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Probing the holomorphic anomaly of the $\mathcal{N}=2, D=2$ WZ model on the lattice

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Supersymmetries and Quantum Symmetries, Dubna 2017

1. Work based, essentially, on arXiv : 1302.2361[hep-th], 1606.08284[hep-th] and in progress

Outline

1. Introduction
2. How particles and fields probe spacetime
3. Closing the system
4. The lattice framework
5. Conclusions

Supersymmetry : hidden in plain sight

- Supersymmetry was invented for very specific reasons. But its scope has grown. Its realization, within the framework of the Standard Model, is but one example of its relevance for physics and doesn't exhaust its relevance. The purpose of this talk is to show how it can be useful for providing insights to a much broader class of physical models, by providing effective strategies for describing non-perturbative effects through lattice techniques.

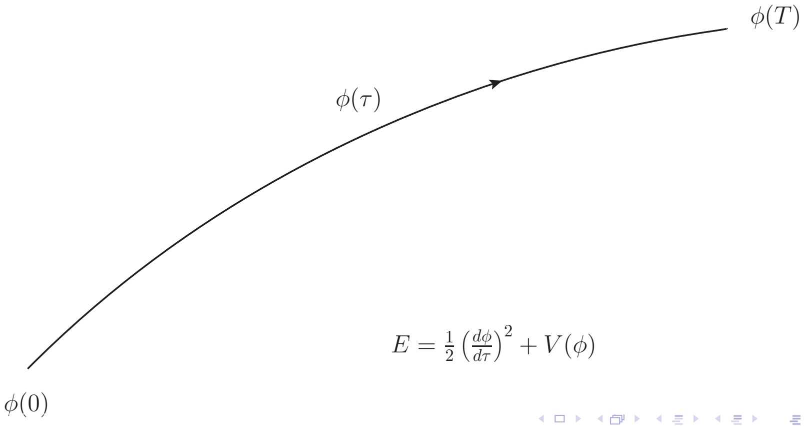
Supersymmetry : hidden in plain sight

- In particular, we wish to stress that supersymmetry provides the only consistent definition of quantum field theory, beyond perturbation theory, since it expresses the fact that "dynamical system+fluctuations" is consistently closed and doesn't depend on how the separation between the two is realized.

Supersymmetry : hidden in plain sight

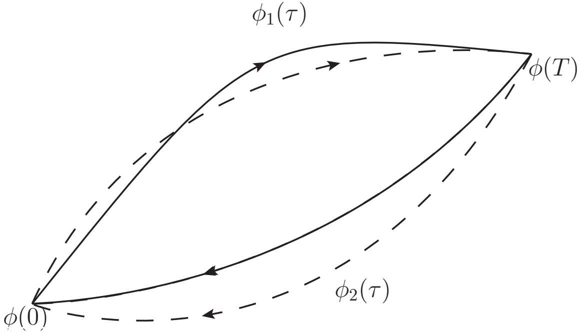
- It, also, will motivate the idea that, since it puts bosons and fermions in the same multiplet, it can be used to deduce properties of fermions by studying actions that contain only bosons–but computing the identities satisfied by functions of the bosonic fields, that can probe the properties of fermions.
  While, for analytical calculations, fermions are more convenient than bosons, for numerical calculations, bosons are more convenient. Supersymmetry–even when broken–allows to use bosons for deducing properties of fermions (and vice versa, of course), in a local way–it's in this respect that it differs from bosonization/fermionization.

How a classical, non–relativistic, particle probes spacetime



$$E = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + V(\phi)$$

How a quantum, non–relativistic, particle probes spacetime



Periodic motion ! Sum over all paths, with an appropriate weight, $\exp(\mathrm{i}S[\phi(\tau)]/\hbar)$.

The sum over all paths

$$\langle 0|0\rangle = \int [\mathscr{D}\phi(t)]\, e^{\mathrm{i}\frac{1}{\hbar}S[\phi]} = \int [\mathscr{D}\phi(t)]\, e^{\mathrm{i}\frac{1}{\hbar}\left(\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi)\right)} = e^{\mathrm{i}\frac{1}{\hbar}\mathcal{W}[\phi]}$$

$$\Rightarrow \int [\mathscr{D}\phi(\tau)]\, e^{-\frac{1}{\hbar}S_{\mathrm{E}}[\phi]} = \int [\mathscr{D}\phi(t)]\, e^{-\frac{1}{\hbar}\left(\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi)\right)} = Z_{\mathrm{E}}$$

Can be reconstructed from the correlation functions, $\langle\phi(\tau_1)\phi(\tau_2)\cdots\phi(\tau_n)\rangle$.

