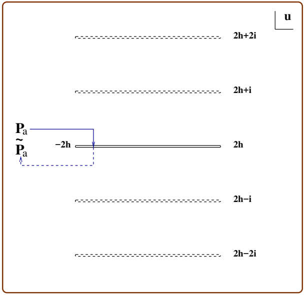
**Vector form ($CP^3$ isometry group $SO(6) \simeq SU(4)$):**

$$\mathbf{P}_A(u)\Big|_{A=1,\ldots,6}, \quad \mu_{AB}(u) = -\mu_{BA}(u)\Big|_{A,B=1,\ldots,6}$$

$$\widetilde{\mathbf{P}}_A - \mathbf{P}_A = \mu_{AB}\,\eta^{BC}\,\mathbf{P}_C, \qquad \widetilde{\mu}_{AB} - \mu_{AB} = \mathbf{P}_A\widetilde{\mathbf{P}}_B - \mathbf{P}_B\widetilde{\mathbf{P}}_A.$$

$$\mathbf{P}_5\mathbf{P}_6 - \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_1\mathbf{P}_4 = 1, \quad \mu_{AB}\,\eta^{BC}\,\mu_{CD} = 0, \quad \widetilde{\mu}_{AB}(u) = \mu_{AB}(u+i)$$




Gromov, Kazakov, Leurent, Volin, 2013
Cavaglia, Fioravanti, Gromov, Tateo, 2014

# ABJM QSC and $\mathbf{P}\mu$ system

**Spinor form ($CP^3$ isometry group $SO(6) \simeq SU(4)$):** the matrix $\mu_{AB}(u)$ is decomposed in terms of $4+4$ functions $\nu_a$, $\nu^a$ as

$$\mu_{AB} = \nu^a \, (\sigma_{AB})_a^{\;b} \, \nu_b \,, \quad \nu^a \, \nu_a = 0.$$

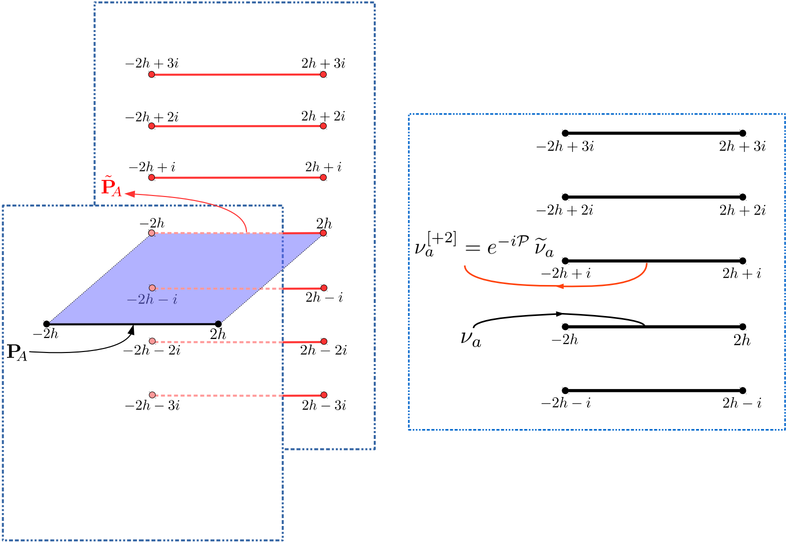
$$\widetilde{\nu}_a(u) = e^{i\mathcal{P}} \, \nu_a(u+i), \quad \widetilde{\nu}^a(u) = e^{-i\mathcal{P}} \, \nu^a(u+i)$$

**Riemann-Hilbert problem to solve:**

$$\widetilde{\mathbf{P}}_{ab} - \mathbf{P}_{ab} = \nu_a \tilde{\nu}_b - \nu_b \tilde{\nu}_a, \quad \widetilde{\mathbf{P}}^{ab} - \mathbf{P}^{ab} = -\nu^a \tilde{\nu}^b + \nu^b \tilde{\nu}^a,$$
$$\tilde{\nu}_a = -\mathbf{P}_{ab} \, \nu^b, \quad \tilde{\nu}^a = -\mathbf{P}^{ab} \, \nu_b.$$

$$\mathbf{P}_{ab} = \mathbf{P}_A \sigma^A_{ab} = \begin{pmatrix} 0 & -\mathbf{P}_1 & -\mathbf{P}_2 & -\mathbf{P}_5 \\ \mathbf{P}_1 & 0 & -\mathbf{P}_6 & -\mathbf{P}_3 \\ \mathbf{P}_2 & \mathbf{P}_6 & 0 & -\mathbf{P}_4 \\ \mathbf{P}_5 & \mathbf{P}_3 & \mathbf{P}_4 & 0 \end{pmatrix}, \quad \mathbf{P}^{ab} \text{ is inverse matrix}$$

