Bounds on Scaling Dimension in Conformal Quantum Mechanics

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QFT = Quantum Mechanics + Special Relativity

State vector in Hilbert space + Pincare sym.

Fields => Casimir of Poincare alg

Spin & **Mass**

‘1915 Einstein

Maxwell eq. is Lorentz inv.

QFT = Quantum Mechanics + Special Relativity

State vector in Hilbert space + Pincare sym.

Fields => Casimir of Poincare alg

Spin & **Mass**

'1915 Einstein

Maxwell eq. is conformal inv.

'1915 Bateman, Cunningham

CFT = Quantum Mechanics + Conformal sym.

Operators => Casimir of conf alg

Spin & **Scaling dimension**

anomalous dimension,
critical dimension etc.

CFT can be defined via sym properties of the ***correlation functions*** !

$$\left\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle$$

No need Lagrangian !

CQM = Quantum Mechanics + Conformal Symmetry of Time

State vector in Hilbert space + $SL(2,\mathbb{R})$

Operators => Casimir of conf alg

Scaling dimension

⇒ **Energy & Scaling dimension !**

Hamiltonian ≠ dilatation
in CQM

Q. What can we learn CQM from **correlation function** ?

$$\langle \mathcal{O}_{\Delta_1}(t_1)\cdots\mathcal{O}_{\Delta_n}(t_n)\rangle$$

w/o Lagrangian description

Outline

review

I. **Conformal Symmetry of Time**

II. **Lagrangian CQM**

III. **Non-Lagrangian Approach for CQM**

   i C-function in CQM

   ii No-Go thm in CQM

Outline

I. **Conformal Symmetry of Time**

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III. **Non-Lagrangian Approach for CQM**

i C-function in CQM

ii No-Go thm in CQM

new work

Main Results

i. C-function in CQM

cf.

$$\left(c = \int_{S^{2k}} \langle T^{\mu}{}_{\mu} \rangle \right)$$

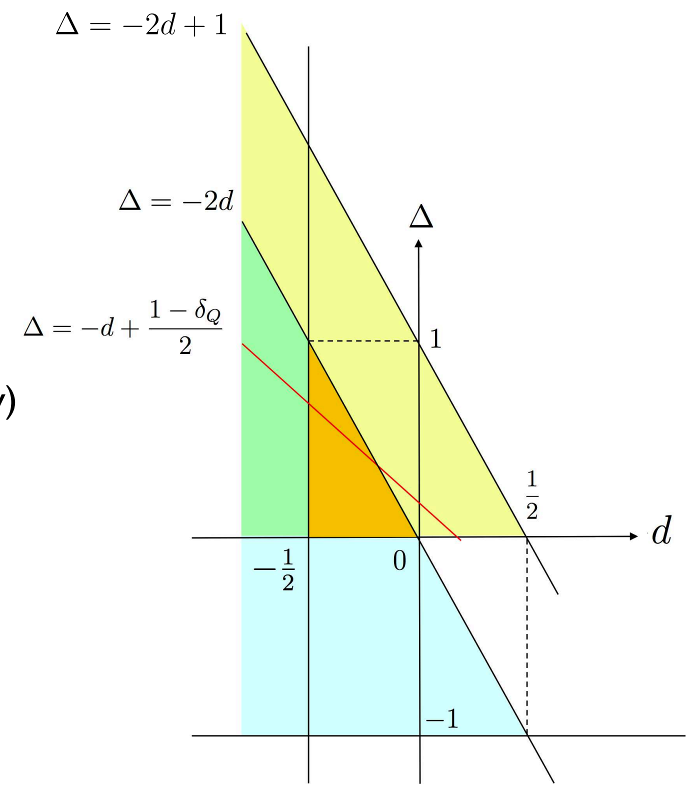
$$D(E) := i\langle E|D|E\rangle$$

Bounds on dimensions of reference state $\boxed{-\frac{1}{2} \le d \le \frac{1}{2}}$ (unitarity)

II. No-Go theorem in CQM

Bounds on dimensions of physical state $\boxed{-\frac{1}{2} \le d + \Delta \le \frac{1}{2}}$ (positivity)