$S_{\mathrm{E}}[\phi]$ is the classical (Euclidian) action and the quantum corrections are combinations of analytic–and non–analytic functions of $\hbar$. How can we take them into account ? This partition function doesn't describe a closed system, since it doesn't specify the degrees of freedom of the bath, in the same detail it does for the "physical" degrees of freedom.

Many particles–or many dimensions in target space

Many, non–relativistic, particles (or a particle, that moves in a target space of more than one dimensions) :

$$Z_{\mathrm{E}} = \int [\mathscr{D}\phi^I(\tau)]\, e^{-\int d\tau \left\{\frac{1}{2}\left(\dot{\phi}^I\right)^2 + V(\phi^I)\right\}}$$

Here, too, the system isn't closed–the fluctuations are implicitly described. We would like to describe them explicitly.

The fluctuations as degrees of freedom

We want to describe the dynamics of a *closed* system, in a way that doesn't depend on how we pick out system from the fluctuations, it's in equilibrium with.

A simple example is that of Gaussian, white, additive, noise. (Generalization to colored, multiplicative, noise will be discussed at the end.)

White additive noise

$\{\eta(\tau)\}$ is a Gaussian process :

$$
\begin{aligned}
&\langle \eta(\tau) \rangle = 0 \\
&\langle \eta(\tau_1)\eta(\tau_2) \rangle = \nu\delta(\tau_1 - \tau_2) \\
&\langle \eta(\tau_1)\eta(\tau_2)\cdots\eta(\tau_{2n}) \rangle = \\
&\qquad \sum_{\pi} \left\langle \eta(\tau_{\pi(1)})\eta(\tau_{\pi(2)}) \right\rangle \cdots \left\langle \eta(\tau_{\pi(2n-1)})\eta(\tau_{\pi(2n)}) \right\rangle
\end{aligned}
$$

$$
\nu = \begin{cases} \hbar & \text{quantum} \\ k_B T & \text{thermal} \\ \sigma & \text{annealed disorder} \end{cases}
$$

Choose units such that $1 = \hbar\,(= k_B T)\,(= \sigma)$.

(non–relativistic) QM in the Stochastic Formulation

$$\eta(\tau) = \frac{d\phi}{d\tau} + \frac{dW}{d\phi(\tau)}$$

$$\eta^I(\tau^A) = s_A^{IJ}\frac{\partial\phi^J}{\partial\tau^A} + \frac{\partial W}{\partial\phi_I}$$

with $[s_A, s_B] = 0$–we'll see, shortly, what happens, when the s_A don't all commute.

From the known correlation functions of $\eta(\tau)$ deduce the correlation functions of $\phi(\tau)$.

Noncommutative worldvolume

When $[\boldsymbol{s}_A, \boldsymbol{s}_B] \neq 0$, the worldvolume can't be reduced to one dimension. A particularly interesting choice for the matrices $\boldsymbol{s}_A$ is to have them generate a Clifford algebra, e.g.

$$\{\boldsymbol{s}_A, \boldsymbol{s}_B\} = 2\delta_{AB}$$

The simplest such example is when $\boldsymbol{s}_A \equiv \sigma_A$, the Pauli matrices and the worldvolume taken to be two–dimensional, Euclidian, space, the global invariance is $SO(2)$ and the two scalars, ϕ^I, label the target space.
The reason we would like to choose the $\boldsymbol{s}_A$ this way is because we would like to describe target space Lorentz invariance.

The (Langevin) partition function

$$Z = 1 \equiv \int [\mathscr{D}\eta(\tau)]\, e^{-\int d\tau\, \frac{1}{2}\eta(\tau)^2} \int [\mathscr{D}\phi(\tau)]\delta\left(\frac{d\phi(\tau)}{d\tau} + \frac{dW}{d\phi(\tau)} - \eta(\tau)\right)$$

$$= \int [\mathscr{D}\phi(\tau)]\, e^{-\int d\tau\, \frac{1}{2}\left(\frac{d\phi}{d\tau}+\frac{dW}{d\phi(\tau)}\right)^2} \left|\mathrm{det}\left(\delta(\tau-\tau')\left[\frac{d}{d\tau} + \frac{d^2W}{d\phi(\tau)^2}\right]\right)\right|$$

These are not well–defined expressions! Use lattice techniques to *define* them (cf. arXiv :1302.2361 and 1606.08284). These expressions define the "Langevin" partition function, Z_{L}. It describes the consistent closure of the system, if it can be well defined.