Bombardelli, Cavaglia, Fioravanti, Gromov, Tateo, 2017




Bombardelli, Cavaglia, Fioravanti, Gromov, Tateo, 2017

**Boundary conditions in $sl(2)$ sector (large $u$):**

$$\mathbf{P}_a \simeq (A_1 u^{-L},\ A_2 u^{-L-1},\ A_3 u^{+L+1},\ A_4 u^{+L},\ A_0 u^0),$$

$$-A_1 A_4 = \frac{(-\Delta + L - S)(-\Delta + L + S - 1)(\Delta + L - S + 1)(\Delta + L + S)}{L^2(2L+1)},$$

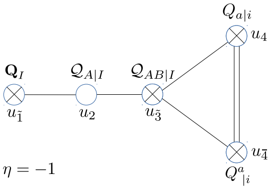
$$-A_2 A_3 = \frac{(-\Delta + L - S + 1)(-\Delta + L + S)(\Delta + L - S + 2)(\Delta + L + S + 1)}{(L+1)^2(2L+1)},$$

$$\nu_a \sim \left(u^{\Delta - L}, u^{\Delta+1}, u^{\Delta}, u^{\Delta+L+1}\right).$$

$L \in \mathbb{N}^+$ (twist), $S \in \mathbb{N}^+$ (spin) and $\Delta$ is the conformal dimension.
The anomalous dimension $\gamma$ is given by $\gamma = \triangle - L - S$.

Cavaglia, Fioravanti, Gromov, Tateo, 2014

## Dynkin diagram for $\eta = -1$ grading




## Q-system relations

$$F_1 : \quad \mathcal{Q}^+_{2|2} - \mathcal{Q}^-_{2|2} = \mathbf{P}_2\,\mathbf{Q}_2,$$

$$B_{2*} : \quad \mathcal{Q}^+_{1|2}\,\mathcal{Q}^-_{2|2} - \mathcal{Q}^+_{2|2}\,\mathcal{Q}^-_{1|2} = \mathcal{Q}_{12|2}\,\mathbf{Q}_2,$$

$$F_3 : \quad (Q_{1|1}\,Q^4{}_{|1})^+\,\mathcal{Q}^-_{2|2} - (Q_{1|1}\,Q^4{}_{|1})^-\,\mathcal{Q}^+_{2|2} = \mathcal{Q}_{12|2}\,\mathcal{Q}_{2|12},$$

$$F_4 : \quad (Q^4{}_{|1})^+\,Q^-_{1|3} - (Q^4{}_{|3})^+\,Q^-_{1|1} = \mathcal{Q}_{12|2},$$

$$F_{\bar{4}} : \quad (Q^4{}_{|1})^-\,Q^+_{1|3} - (Q^4{}_{|3})^-\,Q^+_{1|1} = \mathcal{Q}_{12|2}$$

Cavaglia, Fioravanti, Gromov, Tateo, 2014

Twisting similar to $\mathcal{N}=4$ QSC Kazakov, Leurent, Volin, 2015

$$\mathcal{Q}_{A|I} \sim \left(\frac{\prod_{a\in A} x_a}{\prod_{i\in I} y_i}\right)^{-i\,u} \mathcal{Q}_{A|I}\,, \quad Q^4_{|1} \sim \alpha^{-i\,u} Q^4_{|1}\,, \quad Q_{1|1} \sim \beta^{-i\,u} Q_{1|1}$$

Twisted exact Bethe equations similar $\eta=-1$ grading

$$1 = \frac{\alpha^2 y_2}{x_1 x_2} \frac{Q^4|^{++}_{|1}}{Q^4|^{--}_{|1}} \frac{\mathcal{Q}^-_{12|2}}{\mathcal{Q}^+_{12|2}}\Bigg|_{u_{4,k}}, \qquad \text{with } Q_{1|1}(u_{4,k})=0,$$

$$1 = \frac{\beta^2 y_2}{x_1 x_2} \frac{Q^{++}_{1|1}}{Q^{--}_{1|1}} \frac{\mathcal{Q}^-_{12|2}}{\mathcal{Q}^+_{12|2}}\Bigg|_{u_{\bar{4},k}}, \qquad \text{with } Q^4_{|1}(u_{\bar{4},k})=0,$$

$$1 = \frac{\alpha\beta y_2}{x_2} \frac{Q^+_{1|1}}{Q^-_{1|1}} \frac{Q^4|^+_{|1}}{Q^4|^-_{|1}} \frac{\mathcal{Q}^-_{2|2}}{\mathcal{Q}^+_{2|2}}\Bigg|_{u_{\tilde{3},k}}, \qquad \text{with } \mathcal{Q}_{12|1}(u_{\tilde{3},k})=0,$$

$$-1 = \frac{x_1}{x_2} \frac{\mathcal{Q}^{--}_{2|2}}{\mathcal{Q}^{++}_{2|2}} \frac{\mathcal{Q}^+_{12|2}}{\mathcal{Q}^-_{12|2}} \frac{\mathbf{Q}^+_2}{\mathbf{Q}^-_2}\Bigg|_{u_{2,k}}, \qquad \text{with } \mathcal{Q}_{2|2}(u_{2,k})=0,$$

$$1 = \frac{y_2}{x_2} \frac{\mathcal{Q}^-_{2|2}}{\mathcal{Q}^+_{2|2}}\Bigg|_{u_{\tilde{1},k}}, \qquad \text{with } \mathbf{Q}_2(u_{\tilde{1},k})=0.$$