Bounds on dimensions of gauge operator $\boxed{0 < \delta_Q \le 2}$



I. Conformal Symmetry of Time

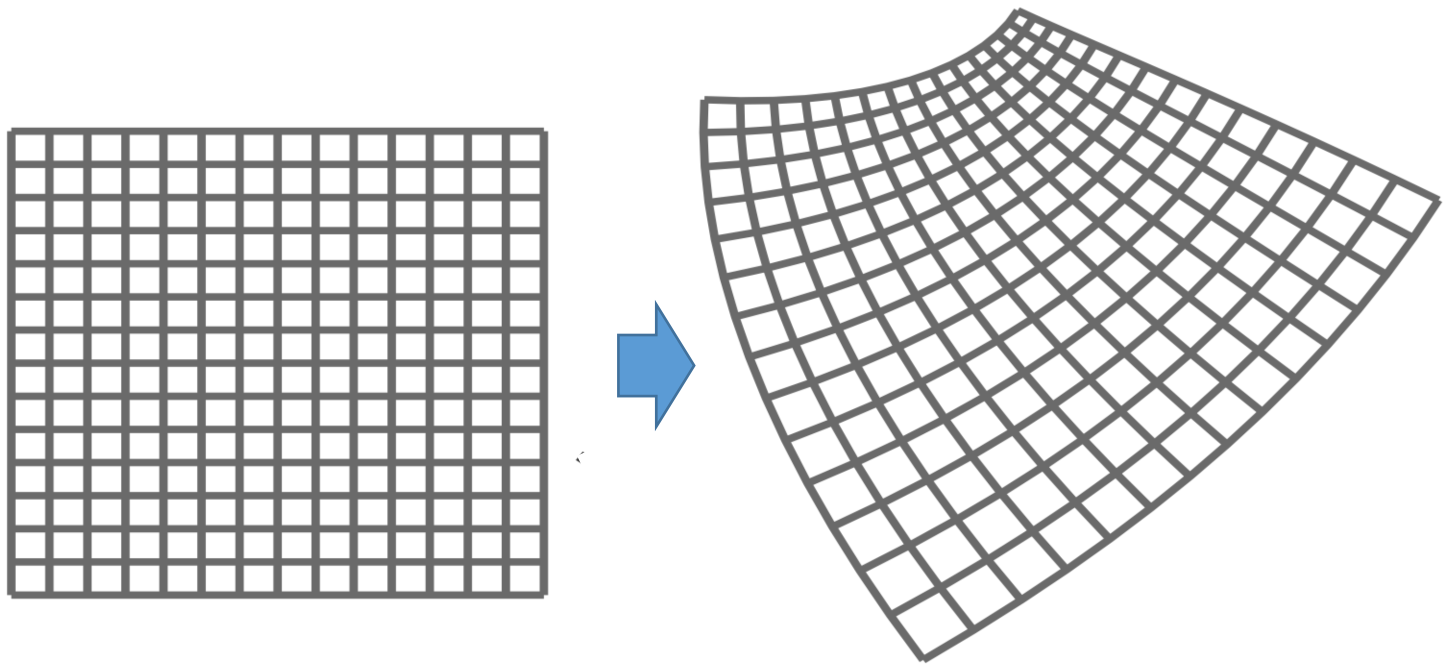
Conformal Symmetry

Conformal Transf = Invertible coordinate map that leaves metric invariant up to scale

$$x \to x' \qquad\qquad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g(x)$$

	<u>finite transf.</u>	<u>infinitesimal transf.</u>
<u>1. Translation</u>	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_{\mu} = -i\partial_{\mu}$
<u>2. Dilatation</u>	$x'^{\mu} = \alpha x^{\mu}$	$D = -ix^{\mu}\partial_{\mu}$
<u>3. Rotation</u>	$x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$	$L_{\mu\nu} = i(x_{\mu}x_{\nu} - x_{\nu}x_{\mu})$
<u>4. SCT</u>	$x'^{\mu} = \dfrac{x^{\mu} - b^{\mu}x^2}{1 - 2 \cdot x + b^2 x^2}$	$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu})$

Extension of Poincare sym.



Locally preserving angle
between 2 distinct points

Conformal Transformation of Time

$$x^\mu \to x^\mu + \epsilon^\mu$$

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$$

$$f(x) g_{\mu\nu} \xrightarrow{\text{solve}} f(x) = \frac{2}{d} \partial_\mu \epsilon^\mu$$

$$\partial_\mu \epsilon^\mu = A + B_\mu x^\mu$$

d=1

$$\Rightarrow \begin{cases} (d-1)\partial^2 f = 0 \\ 2\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f \end{cases}$$

no constraint

$$\Rightarrow \quad \mathrm{Conf}(\mathbb{R}) \cong \mathrm{Diff}(\mathbb{R})$$

$$\mathfrak{conf}(\mathbb{R}) = C^\infty(\mathbb{R})$$

But…

- Definition through **metric tensor** & absence of **angle** shows that d=1 is completely different.

- $\mathrm{Diff}(\mathbb{R})$ essentially requires the presence of **gravity**.

$\Downarrow$

Appearance of D & K_μ
is intrinsically motivated time symmetry !

Infinitesimal transf.

1. Translation $\quad P_{\cancel{\mu}} = -i\partial_{\cancel{\mu}}$

2. Dilatation $\quad D = -ix^{\cancel{\mu}}\partial_{\cancel{\mu}}$

~~3. Rotation~~ $\quad \cancel{L_{\mu\nu} = i(x_\mu x_\nu - x_\nu x_\mu)}$

4. SCT $\quad K_{\cancel{\mu}} = -i(\cancel{2x_\mu x^\nu \partial_\nu} - x^2\partial_{\cancel{\mu}})$

1. Translation $\quad H = i\dfrac{\partial}{\partial t}$

2. Dilatation $\quad D = it\dfrac{\partial}{\partial t}$

3. SCT $\quad K = it^2\dfrac{\partial}{\partial t}$

$$[H, D] = iH \qquad [K, D] = -iK \qquad [H, K] = 2iD$$

$$[T_i, T_j] = i\epsilon_{ijk}T^k \qquad T_0 = \frac{1}{2}\left(\frac{K}{a} + aH\right) \quad T_1 = D \quad T_2 = \frac{1}{2}\left(\frac{K}{a} - aH\right)$$

$$\boxed{\mathfrak{conf}(\mathbb{R}) := \mathfrak{so}(1,2) \cong \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{sp}(2)}$$

Conformal Symmetry of Time

New time

$$G = uH + vD + wK \qquad\qquad d\tau = \frac{dt}{u + vt + wt^2}$$

$$G|\Psi\rangle = i\frac{\partial}{\partial\tau}|\Psi\rangle$$

Schrodinger Equation

=> G is the new Hamiltonian of new time τ

Discriminant of new time

$$\Delta = v^2 - 4uw$$

$$\tau = \int d\tau = \int_{t_0}^{t} \frac{dt'}{u + vt' + wt'^2} + \tau_0$$

$$= \begin{cases} \frac{1}{\sqrt{\Delta}} \left(\ln \left| \frac{2wt+v-\sqrt{\Delta}}{2wt+v+\sqrt{\Delta}} \right| - \ln \left| \frac{v-\sqrt{\Delta}}{v+\sqrt{\Delta}} \right| \right) & \text{for } \Delta > 0 \\ \frac{2}{\sqrt{|\Delta|}} \left(\tan^{-1} \frac{2wt+v}{\sqrt{|\Delta|}} - \tan^{-1} \frac{v}{\sqrt{|\Delta|}} \right) & \text{for } \Delta < 0 \\ - \left(\frac{2}{2wt+v} - \frac{2}{v} \right) & \text{for } \Delta = 0 \end{cases}$$

3 different classes of new time

finite transf. of time

$$t' = \frac{at+b}{ct+d} \qquad A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \; PSL(2,\mathbb{R})$$

$$\mathrm{det} A = 1$$

1. Translation $\quad t' = t - \epsilon_1 \quad A = \begin{pmatrix} 1 & 0 \\ -\epsilon_1 & 1 \end{pmatrix}$

2. Dilatation $\quad t' = e^{-\epsilon_2} t \quad A = \begin{pmatrix} e^{-\frac{\epsilon_2}{2}} & 0 \\ 0 & e^{\frac{\epsilon_2}{2}} \end{pmatrix}$