Surface terms and their avatars

When $[s_A, s_B] = 0$, the s_A can all be diagonalized simultaneously : $s_A^{IJ} = \lambda_A \delta^{IJ}$. When we expand out the action, we find

$$\frac{1}{2}\left(\lambda_A \delta^{IJ}\dot{\phi}^I + \frac{\partial W}{\partial \phi_I}\right)^2 = \frac{1}{2}\lambda_A^2\left(\dot{\phi}^I\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial \phi_I}\right)^2 + \lambda_A \delta^{IJ}\dot{\phi}^I \frac{\partial W}{\partial \phi_I}$$

In this case we notice that the last term,

$$\lambda_A \delta^{IJ}\dot{\phi}^I \frac{\partial W}{\partial \phi_I} = \lambda_A \frac{d}{d\tau^A} W(\phi)$$

is a total derivative ; if we impose periodic boundary conditions, it doesn't contribute.

Surface terms and their avatars–II

When $[s_A, s_B] \neq 0$, the s_A can't all be diagonalized simultaneously. In that case

$$s_A^{IJ} \frac{\partial \phi^J}{\partial \tau^A} \frac{\partial W}{\partial \phi_I}$$

isn't, manifestly, a total derivative, so does contribute, apparently, to the classical equations of motion, even if we impose periodic boundary conditions.
The worldvolume is a non–commutative manifold and this term resembles a spin–orbit coupling, that describes long–range interactions. But could these, in fact, be an illusion ?

How holomorphy eliminates the spin–orbit term

When $s_A = \sigma_A$, something interesting happens : The cross–term takes the form

$$\sigma_x^{IJ} \frac{\partial \phi^J}{\partial x} \frac{\partial W}{\partial \phi_I} + \sigma_z^{IJ} \frac{\partial \phi^J}{\partial y} \frac{\partial W}{\partial \phi_I} = \left(\frac{\partial \phi^2}{\partial x} + \frac{\partial \phi^1}{\partial y} \right) \frac{\partial W}{\partial \phi_1} + \left(\frac{\partial \phi^1}{\partial x} - \frac{\partial \phi^2}{\partial y} \right) \frac{\partial W}{\partial \phi_2}$$

and, if the $\phi_{1,2}$ satisfy the Cauchy–Riemann equations, the term, in fact, vanishes. The question is, whether the fluctuations respect this, too. Let's have a look at them–they're described by the insertion of the absolute value of the determinant in the expectation values. If this isn't well–defined, this is a manifestation of the holomorphic anomaly.

The determinant

Assumption : W is ultra local :

$$\frac{\partial^2 W}{\partial\phi(\tau)\phi(\tau')} = \delta(\tau-\tau')\frac{d^2 W}{d\phi(\tau)^2}$$

Then, in one dimension (or a worldvolume with abelian isometries)

$$\left|\det\left(\delta(\tau-\tau')\frac{d}{d\tau} + \frac{\partial^2 W}{\partial\phi(\tau)\partial\phi(\tau')}\right)\right| = |\det\delta(\tau-\tau')| \left|\det\left(\frac{d}{d\tau} + \frac{d^2 W}{d\phi(\tau)^2}\right)\right|$$

The first factor can be taken outside the path integral and contributes the same constant to numerator and denominator, to any expectation value–it can be dropped.

The determinant

The absolute value of the determinant can be expressed as a product of its value and its phase, in general (assuming the determinant is real, the phase is, just, its sign) :

$$\left|\mathrm{det}\left(\frac{d}{d\tau}+\frac{d^2W}{d\phi(\tau)^2}\right)\right| = e^{-\mathrm{i}\phi_{\mathrm{det}}}\mathrm{det}\left(\frac{d}{d\tau}+\frac{d^2W}{d\phi(\tau)^2}\right)$$

$$\left|\mathrm{det}\left(s_A^{IJ}\frac{\partial}{\partial\tau^A}+\frac{\partial^2W}{\partial\phi_I\partial\phi_J}\right)\right| = e^{-\mathrm{i}\phi_{\mathrm{det}}}\mathrm{det}\left(s_A^{IJ}\frac{\partial}{\partial\tau^A}+\frac{\partial^2W}{\partial\phi_I\partial\phi_J}\right)$$

We may, thus, write

$$1 = Z_{\mathrm{L}} = \left\langle e^{-\mathrm{i}\phi_{\mathrm{det}}}\right\rangle_{\mathrm{SUSY}} Z_{\mathrm{SUSY}}$$

assuming that Z_{SUSY} exists, and that the average displayed does too. The reason for the qualifier SUSY will be explained presently.