# Solution for $sl(2)$ sector

We will look for solution at weak coupling in the form $(\mathbf{P}_0 = \mathbf{P}_5 = \mathbf{P}_6)$

$$\mathbf{P}_1 = (xh)^{-L}\mathbf{p}_1 = (xh)^{-L}\left(1 + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{1,k}^{(l)} \frac{h^{2l+k}}{x^k}\right), \quad \nu_i(u) = \sum_{l=1}^{\infty} h^{2l-L}\nu_i^{(l)}(u),$$

$$\mathbf{P}_2 = (xh)^{-L}\mathbf{p}_2 = (xh)^{-L}\left(\frac{h}{x} + \sum_{k=2}^{\infty}\sum_{l=0}^{\infty} c_{2,k}^{(l)} \frac{h^{2l+k}}{x^k}\right),$$

$$\mathbf{P}_0 = (xh)^{-L}\mathbf{p}_0 = (xh)^{-L}\left(\sum_{l=0}^{\infty} A_0^{(l)} h^{2l} u^L + \sum_{j=0}^{L-1}\sum_{l=0}^{\infty} m_j^{(l)} h^{2l} u^j + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{0,k}^{(l)} \frac{h^{2l+k}}{x^k}\right),$$

$$\mathbf{P}_3 = (xh)^{-L}\mathbf{p}_3 = (xh)^{-L}\left(\sum_{l=0}^{\infty} A_3^{(l)} h^{2l} u^{2L+1} + \sum_{j=0}^{2L}\sum_{l=0}^{\infty} k_j^{(l)} h^{2l} u^j + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{3,k}^{(l)} \frac{h^{2l+k}}{x^k}\right),$$

where

$$x \equiv x(u) = \frac{u + \sqrt{u^2 - 4h^2}}{2h}$$

$c_{i,k}^{(l)}$ are some functions of spin $S$ only, otherwise they are just constants. The analytically continued though the cut functions are defined as

$$\tilde{\mathbf{P}}_i = \left(\frac{x}{h}\right)^L \tilde{\mathbf{p}}_i\,, \quad \tilde{\mathbf{p}}_i = \mathbf{p}_i\Big|_{x\to 1/x}.$$

**Initial conditions for iterative solution:**

$$\mathbf{p}_{1,0} = 1, \qquad \mathbf{p}_{2,0} = 0,$$
$$\widetilde{\mathbf{p}}_{1,0}(u) \sim 1 + O(u), \qquad \widetilde{\mathbf{p}}_{2,0}(u) \sim u + O(u^2).$$

**Baxter equations to solve** (state quantum numbers are specified by LO Baxter polynomial $Q(u) \sim \nu_1^{[1]}(u)$ ) :

$$\frac{\nu_1^{[3]}}{\mathbf{P}_1^{[1]}} - \frac{\nu_1^{[-1]}}{\mathbf{P}_1^{[-1]}} - \sigma\left(\frac{\mathbf{P}_0^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_0^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_1^{[1]} = -\sigma\left(\frac{\mathbf{P}_2^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_2^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_2^{[1]}.$$

$$\frac{\nu_2^{[3]}}{\mathbf{P}_1^{[1]}} - \frac{\nu_2^{[-1]}}{\mathbf{P}_1^{[-1]}} + \sigma\left(\frac{\mathbf{P}_0^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_0^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_2^{[1]} = \sigma\left(\frac{\mathbf{P}_3^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_3^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_1^{[1]}.$$

where

$$\sigma \equiv e^{i\mathcal{P}} = Q^{[1]}(0)/Q^{[-1]}(0)\,, \quad Q \text{ is LO Baxter polynomial}$$

Marboe, Volin, 2014; Anselmetti, Bombardelli, Cavaglia, Tateo, 2015

**Coefficients are fixed from equations:**

$$\left.\begin{aligned} \nu_a(u)+\widetilde{\nu}_a(u) &= \nu_a(u)+\sigma\,\nu_a^{[2]}(u) \\ \frac{\nu_a(u)-\widetilde{\nu}_a(u)}{\sqrt{u^2-4h^2}} &= \frac{\nu_a(u)-\sigma\,\nu_a^{[2]}(u)}{\sqrt{u^2-4h^2}} \end{aligned}\right\} \quad \text{free of cuts on real axis}$$

$$\begin{aligned} \left(\nu_1+\sigma\,\nu_1^{[2]}\right)\left(\mathbf{p}_0-(hx)^L\right) &= \mathbf{p}_2\,\left(\nu_2+\sigma\,\nu_2^{[2]}\right)-\mathbf{p}_1\,\left(\nu_3+\sigma\,\nu_3^{[2]}\right), \\ \left(\nu_2+\sigma\,\nu_2^{[2]}\right)\left(\mathbf{p}_0+(hx)^L\right) &= \mathbf{p}_3\,\left(\nu_1+\sigma\,\nu_1^{[2]}\right)+\mathbf{p}_1\,\left(\nu_4+\sigma\,\nu_4^{[2]}\right). \end{aligned}$$

$$\sigma\,\nu_1^{[2]} = \mathbf{P}_0\,\nu_1-\mathbf{P}_2\,\nu_2+\mathbf{P}_1\,\nu_3, \quad \widetilde{\mathbf{P}}_2-\mathbf{P}_2=\sigma\,\left(\nu_3\nu_1^{[2]}-\nu_1\nu_3^{[2]}\right),$$

$$\sigma\,\nu_2^{[2]} = -\mathbf{P}_0\,\nu_2+\mathbf{P}_3\,\nu_1+\mathbf{P}_1\,\nu_4, \quad \widetilde{\mathbf{P}}_1-\mathbf{P}_1=\sigma\,\left(\nu_2\nu_1^{[2]}-\nu_1\nu_2^{[2]}\right).$$