3. SCT $\quad t' = \dfrac{t}{\epsilon_3 t + 1} \quad A = \begin{pmatrix} 1 & \epsilon_3 \\ 0 & 1 \end{pmatrix}$

infinitesimal transf. of time

$$\delta t = \epsilon_1 + \epsilon_2 t + \epsilon_3 t^2$$

Difference between CQM & CFT

I. **Time symmetry** taken seriously

(H ≠ D, energy & dimension, no Wick rotation, no radial quantization)

II. **Hilbert space** rather than **Fock space**

(zero point energy, ground state ≠ vacuum) '08 Sen '11 Chamon et al

III. **AdS_2/CFT_1** correspondence

(one-dimensional disconnected bdy, AdS_2 factor in BH)

II. Lagrangian CQM

Lagrangian CQM (DFF model)

'76 de Alfaro Fubini Furlan

d=1

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - g\phi^{\frac{2d}{d-2}}$$

scale inv. scalar field theory

$$S = \frac{1}{2}\int dt \left(\dot{x}^2 - \frac{g}{x^2}\right)$$

$$t' = \frac{at+b}{ct+d} \qquad x'(t') = \frac{x(t)}{ct+d}$$

Hamiltonian $\quad H = \frac{1}{2}(p^2 + \frac{g}{x^2})$

Dilatation generator $\quad D = tH - \frac{1}{4}(xp + px)$

SCT generator $\quad K = t^2 H - \frac{1}{2}t(xp + px) + \frac{1}{2}x^2$

Energy eigenstate $\quad H|E\rangle = E|E\rangle$

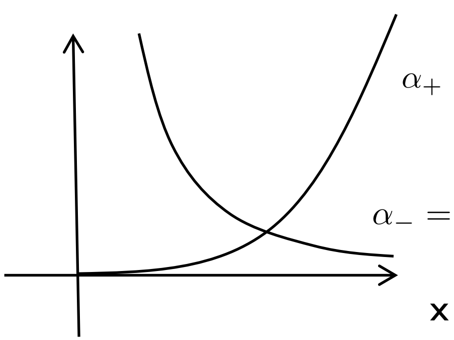
- The spectrum is **continuous**
- The spectrum is **bounded below**
- For E>0, there exists a plane wave **normalizable** state

$$\phi_E(x) = \sqrt{x} J_{\sqrt{g+\frac{1}{4}}}\left(\sqrt{2E}x\right)$$

- For E=0 (vacuum state), the eigenstate is **non-normalizable**

$$\left(-\frac{d^2}{dx^2} + \frac{g}{x^2}\right)\phi(x) = 0$$

$$\phi(x) = x^\alpha$$



$$\alpha_+ = \frac{1+\sqrt{1+4g}}{2}$$

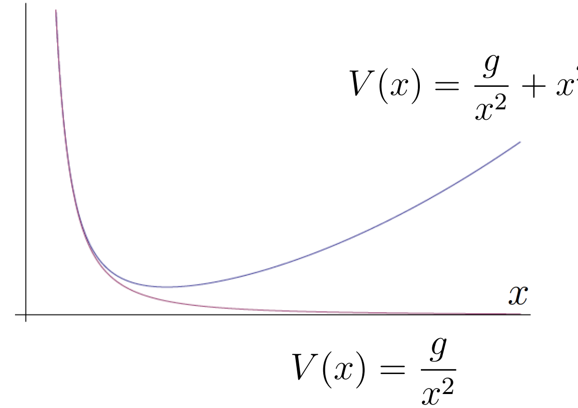
$$\alpha_- = \frac{1-\sqrt{1+4g}}{2}$$

DFF's proposal

Change time and Hamiltonian

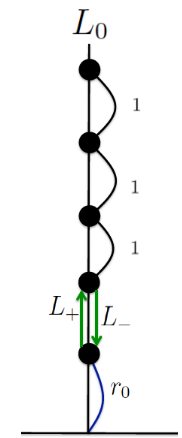
$$\tau = \tan^{-1} t - \tan^{-1} t_0$$

$$G = L_0 = \frac{1}{2}(H + K) \quad \propto p^2 + \frac{g}{x^2} + x^2$$



$$L_\pm = \frac{1}{2}\left(\frac{K}{a} - aH \pm 2iD\right)$$

$$[L_n, L_m] = (m - n)L_{m+n}$$



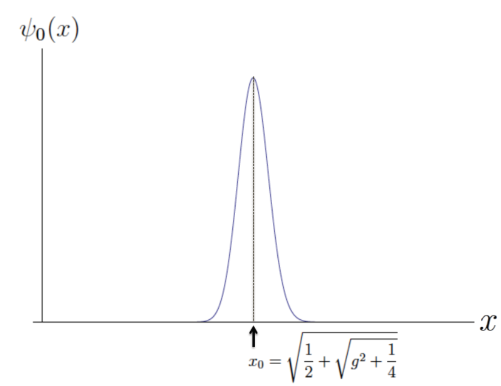
discrete spectrum $\quad L_0|n\rangle = r_n|n\rangle \qquad r_n = r_0 + n, \quad n = 0, 1, 2, \cdots$

$$\mathcal{C}_2 = r_0(r_0 - 1) \qquad r_0 = \frac{1}{2}\left(1 + \sqrt{g^2 + \frac{1}{4}}\right)$$

normalized wave function

$$\phi_0 = \sqrt{\frac{2}{\Gamma(2r_0)}} e^{-\frac{x^2}{2}} x^{\frac{1}{2}+\sqrt{g+\frac{1}{4}}}$$

$$\phi_n(x) = \sqrt{\frac{\Gamma(n+1)}{2\Gamma(n+2r_0)}} x^{-\frac{1}{2}} x^{2r_0} e^{-\frac{x^2}{2}} L_n^{2r_0-1}(x^2)$$



n-point function

'76 de Alfaro et al.