Grassmann variables for a local action

The determinant–which is a non–local quantity–can be introduced in the action in a local way, using Grassmann fields, $\psi_I(\tau^A), \chi_J(\tau^A)$:

$$\det\left(s_A^{IJ}\frac{\partial}{\partial\tau^A}+\frac{\partial^2 W}{\partial\phi_I\partial\phi_J}\right)= \int [\mathscr{D}\psi_I(\tau^A)][\mathscr{D}\chi_J(\tau^A)]\, e^{\int d^D\tau\,\psi_I\left(s_A^{IJ}\frac{\partial}{\partial\tau^A}+\frac{\partial^2 W}{\partial\phi_I\partial\phi_J}\right)\chi_J}$$

(In Euclidian signature the fields ψ_I and χ_J are independent.)

SUSY

Assuming that $\{s_A, s_B\} = 2\delta_{AB}$, the Euclidian action

$$S_{\mathrm{E}} = \int d^D\tau \left[\frac{1}{2}\left(\frac{\partial\phi^I}{\partial\tau^A}\right)^2 - \frac{1}{2}F_I^2 + F_I\frac{\partial W}{\partial\phi_I} - \psi_I\left(s_A^{IJ}\frac{\partial}{\partial\tau^A} + \frac{\partial^2 W}{\partial\phi_I\partial\phi_J}\right)\chi_J\right]$$

can be shown to be invariant–up to total derivatives–under supersymmetric transformations. The absence of the "spin–orbit" term is crucial in this regard. The practical question is how to test this–as well as the equivalence between the three partition functions that seem to describe the same system.

How to use the lattice regularization

We seem, therefore, to have three partition functions, that describe the same system : Z_{L}, Z_{SUSY} and Z_{QM} ; the latter is defined by

$$Z_{\mathrm{QM}} = \int [\mathscr{D}\phi^I(\tau^A)]\, e^{-\int d^D\tau \left\{ \frac{1}{2}\left(\frac{\partial\phi^I}{\partial\tau^A}\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial\phi_I}\right)^2 \right\}}$$

It's clear that Z_{QM} is the easiest to study numerically–and has, indeed, been studied ; only not from this angle.
The appropriate observables seem to be the noise fields :

$$\eta^I(\tau^A) = s_A^{IJ}\frac{\partial\phi^J}{\partial\tau^A} + \frac{\partial W}{\partial\phi_I}$$

The observables : general considerations

These should satisfy

$$\left\langle \eta^I(\tau^A)\eta^J(\tau'^B)\right\rangle_{\mathrm{QM}} = 2\delta^{IJ}\delta(\tau^A - \tau'^B)$$

while the higher connected correlation functions should be compatible with zero.
If the 1–point function vanishes, SUSY is realized in the Wigner mode ; if it's non–zero, SUSY is realized in the Nambu–Goldstone mode.
These properties have, already, been checked for the case where $[s_A, s_B] = 0$. The question is to understand, whether there could appear any obstruction to their holding, also, in the case where $\{s_A, s_B\} = 2\delta_{AB}$.

The observables : general considerations

For two–dimensional models such an obstruction, naïvely isn't expected. The reason is that tunneling, also, occurs, as in the one–dimensional case. What can occur is that the superpotential can change quite dramatically, due to the quantum fluctuations, since the multiple classical minima will, always, be eliminated.
A caveat seems to be that it is, rather, the infrared divergences of massless particles, rather than tunneling, that may be the relevant property–and, whether a Berezinskii–Kosterlitz–Thouless (i.e. topology–changing) transition can occur and how it might be detected from the correlations of the noise field, remains to be clarified.