Marboe, Volin, 2014; Anselmetti, Bombardelli, Cavaglia, Tateo, 2015

**Take as example first Baxter equation** $(q_1 = \nu_1^{[1]})$:

$$(u + i/2)^L\, q_1^{[2]} - (u - i/2)^L\, q_1^{[-2]} - T_0\, q_1 = -U_1^{[-1]},$$

**using an ansatz** $q_1 = Q\, f_1^{[1]}$ **we have** $(\nabla_\pm\, g = g \mp g^{[2]})$:

$$\nabla_- \left( u^L Q^{[1]} Q^{[-1]} \nabla_+ (f_1) \right) = U_1\, Q^{[1]},$$

**introducing inverse operators** $\nabla_\pm \Psi_\pm g = g$ **we get**

$$f_{1,\text{inhomo}} = \Psi_+ \left( \frac{1}{u^L Q^{[1]} Q^{[-1]}} \Psi_- \left( U_1\, Q^{[1]} \right) \right).$$

**full solution**

$$q_1^{[-1]} = \Phi_{1,\text{per}} Q^{[-1]} + \Phi_{1,\text{anti}}\, \mathcal{Z}^{[-1]} + \Psi_+ \left( \frac{1}{u^L Q^{[1]} Q^{[-1]}} \Psi_- \left( U_1\, Q^{[1]} \right) \right),$$

$$\mathcal{Z}^{[-1]} = Q^{[-1]}\, \Psi_- \left( \frac{1}{u^L Q^{[1]} Q^{[-1]}} \right),$$

Marboe, Volin, 2014; Anselmetti, Bombardelli, Cavaglia, Tateo, 2015

**Next, introducing polynomials** $A$, $B$ :

$$A\,Q^{[1]} + B\,Q^{[-1]} = 1.$$

**from Baxter equation we get** ($R$ is polynomial of degree $L-2$):

$$-A^{[1]}(u^{[-1]})^L + B^{[-1]}(u^{[1]})^L = QR,$$

$$\frac{R}{(u^{[1]}u^{[-1]})^L} = \sum_{k=1}^{L}\left(\frac{r_{k,+}}{(u^{[1]})^k} + \frac{r_{k,-}}{(u^{[-1]})^k}\right), \quad C = \frac{A}{u^L} - Q^{[-1]}\sum_{k=1}^{L}\frac{r_{k,+}}{u^k},$$

**Then the solution reads**

$$\mathcal{Z}^{[-1]} = \left(C + Q^{[-1]}\sum_{k=1}^{L}(-r_{k,+} + r_{k,-})\,\eta_{-k}(u)\right),$$

$$f_{1,\text{inhomo}} = \Psi_-\left(U_1 Q^{[1]}\right) C + Q^{[-1]}\Psi_+\left(\Psi_-\left(U_1 Q^{[1]}\right)\sum_{k=1}^{L}\frac{-r_{k,+} + r_{k,-}}{u^k} + C^{[2]}U_1\right)$$

**Solution is expressed in terms of polynomials, rational functions and generalized Hurwitz functions**

$$\eta_{a_1,a_2,\ldots,a_k}(u) = \sum_{n_k > n_{k-1} > \cdots > n_1 \geq 0}\ \prod_{i=1}^{k}\frac{(\text{sgn}(a_i))^{n_i - n_{i-1} - 1}}{(u + i n_i)^{|a_i|}},$$

**Studies with Mellin space techniques:**

Faddeev, Korchemsky, 1995
Kotikov, Rej, Zieme, 2008
Beccaria, Belitsky, Kotikov, Zieme, 2010
Lee, Onishchenko, 2017

**Baxter equations for twist $L = 1$:**

$$(u + i/2)q_1(u + i) - i(2S + 1)q_1(u) - (u - i/2)q_1(u - i) = V_1$$

$$(u + i/2)q_2(u + i) + i(2S + 1)q_2(u) - (u - i/2)q_2(u - i) = V_2$$

$$q_1(u) = \nu_1^{[1]}(u)\,, \quad q_2(u) = \nu_2^{[1]}(u)$$

**As coefficients are linear functions, then Mellin transform will result in a first order differential equation!**

**Mellin transformation:**

$$f(u) = \frac{1}{K_1(u)} \int_0^\infty d\omega\, \omega^{iu-1} \frac{\sqrt{\omega}}{1+\omega} \tilde{f}(\omega)$$

$$\tilde{f}(\omega) = \frac{1+\omega}{2\pi} \int_{-\infty}^{\infty} du\, \omega^{-iu-\frac{1}{2}} K_1(u) f(u)$$