They assume the existence of

$|t\rangle$

$$H|t\rangle = -i\frac{d}{dt}|t\rangle,$$

$$D|t\rangle = -i\left(t\frac{d}{dt}+r_0\right)|t\rangle,$$

$$K|t\rangle = -i\left(t^2\frac{d}{dt}+2r_0 t\right)|t\rangle$$

$$L_0|t\rangle = -\frac{i}{2}\left[\left(a+\frac{t^2}{a}\right)\frac{d}{dt}+2r_0\frac{t}{a}\right]|t\rangle$$

Define $\beta_n := \langle t|n\rangle$

$$\frac{i}{2}\left[\left(a+\frac{t^2}{a}\right)\frac{d}{dt}+2r_0\frac{t}{a}\right]\beta_n = r_n\beta_n$$

$$\beta_n(t) = (-1)^n\left[\frac{\Gamma(2r_0+n)}{n!}\right]^{\frac{1}{2}}\left(\frac{a-it}{a+it}\right)^{r_n}\frac{1}{\left(1+\frac{t^2}{a^2}\right)^{r_0}}.$$

$$\begin{aligned} F_2(t_1,t_2) &= \langle t_1|t_2\rangle \\ &= \sum_n \beta_n(t_1)\beta_n^*(t_2) \\ &= \frac{\Gamma(2r_0)a^{2r_0}}{[2i(t_1-t_2)]^{2r_0}} \propto \frac{1}{(t_1-t_2)^{2r_0}} \end{aligned}$$

'11 Chamon, Jackiw, Santos

Explicit construction of $|t\rangle$

$$|t\rangle = \mathcal{O}(t)|n=0\rangle$$

$$\mathcal{O}(t) = N(t)e^{-i\omega(t)L_+}$$

$$N(t) = \sqrt{\Gamma(2r_0)}\left(\frac{\omega(t)+1}{2}\right)^{2r_0}$$

$$\omega(t) = \frac{a+it}{a-it}$$

- $|t\rangle$ is almost coherent states $\quad (L_- + \omega L_0)|t\rangle = -r_0\omega|t\rangle$
- $|t\rangle$ gives operator-state correspondence $\quad |\Psi\rangle := e^{-Ha}|t=0\rangle = e^{-Ha}e^{-L_+}|n=0\rangle$

$$L_0|\Psi\rangle = r_0|\Psi\rangle$$

$$|\Psi\rangle \propto |n=0\rangle$$

operator		state
e^{-L_+}	$\Leftrightarrow$	$\|n=0\rangle$

operator is not primary & state is not conformal inv.

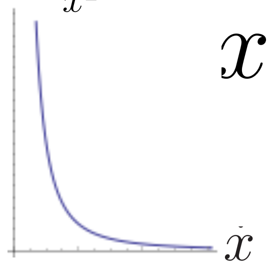
III. No-Lagrangian Approach for CQM

Reconsider the original situation in DFF-model

I. The energy spectrum is **continuous**
II. The vacuum state is **non-normalizable**

They rather indicate that
the quantization needs subtle treatment due to
the constraints on canonical variables

$V(x) = \frac{1}{x^2}$



$$x > 0 \quad \Rightarrow \quad \cancel{[x, p] = i}$$

'16 Strocchi

The situation may not be essentially problem !

The old issues in CQM could be due to an inappropriate quantization manner which naively assume that all variables in Lagrangian are physical variables.

Here we will skip the issue on Lagrangian CQM, but address the Non-Lagrangian CQM as follows

Step1

Introduce 2 ingredients which stem from conformal symmetry

Vacuum State $|\Omega\rangle$ **Primary Operators** $\mathcal{O}_\Delta$

Step2

Construct physical states as

$$\mathcal{O}_{\Delta_1}(t_1)\cdots\mathcal{O}_{\Delta_n}(t_n)|\Omega\rangle$$

Vacuum state $|\Omega\rangle$

Def

$$\boxed{H|\Omega\rangle = 0}$$

Due to the constraints, vacuum is not generically physical state,
but rather the reference state

$$|\Omega\rangle \notin \mathcal{H}$$

Hilbert space

Bra (dual states)

Def

$$\langle \mathrm{bra}| : \quad |\mathrm{ket}\rangle \mapsto \mathbb{C}$$

$SL(2,\mathbb{R})$ Casimir operator $\mathcal{C}_2 = KH + iD - D^2$

imposes constraints on construction of bra

$$\langle \mathrm{bra}|K \Leftrightarrow H|\mathrm{ket}\rangle$$

$$\langle \mathrm{bra}|D \Leftrightarrow D|\mathrm{ket}\rangle$$

$$\langle \mathrm{bra}| \Leftrightarrow D|\mathrm{ket}\rangle$$

We take the normalization $\langle \Omega|\Omega\rangle = 1$

non-normalizable vacuum ?

$\Rightarrow$ No !

such vacuum is generically not physical state,

it cannot be written as wfn of physical obserbables as in DFF.

$d \in \mathbb{R}$

$$D|\Omega\rangle = id|\Omega\rangle$$

scaling dimension of vacuum

$$\langle\Omega|D|\Omega\rangle = id$$

$$\langle\Omega|D^2|\Omega\rangle = -(d + \mathcal{C}_2) = -d^2$$

$$\mathcal{C}_2 = d(d-1)$$

Casimir characterizes the scaling dimension of vacuum of CQM !

※ Conformal boost K increases d by 1

$$D(K^n|\Omega\rangle) = i(d+n)K|\Omega\rangle$$

Energy eigenstate

$$H|E\rangle = E|E\rangle$$

The energy eigenstates in CQM generally have continuous spectrum.

$$H(e^{i\alpha D}|E\rangle) = e^{-\alpha}E(e^{i\alpha D}|E\rangle)$$

Dilatation operator makes such energy eigenstate.

However, again we do not identify generic energy eigenstates $|E\rangle$ with the physical states !

We have to project out non-physical states from $|E\rangle$

Thus far we have no physical input.
We have just viewed the SL(2,R) as conformal symmetry
and introduced reference states for CQM.