The lattice action

The lattice action is given by the expression

$$S_{\rm latt} = -\sum_{n_1,n_2} \Big[\\ \phi_{n_1,n_2} \cdot (\phi_{n_1+1,n_2} + \phi_{n_1-1,n_2} + \phi_{n_1,n_2+1} + \phi_{n_1,n_2-1}) + \\ \frac{1}{2}||\phi_{n_1,n_2}||^2 + g^2_{\rm latt} v(||\phi_{n_1,n_2}||^2) \Big]$$

where $g_{\rm latt} \equiv ga$ and it can be readily verified that it is equal to $||\eta_{n_1,n_2}||^2/2$ up to surface terms and terms that are suppressed by positive powers of the lattice spacing–as occurs in $D = 1$, as well.

Noise on the lattice

The discretization of the noise field on the lattice is given by the expressions

$$\begin{array}{l} \eta^1_{n_1,n_2} = \frac{1}{2}\left(\phi^2_{n_1+1,n_2} - \phi^2_{n_1-1,n_2} + \phi^1_{n_1,n_2+1} - \phi^1_{n_1,n_2-1}\right) + g_{\mathrm{latt}}\frac{\partial w}{\partial\phi_1} \\ \eta^2_{n_1,n_2} = \frac{1}{2}\left(\phi^1_{n_1+1,n_2} - \phi^1_{n_1-1,n_2} - \phi^2_{n_1,n_2+1} + \phi^2_{n_1,n_2-1}\right) + g_{\mathrm{latt}}\frac{\partial w}{\partial\phi_1} \end{array}$$

where we have written $W(\phi_1,\phi_2) \equiv gw(\phi_1,\phi_2)$. We remark that, for a cubic superpotential, $\partial w/\partial\phi$ is quadratic in the fields, therefore, generically, its vev doesn't vanish–so SUSY, generically, will be broken, in this case.

Conclusions–Perspectives

- It is possible to describe consistently the fluctuations, to which physical systems are coupled, by quantum, thermal effects, or disorder, in a way that doesn't depend on how the physical degrees of freedom are selected among the fluctuations–and the reason is that the fluctuations and the physical degrees of freedom are but components of the same supermultiplet. The fluctuations are mutually non–local wrt the physical degrees of freedom.

Conclusions–Perspectives

- The difference between quantum, thermal and disorder fluctuations is that, while, in the latter two cases, the effective anticommuting degrees of freedom can be expressed in terms of more fundamental degrees of freedom, that can be commuting at that scale, quantum fluctuations cannot be expressed as local, commuting, degrees of freedom–but as local, anticommuting, degrees of freedom, only.

Conclusions–Perspectives

The most direct building blocks of a quantum field theory, therefore, seem to be the noise fields

$$\eta^I(\tau^A) = s_A^{IJ}\frac{\partial\phi^J}{\partial\tau^A} + \frac{\partial W}{\partial\phi_I} \equiv s_A^{IJ}\frac{\partial\phi^J}{\partial\tau^A} + F^I$$

From the known properties of the noise it is possible to deduce the properties of the physical fields and, in particular, describe how the bath reacts to the presence of the physical degrees of freedom. A consistent reaction is described by the fact that the 1–point function can take a non–zero value ; and that the potential can change, due to tunneling effects.

Conclusions–Persepctives

- The framework for studying hypermultiplets by themselves seems well–defined. For studying (super)particles and the corresponding fields in curved target spaces (or with torsion)–relevant for non–linear $\sigma-$models, what's needed is multiplicative noise, e.g.

$$E^I_M(\phi)\eta^M(\tau^A) = \boldsymbol{s}^{IJ}_A \frac{\partial\phi^J}{\partial\tau^A} + \frac{\partial W}{\partial\phi_I}$$

here $E^I_M(\phi)$ is the vielbein, that expresses the fact that the target space is curved. Cf. arXiv : 1610.01622 for an application to magnets.

Conclusions–Perspctives

- In the stochastic approach the auxiliary fields are the starting point–so it would be of interest to study the properties of the multiplets of the form $(4, 4, 0)$, for instance, in the stochastic approach.

Conclusions–Perspectives

- A non–trivial problem is how to describe gauge theories, i.e. theories, whose fields are target space vectors. The reason is gauge invariance. In two dimensions, where gauge fields don't propagate, it may be possible to work around this obstacle by working with the dual scalars. For higher dimensions, a concrete proposal for a noise field that describes the fluctuations of the gauge field, beyond perturbation theory, remains to be worked out.