**where** $K_1(u) = \Gamma\left(\frac{1}{2}+iu, \frac{1}{2}-iu\right) = \frac{\pi}{\cosh(\pi u)}$, **or with** $z = \frac{\omega}{1+\omega}$ **as**

$$f(u) = \frac{1}{K_1(u)} \int_0^1 dz\, z^{iu-\frac{1}{2}} \bar{z}^{-iu-\frac{1}{2}} \tilde{f}(z)$$

$$\tilde{f}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du\, z^{-iu-\frac{1}{2}} \bar{z}^{iu-\frac{1}{2}} K_1(u) f(u)$$

## Mellin transform properties:

$$\widetilde{uf(u)}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, z^{-iu-\frac{1}{2}}\bar{z}^{iu-\frac{1}{2}} K_1(u)\, uf(u) = \frac{i}{2}\left(\bar{z} - z + 2z\bar{z}\partial_z\right)\tilde{f}(z)$$

$$f\widetilde{(u+i)}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, z^{-iu-\frac{1}{2}}\bar{z}^{iu-\frac{1}{2}} K_1(u)\, f(u+i)$$

$$= -\frac{\bar{z}}{z}\tilde{f}(z) + \frac{1}{z} f(i/2) \text{ (if } f(u) \text{ does not have poles for } 0 < \Im u_n < i).$$

$$f\widetilde{(u-i)}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, z^{-iu-\frac{1}{2}}\bar{z}^{iu-\frac{1}{2}} K_1(u)\, f(u-i)$$

$$= -\frac{z}{\bar{z}}\tilde{f}(z) + \frac{1}{\bar{z}} f(-i/2) \text{ (if } f(u) \text{ does not have poles for } -i < \Im u_n < 0).$$

## Mellin transform properties continued:

$$\frac{1}{K_1(u)}\int_0^1 dz\, z^{iu-\frac{1}{2}}\bar{z}^{-iu-\frac{1}{2}}z^k = \frac{\Gamma\left(\frac{1}{2}+iu+k,\frac{1}{2}-iu\right)}{K_1(u)\,k!} = \frac{\left(\frac{1}{2}+iu\right)_k}{k!}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} du\, z^{-iu-\frac{1}{2}}\bar{z}^{iu-\frac{1}{2}}K_1(u)\,u^k = \frac{i^k}{\bar{z}}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^k\left(-\frac{z}{\bar{z}}\right)^n = \frac{i^k}{\bar{z}}\left(\frac{1}{2}+z\bar{z}\partial_z\right)^k\bar{z}$$

**and** $\left(\hat{u} = \frac{i}{2}\left(\bar{z}-z+2z\bar{z}\partial_z\right) = \sqrt{z\bar{z}}\, i\partial_z\sqrt{z\bar{z}}\right)$:

$$\hat{u}\left(\bar{z}-z\right)^S = \frac{i}{2}\left[(S+1)\left(\bar{z}-z\right)^{S+1} - S\left(\bar{z}-z\right)^{S-1}\right]$$

$$\hat{u}^2\left(\bar{z}-z\right)^S = -\frac{1}{4}\Big[(S+2)(S+1)\left(\bar{z}-z\right)^{S+2} - \left((S+1)^2+S^2\right)\left(\bar{z}-z\right)^S + S(S-1)\left(\bar{z}-z\right)^{S-2}\Big]$$

# Solution of Baxter equations in Mellin space

**Mellin transformed Baxter equations take the form:**

$$(\bar{z}-z)\,\Psi_1'(z)+2S\Psi_1(z)=i\widetilde{V}_1(z)\,,\quad (\bar{z}-z)\,\Psi_2'(z)-2(S+1)\,\Psi_2(z)=i\widetilde{V}_2(z)$$

**where** $\Psi_{1,2}(z)=\widetilde{q_{1,2}(u)}$ **and** $\left(\hat{u}=\frac{i}{2}(\bar{z}-z+2z\bar{z}\partial_z)=\sqrt{z\bar{z}}\,i\partial_z\sqrt{z\bar{z}}\right)$ :

$$\widetilde{V}_1(z)=\sigma\left[\left(\frac{\mathbf{P}_2}{\mathbf{P}_1}\right)_{u\to\hat{u}-\frac{i}{2}}-\left(\frac{\mathbf{P}_2}{\mathbf{P}_1}\right)_{u\to\hat{u}+\frac{i}{2}}\right]\Psi_2$$

$$+\left[\frac{\bar{z}}{z}\left(\frac{1}{\mathbf{P}_1}\right)_{u\to\hat{u}-\frac{i}{2}}-\frac{z}{\bar{z}}\left(\frac{1}{\mathbf{P}_1}\right)_{u\to\hat{u}+\frac{i}{2}}+\sigma\left(\frac{\mathbf{P}_0}{\mathbf{P}_1}\right)_{u\to\hat{u}+\frac{i}{2}}-\sigma\left(\frac{\mathbf{P}_0}{\mathbf{P}_1}\right)_{u\to\hat{u}-\frac{i}{2}}\right]\Psi_1$$