Q. How can we construct CQM by imposing physical requirement?

unitary evolution of energy eigenstates $|E\rangle$

D-function

Define $$D(E) := \frac{1}{i}\langle E|D|E\rangle$$

completeness of $|E\rangle$ $$1 = \int dE|E\rangle\langle E|$$

normalization $$\langle E_1|E_2\rangle = \delta(E_1 - E_2)$$

$$D(E) = \frac{1 \pm \sqrt{1 + 4(\mathcal{C}_2 - E^2)}}{2}$$

quantum scaling dimension of $|E\rangle$

Unitarity of the evolution operator for $|E\rangle$
can be realized when $E^2 = \mathcal{C}_2$ as $|E\rangle \sim t^{D(E)=0}$
for a choice of negative sign

$$D(E) = \frac{1 - \sqrt{1 + 4(\mathcal{C}_2 - E^2)}}{2}$$

Q. When can we choose negative sign?

AdS/CFT & BF bound

AdS$_{d+1}$ free scalar with mass m

$$S = \int d^{d+1}x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right)$$

AdS/CFT correspondence claims that
bulk mass m of scalar in AdS$_{d+1}$ is related to
scaling dimension of corresponding boundary CFT$_d$ operator

’98 GKP; Witten

$$\Delta_\pm = \frac{d \pm \sqrt{d^2 + 4m^2}}{2}$$

$-\frac{d^2}{4} + 1 < m^2$ 1 admissible BC of wfn and quantization => Δ_+

$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1$ 2 admissible BC of wfn and 2 quantizations => $\Delta_\pm$

’82 Breitenlohner, Freedman

For AdS_2

$$\Delta_\pm = \frac{d \pm \sqrt{d^2 + 4m^2}}{2} \quad \Rightarrow \quad \Delta_m^\pm = \frac{1 \pm \sqrt{1 + 4m^2}}{2}$$

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1 \quad \Rightarrow \quad -\frac{1}{4} < m^2 < \frac{3}{4}$$

$$\Downarrow$$

$$D(E) = \frac{1 \pm \sqrt{1 + 4(\mathcal{C}_2 - E^2)}}{2}$$

quantum scaling dimension of $|E\rangle$

$$E^2 - \tfrac{1}{4} \leq \mathcal{C}_2 \leq E^2 + \tfrac{3}{4}$$

$$\Downarrow$$

existence of vacuum of scaling dimension d

$$-\tfrac{1}{4} \leq \mathcal{C}_2 \leq \tfrac{3}{4}$$

$$\boxed{-\frac{1}{2} \leq d \leq \frac{1}{2}}$$

Bounds on dimensions
of the reference vacuum state

C-function and C-theorem

A function of coupling constant g and energy scale E
which is defined on the space of theory

- positive & real
- decreasing monotonically along RG-flow
- stationary at RG-fixed pt and equal to crucial parameter in CFT_d

decreasing energy scale

conformal anomaly in d=2k

$$\int_{S^{2k}} \langle T^\mu{}_\mu \rangle$$

'88 Cardy

trace of EM tensor
=> dilatation current

$$\langle D \rangle$$

$$\cancel{\frac{d}{d\alpha} \log Z[S^{2k}] = \frac{V_{2k}}{V_{2k-1}} r^{2k} \langle T^\mu{}_\mu \rangle}$$

$$\cancel{g_{\mu\nu} \to e^{2\alpha} g_{\mu\nu}}$$

not applicable for d=1
w/o metric tensor

c-function: c (2d)
free energy: F (3d)
a-function: a (4d)

counting massless dof
on that energy scale

Actually the D-function $D(E) := \frac{1}{i}\langle E|D|E\rangle$ follow the properties of C-function !

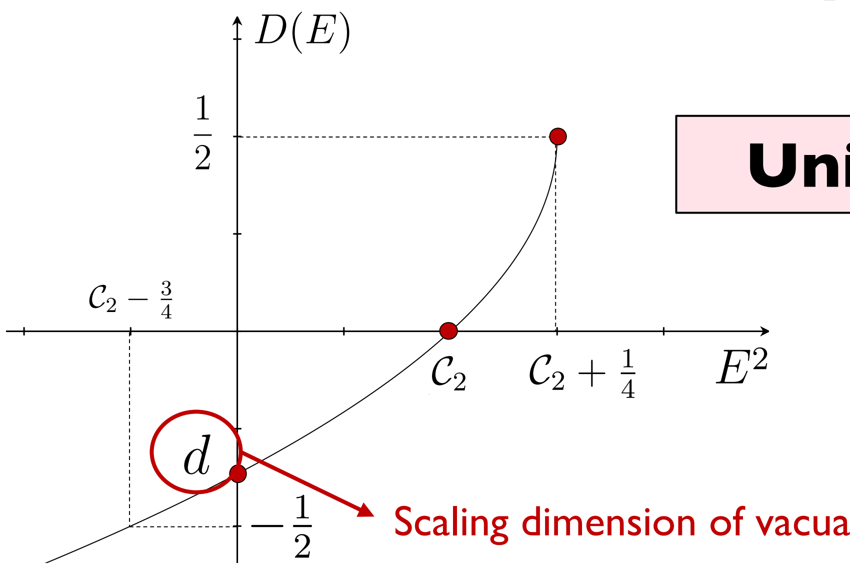
A function of coupling constant g and energy scale E which is defined on the space of theory

e.g. DFF model

$$D(E) = \frac{1-\sqrt{g-4E^2+\frac{1}{4}}}{2}$$

- real
- decreasing monotonically along descrease of energy scale $\frac{dD(E)}{dE} = \frac{2E}{\sqrt{1+4(\mathcal{C}_2-E^2)}} \geq 0$
- stationary at RG-fixed pt and equal to crucial parameter in CQM

$$D(E=0) = d = \frac{1-\sqrt{1+4\mathcal{C}_2}}{2}$$



Unitarity <=> C-theorem

Primary Operators

To describe physical system,
we postulate the presence of **primary operators**

$$\mathcal{O}_\Delta(t) \to \left(\frac{\partial t'}{\partial t}\right)^\Delta \mathcal{O}_\Delta(t')$$
$$= \frac{1}{(ct+d)^{2\Delta}} \mathcal{O}_\Delta(t')$$

$$t' = \frac{at+b}{ct+d}$$

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in PSL(2,\mathbb{R})$$

$$H\mathcal{O}_\Delta(t) = i\dot{\mathcal{O}}_\Delta(t)$$
$$D\mathcal{O}_\Delta(t) = \left(-it\frac{\partial}{\partial t} + \Delta\right)\mathcal{O}_\Delta(t)$$
$$K\mathcal{O}_\Delta(t) = \left(it^2\frac{\partial}{\partial t} - 2t\Delta\right)\mathcal{O}_\Delta(t)$$

Q. What can we learn CQM from **correlation function** ?

$$\langle \mathcal{O}_{\Delta_1}(t_1)\cdots\mathcal{O}_{\Delta_n}(t_n)\rangle \qquad |\mathcal{O}_{\Delta_1}(t_1)\cdots\mathcal{O}_{\Delta_n}(t_n)\rangle = \mathcal{O}_{\Delta_1}(t_1)\cdots\mathcal{O}_{\Delta_n}(t_n)|\Omega\rangle$$

w/o Lagrangian description

Weinberg-Witten Theorem

'80 Weinberg-Witten

I. If theory has Lorentz covariant conserved current, massless charged **spin > 1/2** particle does not exist. <=> **Charge conservation**

II. If theory has Lorentz covariant EM tensor, massless **spin > 1** does not exist. <=> **Energy conservation**

Given the conditions, the following matrix elements should never vanish

$$\langle p', \pm j | J^\mu | p, \pm j \rangle = 0 \qquad j > \frac{1}{2}$$

$$\langle p', \pm j | T^{\mu\nu} | p, \pm j \rangle = 0 \qquad j > 1$$

However, states w/ spin j transform as

$$|p, \pm j\rangle \to e^{\pm i\theta j} |p, \pm j\rangle$$
$$|p', \pm j\rangle \to e^{\mp i\theta j} |p', \pm j\rangle$$

$\Rightarrow$

$$e^{\pm 2i\theta j} \langle p', \pm j | J^\mu | p, \pm j \rangle = R(\theta)^\mu{}_\rho \langle p', \pm j | J^\rho | p, \pm j \rangle$$
$$e^{\pm 2i\theta j} \langle p', \pm j | T^{\mu\nu} | p, \pm j \rangle = R(\theta)^\mu{}_\rho R(\theta)^\nu{}_\sigma \langle p', \pm j | T^{\rho\sigma} | p, \pm j \rangle$$

$$R(\theta)^\mu{}_\rho \Rightarrow e^{i\theta}, e^{-i\theta}$$

*Although the statement is just simple **group theoretical analysis**, the it gives powerful constraint on **spin** and physically meaningful.*

Casimir of Poincare group

No-Go Theorem in CQM

Now consider similar constraints in CQM on **scaling dimension**

Casimir of SL(2,R) conformal group

We take the normalization

$$\langle \mathcal{O}_\Delta | \mathcal{O}_\Delta \rangle = 1$$

$$|\mathcal{O}_{\Delta_1}(t_1)\cdots\mathcal{O}_{\Delta_n}(t_n)\rangle = \mathcal{O}_{\Delta_1}(t_1)\cdots\mathcal{O}_{\Delta_n}(t_n)|\Omega\rangle \qquad \mathcal{O}_\Delta := \mathcal{O}_\Delta(0)$$

Consier

Crucial parameter fixed by the theory that we consider $\mathcal{C}_2 = d(d-1)$

$$\langle \mathcal{O}_\Delta | \mathcal{C}_2 | \mathcal{O}_\Delta \rangle = d(d-1)$$

$$\begin{aligned} \mathcal{C}_2 &= KH + iD - D^2 \\ &= HK - iD - D^2 \end{aligned}$$

$$\left.\begin{aligned} &\langle \mathcal{O}_\Delta | HK | \mathcal{O}_\Delta \rangle + (d+\Delta)(d+\Delta+1) \\ &\langle \mathcal{O}_\Delta | KH | \mathcal{O}_\Delta \rangle + (d+\Delta)(d+\Delta-1) \end{aligned}\right\} = d(d-1)$$

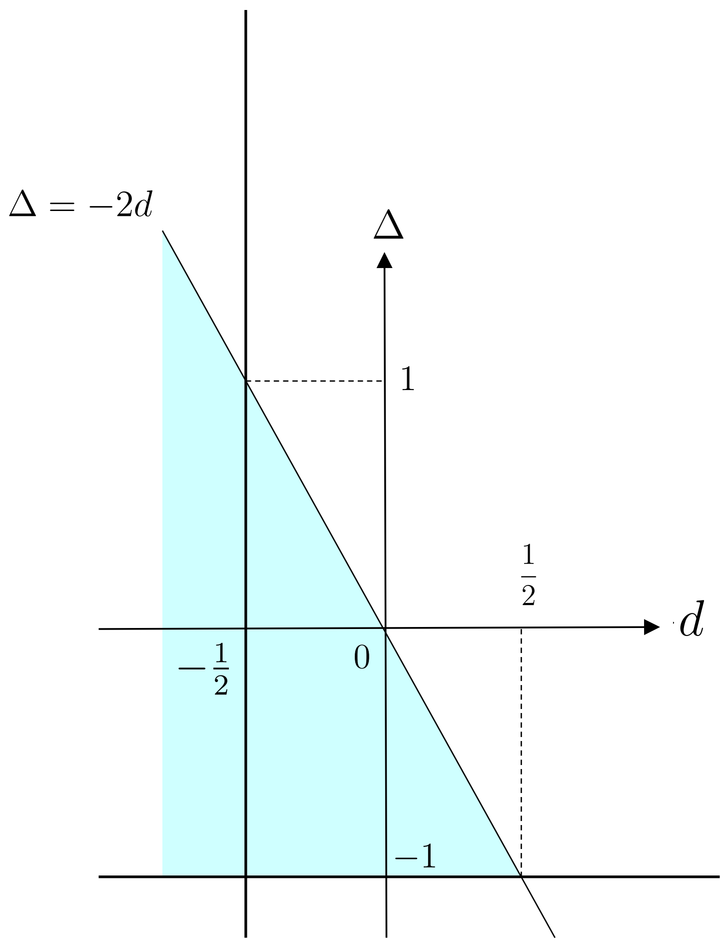
$$\langle \mathrm{bra} | K \Leftrightarrow H | \mathrm{ket} \rangle$$

$$\left.\begin{aligned} \langle \mathcal{O}_\Delta | KH | \mathcal{O}_\Delta \rangle &= |H|\mathcal{O}_\Delta\rangle|^2 \\ \langle \mathcal{O}_\Delta | HK | \mathcal{O}_\Delta \rangle &= |K|\mathcal{O}_\Delta\rangle|^2 \end{aligned}\right\} \geq 0$$