$$-\frac{1}{z}\frac{q_1(i/2)}{\mathbf{P}_1(0)}+\frac{1}{\bar{z}}\frac{q_1(-i/2)}{\mathbf{P}_1(0)}$$

$$\widetilde{V}_2(z)=-\sigma\left(\left(\frac{\mathbf{P}_3}{\mathbf{P}_1}\right)_{u\to\hat{u}-\frac{i}{2}}-\left(\frac{\mathbf{P}_3}{\mathbf{P}_1}\right)_{u\to\hat{u}+\frac{i}{2}}\right)\Psi_1$$

$$+\left[\frac{\bar{z}}{z}\left(\frac{1}{\mathbf{P}_1}\right)_{u\to\hat{u}-\frac{i}{2}}-\frac{z}{\bar{z}}\left(\frac{1}{\mathbf{P}_1}\right)_{u\to\hat{u}+\frac{i}{2}}-\sigma\left(\frac{\mathbf{P}_0}{\mathbf{P}_1}\right)_{u\to\hat{u}+\frac{i}{2}}+\sigma\left(\frac{\mathbf{P}_0}{\mathbf{P}_1}\right)_{u\to\hat{u}-\frac{i}{2}}\right]\Psi_2$$

$$-\frac{1}{z}\frac{q_2(i/2)}{\mathbf{P}_1(0)}+\frac{1}{\bar{z}}\frac{q_2(-i/2)}{\mathbf{P}_1(0)}$$

## Solution of Baxter equations in Mellin space:

$$\left((\bar{z}-z)^{-S}\,\Psi_1(z)\right)' = i\,(\bar{z}-z)^{-S-1}\,\widetilde{V}_1\,(z)$$

$$\Psi_1(z) = i\,(\bar{z}-z)^{S}\int(\bar{z}-z)^{-S-1}\,\widetilde{V}_1\,(z)\,dz$$

$$\left((\bar{z}-z)^{S+1}\,\Psi_2(z)\right)' = i\,(\bar{z}-z)^{S}\,\widetilde{V}_2\,(z)$$

$$\Psi_2(z) = i\,(\bar{z}-z)^{-S-1}\int(\bar{z}-z)^{S}\,\widetilde{V}_2\,(z)\,dz$$

## At LO

$$\widetilde{V}_1^{(0)}\,(z) = 0, \quad \Psi_1^{(0)}(z) = \alpha\,(\bar{z}-z)^{S}\ ,$$

$$\widetilde{V}_2^{(0)}\,(z) == i\sigma\left[A_3^{(0)}\left(3\hat{u}^2-\frac{1}{4}\right)+2k_2^{(0)}\hat{u}+k_1^{(0)}\right]\Psi_1^{(0)}(z)$$

$$\Psi_2^{(0)}(z) = \sigma\frac{1}{2}\alpha\Bigg[-\frac{1}{4}A_3^{(0)}\Bigg(3\,(S+2)\,(S+1)\,\frac{(\bar{z}-z)^{S+2}}{2S+3}$$

$$-2\left[3S^2+3S+1\right]\frac{(\bar{z}-z)^{S}}{2S+1}+3S\,(S-1)\,\frac{(\bar{z}-z)^{S-2}}{2S-1}\Bigg)+k_1^{(0)}\frac{(\bar{z}-z)^{S}}{2S+1}\Bigg]$$

**At NLO for first Baxter equation we have**

$$V_1^{(1)} = \frac{4i\sigma q_2^{(0)}(u)}{1+4u^2} + \left(c_{1,1}^{(0)} + \frac{2}{2u+i}\right) q_1^{(0)}(u+i) - \left(c_{1,1}^{(0)} + \frac{2}{2u-i}\right) q_1^{(0)}(u-i) + \sigma\left\{iA_0^{(1)} - \frac{4i}{1+4u^2}\left(c_{0,1}^{(0)} - c_{1,1}^{(0)} m_0^{(0)}\right)\right\} q_1^{(0)}(u)\,.$$

**Transition to Mellin space is done with formula like this**

$$\widetilde{\frac{q_1^{(0)}(u+i)}{u+i/2}} = \alpha\left[i\bar{z}\partial_z z\right]^{-1}\left[-\frac{\bar{z}}{z}(\bar{z}-z)^S + \frac{1}{z}\right] = i\frac{\alpha}{z}\partial_z^{-1} z^{-1}\left[(\bar{z}-z)^S - \frac{1}{\bar{z}}\right]$$
$$= i\frac{\alpha}{z}\int_0^z \frac{dx}{x}\left[(\bar{x}-x)^S - \bar{x}^{-1}\right] = i\frac{\alpha}{z}\left[\ln\bar{z} + G(S,z)\right]$$

**where**

$$G(S,z) = \int_0^z \frac{dx}{x}\left[(\bar{x}-x)^S - \bar{x}^{-1}\right] - \ln\bar{z} = \int_0^z \frac{dx}{x}\left[(\bar{x}-x)^S - 1\right]$$
$$= \partial_z^{-1} z^{-1}\left[(\bar{z}-z)^S - 1\right] = \sum_{j=1}^{S}\frac{1}{j}\left((\bar{z}-z)^j - 1\right).$$