positivity condition

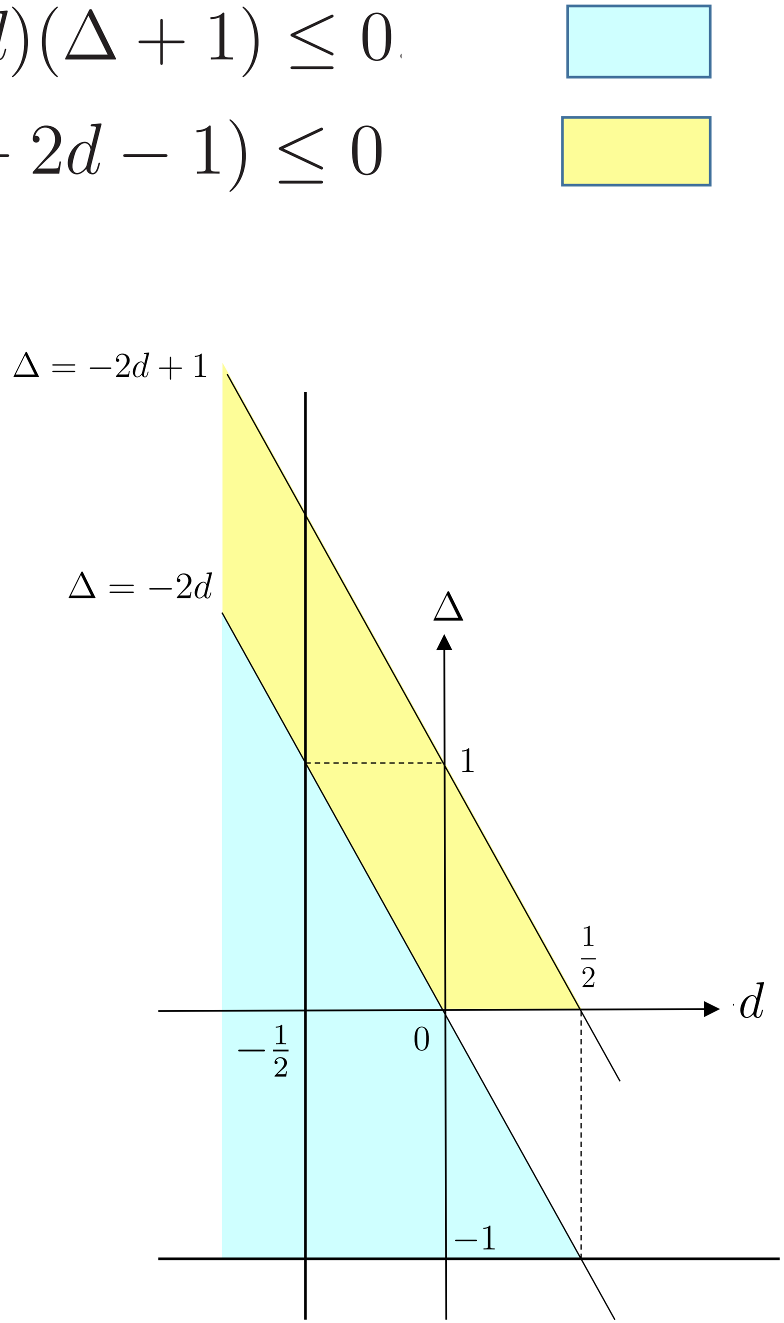
$$(\Delta + 2d)(\Delta + 1) \leq 0.$$

$$\Delta(\Delta + 2d - 1) \leq 0$$



$(\Delta + 2d)(\Delta + 1) \leq 0$

$\Delta(\Delta + 2d - 1) \leq 0$



$$(\Delta + 2d)(\Delta + 1) \leq 0$$

$$\Delta(\Delta + 2d - 1) \leq 0$$

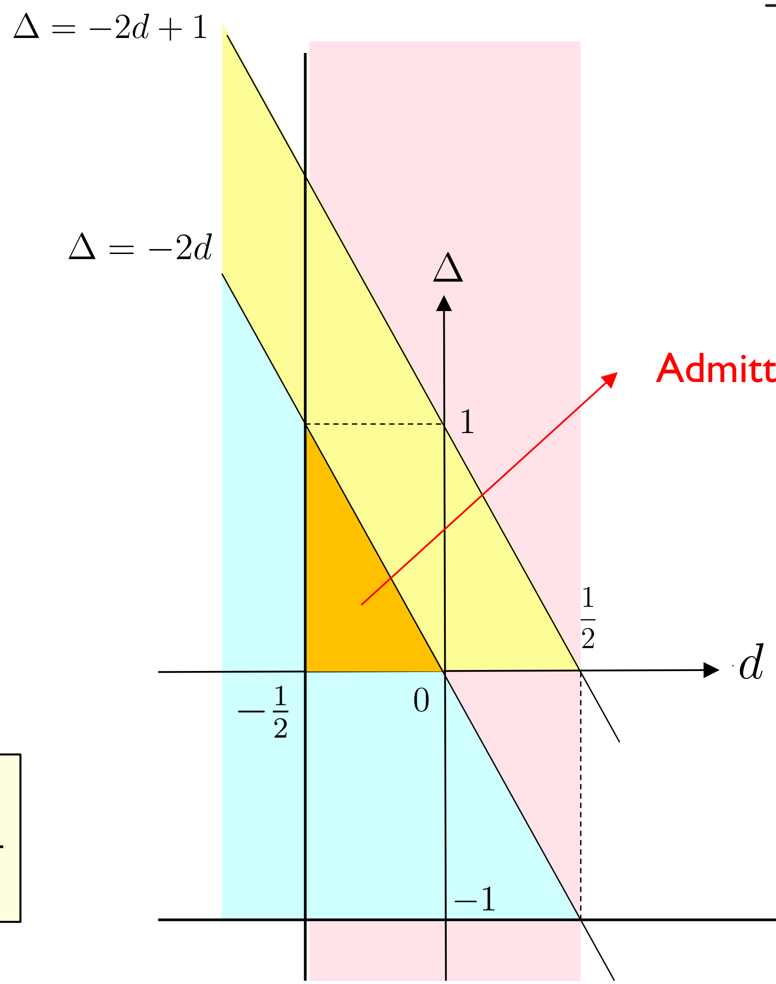
$$-\frac{1}{2} \leq d \leq \frac{1}{2}$$