**Finally, at NLO we have** ($w \equiv \bar{z} - z,\; B_1(S) \equiv S_1(S) - S_{-1}(S)$):

$$\frac{\widetilde{V}_1^{(1)}(z)}{\alpha i} = w^S(\sigma A_0^{(1)} - 2(2S-1)B_1(S)) + 2\sum_{n=0}^{S-1}(1+\sigma(-1)^n)\left\{B_1(S)+B_1(n)\right\}w^n\,,$$

$$\Psi_1^{(1)}(z) = \frac{\alpha}{2}\left(\sigma A_0^{(1)} - 2(2S-1)B_1(S)\right)w^S\log w - \alpha\sum_{n=1}^{S}\frac{1+(-1)^n}{n}w^{S-n}\left\{B_1(S)+B_1(S-n)\right\}\,,$$

**Expressions for $\widetilde{V}_2^{(1)}$ and $\Psi_2^{(1)}(z)$ are too big to be shown on slides.**

**To set up $c_{i,k}^{(l)}$ coefficients we need to go back to $u$-space!**

**Mellin transformation of $(\bar{z}-z)^S \log(\bar{z}-z)$:**

$$I_{\pm} = \frac{\cosh \pi u}{\pi} \int_0^1 dz\, z^{iu-1/2} \bar{z}^{-iu-1/2} \left(\bar{z} - z \pm i0\right)^{\beta} \ ,$$

**uncertainty of analytical continuation:**

$$I_{\pm} = \Gamma(\beta+1) \left[ \frac{{}_2F_1\left(\beta+1, iu+\frac{1}{2}; \beta+iu+\frac{3}{2}; -1\right)}{\Gamma\left(\frac{1}{2}-iu\right)\Gamma\left(\beta+iu+\frac{3}{2}\right)} \right. \left. + e^{\pm i\pi\beta} \frac{{}_2F_1\left(\beta+1, -iu+\frac{1}{2}; \beta-iu+\frac{3}{2}; -1\right)}{\Gamma\left(\frac{1}{2}+iu\right)\Gamma\left(\beta-iu+\frac{3}{2}\right)} \right] ,$$

**where**

$$Q_{\rm up}\left(S,u\right) = (-1)^S\, \Gamma(S+1) \frac{{}_2F_1\left(S+1, -iu+\frac{1}{2}; S-iu+\frac{3}{2}; -1\right)}{\Gamma\left(\frac{1}{2}+iu\right)\Gamma\left(S-iu+\frac{3}{2}\right)}$$

$$Q_{\rm down}\left(S,u\right) = \Gamma(S+1) \frac{{}_2F_1\left(S+1, iu+\frac{1}{2}; S+iu+\frac{3}{2}; -1\right)}{\Gamma\left(\frac{1}{2}-iu\right)\Gamma\left(S+iu+\frac{3}{2}\right)} \ .$$

$$Q_{\rm up}\left(S,u\right) + Q_{\rm down}\left(S,u\right) = Q\left(S,u\right) \ .$$

**are solutions of homogeneous Baxter equation**

**Mellin transformation of arising functions ($w \equiv \bar{z} - z$):**

$$w^S \xrightarrow{M} Q(S,u)\,, \quad Q\left(S,u\right) = \frac{(-1)^S\Gamma\left(\frac{1}{2}+iu\right)}{S!\Gamma\left(\frac{1}{2}+iu-S\right)}\,{}_2F_1\left(-S,\frac{1}{2}+iu;\frac{1}{2}+iu-S;-1\right)\,.$$

$$w^S\log w \xrightarrow{M} -iQ\left(S,u\right)\eta_1\left(u+\frac{i}{2}\right)+\sum_{n=1}^{\left\lfloor\frac{S}{2}\right\rfloor}\frac{1}{n}Q(S-2n,u)+\text{hom. solution}$$

$$w\log w \xrightarrow{M} \frac{2i}{2u+i}-2+2ui\left(1-\log 2-\zeta_1-i\bar{\eta}_1(u+\frac{i}{2})\right)\,,$$

$$\log w \xrightarrow{M} -\frac{2i}{2u+i}+\log 2+\zeta_1+i\bar{\eta}_1(u+\frac{i}{2})\,,$$