Physical State

$$\mathcal{O}_{\Delta_1}(t_1) \cdots \mathcal{O}_{\Delta_n}(t_n)|\Omega\rangle$$

Bounds on dimensions of physical state

$$\boxed{-\frac{1}{2} \leq d + \Delta \leq \frac{1}{2}}$$



Bounds on dimensions
of physical state

$$\boxed{-\frac{1}{2} \leq d + \Delta \leq \frac{1}{2}}$$

$$\left\{\begin{array}{ll} d + \Delta = -\frac{1}{2} & \text{free scalars} \\ d + \Delta = 0 & \text{free fermions} \\ d + \Delta = \frac{1}{2} & \text{auxiliary fields} \end{array}\right.$$

The range is consistent with Lagrangian CQM

cf.) supermultiplet of SUSY QM

$$(p, \mathcal{N}, \mathcal{N} - p)$$

#(free scalars) #(free fermions) #(auxiliary fields)

Furthermore consider charge operator Q

$$Q\mathcal{O}_\Delta = q\mathcal{O}_\Delta$$ Giving charge $q \in \mathbb{R}$ to primary operator

$$Q|\Omega\rangle = 0$$ Giving no charge to reference vacuum state

$$[H, Q] = 0$$ non-dynamical operator (auxiliary field)

$$[D, Q] = i\delta_Q Q$$ Having scaling dimension δ_Q

Consier

$$\langle\mathcal{O}_\Delta|[K, Q]H|\mathcal{O}_\Delta\rangle$$

$$[K, Q]H = 2i\delta_Q QD - \delta_Q^2 Q + \delta_Q Q$$

$$q\delta_Q\left(\delta_Q - 1 + 2(d + \Delta)\right) = 0$$

$$\delta_Q = 0$$

global charge operator

$$d + \Delta = \frac{1 - \delta_Q}{2}$$

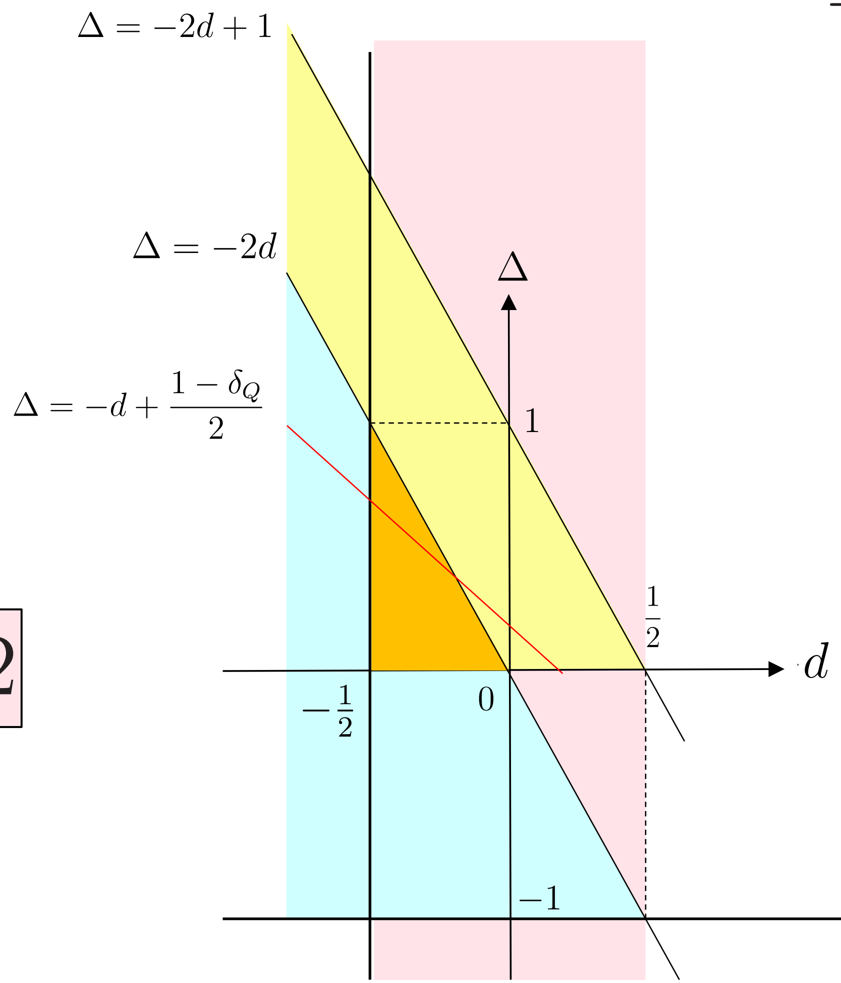
gauge operator

Gauge field may have arbitrary space-time coordinate dependence (local field)

$$(\Delta + 2d)(\Delta + 1) \leq 0$$

$$\Delta(\Delta + 2d - 1) \leq 0$$

$$-\frac{1}{2} \leq d \leq \frac{1}{2}$$



Bounds on dimensions of gauge operator

$$\boxed{0 < \delta_Q \leq 2}$$

Bounds on dimensions
of gauge operator

$$\boxed{0 < \delta_Q \leq 2}$$

$$\left\{ \begin{array}{lll} \delta_Q = 1 & \text{one-form gauge fields} \supset & \text{photon, gluon etc.} \\ \delta_Q = 2 & \text{two-form gauge fields} \supset & \text{graviton} \end{array} \right.$$

$$\Downarrow \qquad d + \Delta = \frac{1 - \delta_Q}{2}$$

fermion may couple to one-form gauge fields

scalar may couple to two-form gauge fields

Main Results

i. C-function in CQM

cf.

$$\left(c = \int_{S^{2k}} \langle T^{\mu}{}_{\mu} \rangle \right)$$

$$D(E) := i\langle E|D|E\rangle$$

Bounds on dimensions of reference state $\boxed{-\frac{1}{2} \le d \le \frac{1}{2}}$ (unitarity)

II. No-Go theorem in CQM

Bounds on dimensions of physical state $\boxed{-\frac{1}{2} \le d + \Delta \le \frac{1}{2}}$ (positivity)

Bounds on dimensions of gauge operator $\boxed{0 < \delta_Q \le 2}$