## Mellin transformation of arising functions continued:

$$w^S \log(1-w) \xrightarrow{M} Q(S,u)\log 2 + \sum_{k=0}^{S} \binom{S}{k} \frac{(-1)^k}{S!} (1/2+iu)_k (1/2-iu)_{S-k}$$
$$\times \left\{ \zeta_1 + i\bar{\eta}_1(u-\frac{i}{2}) - i\sum_{m=1}^{k} \frac{1}{u-im+i/2} - S_1(S) \right\} ,$$

$$w^S \log(1+w) \xrightarrow{M} Q(S,u)\log 2 + \sum_{k=0}^{S} \binom{S}{k} \frac{(-1)^{S-k}}{S!} (1/2-iu)_k (1/2+iu)_{S-k}$$
$$\times \left\{ \zeta_1 - i\eta_1(u+\frac{i}{2}) + i\sum_{m=1}^{k} \frac{1}{u+im-i/2} - S_1(S) \right\} ,$$

$$w\log(1-w) \xrightarrow{M} -1 + 2iu\left(1 - \log 2 - \zeta_1 - i\bar{\eta}_1(u-\frac{i}{2})\right) ,$$
$$w\log(1+w) \xrightarrow{M} 1 - 2iu\left(-1 + \log 2 + \zeta_1 - i\eta_1(u+\frac{i}{2})\right) .$$

**After immediate transition we have only particular solution part, which should be supplemented with general solution of homogeneous equation**

$$q_1^{[-1]} \Leftarrow \Phi_{1,\mathrm{per}} Q^{[-1]} + \Phi_{1,\mathrm{anti}}\, \mathcal{Z}^{[-1]}$$

$$q_2^{[-1]} \Leftarrow \Phi_{2,\mathrm{anti}} Q^{[-1]} + \Phi_{2,\mathrm{per}}\, \mathcal{Z}^{[-1]}$$

**where**

$$\mathcal{Z} = i(-1)^S \sum_{k=0}^{\left\lfloor \frac{S-1}{2} \right\rfloor} \frac{1}{S-k} Q\left(S-1-2k, u\right) + (-1)^S \eta_{-1}(u+\frac{i}{2}) Q\left(S, u\right) \,,$$

$$\eta_{-1}(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{u+in} = \frac{i}{2}\left(\psi\left(-i\frac{u}{2}\right) - \psi\left(-i\frac{u+i}{2}\right)\right) \,.$$

**and**

$$\Phi_{a,\mathrm{per}}(u) = \phi_{a,0}^{\mathrm{per}} + \sum_{j=1}^{\Lambda} \phi_{a,j}^{\mathrm{per}}\, \mathcal{P}_j(u), \qquad \Phi_{a,\mathrm{anti}}(u) = \sum_{j=1}^{\Lambda} \phi_{a,j}^{\mathrm{anti}}\, \mathcal{P}_{-j}(u),$$

$$\mathcal{P}_k(u) = \eta_k(u) + \mathrm{sgn}(k)\, \bar{\eta}_k^{[-2]}(u) = \mathrm{sgn}(k)\, \mathcal{P}_k(u+i), \quad 0 \neq k \in \mathbb{Z},$$

**Values for some of coefficients are given by**

$$
\begin{aligned}
&k_2^{(0)} = 0\,, \quad c_{1,1}^{(0)} = 0\,, \quad A_3^{(0)} = \frac{i(3-4S-4S^2)}{3\alpha^2}\,,\\
&c_{0,1}^{(0)} = \frac{(-1)^S k_1^{(0)}}{2+4S} - 1 - \frac{i(-1)^S S(1+S)}{3(1+2S)\alpha^2}\\
&c_{3,1}^{(0)} = \frac{(-1)^S(2S(1+S)+3i\alpha^2 k_1^{(0)})(12i(1+2S)\alpha^2 - (-1)^S(2S(1+S)+3i\alpha^2 k_1^{(0)}))}{36(1+2S)^2\alpha^4}\,,\\
&\frac{1}{\alpha^2} = -4i(S_1(S) - S_{-1}(S))\,,\\
&A_0^{(1)} = 2(-1)^S(2S+3)B_1(S), \quad \phi_{1,1}^{per} = -2i\alpha B_1(S)\,.
\end{aligned}
$$

**The unsimplified full solution is too big, so we list for the moment only values for some spin values**

$$\gamma^{L=1,S} = 4\,h^2(S_1+S_{-1}) - 16\,h^4\Big(S_{-3}-S_3+S_{-2,-1}-S_{-2,1}+S_{-1,-2}-S_{-1,2}-S_{1,-2} + S_{1,2} - S_{2,-1} + S_{2,1} + S_{-1,-1,-1} - S_{-1,-1,1} - S_{1,-1,-1} + S_{1,-1,1}\Big) + 4\,h^4(S_1 - S_{-1})\,\mathcal{W}(1,S) + O(h^6),$$

**where**

$$\mathcal{W}(1,1) = 4 - 2\zeta_2\,,$$
$$\mathcal{W}(1,2) = \frac{8}{3} - 2\zeta_2\,,$$
$$\mathcal{W}(1,3) = \frac{164}{45} - 2\zeta_2\,,$$
$$\mathcal{W}(1,4) = \frac{932}{315} - 2\zeta_2\,,$$
$$\mathcal{W}(1,5) = \frac{5552}{1575} - 2\zeta_2\,.$$

**in agreement with, for example**

Beccaria, Levkovich-Maslyuk, Macorini, 2010

# Conclusion and future directions

- An extension to six loop and possibly above
- An extension to twist 2 operators
- An extension of computational techniques to twisted ABJM and $\mathcal{N} = 4$ theories
- Solution for arbitrary spin values directly in $u$-space
- Study of untwisting limits

Thank you for your attention!