Integrability of conformal mechanics associated with near horizon extremal Myers-Perry black holes

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Motivation

- Near horizon limits of extremal black holes possess conformal symmetry. Respectively, particle dynamics in this background is described by conformal mechanics.
- In the case of higher-dimensional Myers-Perry black holes with equal rotational parameters this system admits separation of variables in spherical coordinates. It is *superintegrable* in odd dimensions and *almost superintegrable* in even dimensions (A.Galajinsky, A.N., A.Saghatelian, 2013).
- Hence, one could expect that for the Myers-Perry black holes with non-equal rotational parameters the system will be integrable as well.

Goal

- Our goal is the study of conformal mechanics of probe particle moving in the near-horizon limit of extremal Myers-Perry black hole with arbitrary nonzero nonequal rotation parameters.

Main results

- In the most general case with nonequal nonvanishing rotational parameters the system admits separation of variables in higher-dimensional ellipsoidal coordinates.
- The general solution of the corresponding Hamilton-Jacobi equation.
- The explicit expressions of Liouville constants of motion.

I will restrict myself by the odd-dimensional case, though all the results are valid in the even dimensions as well.

Content

- Conformal mechanics: canonical frame and angular part
- Conformal mechanics of particle moving near the horizon of extreme black holes
- Odd-dimensional Perry-Myers BH
- Ellipsoidal coordinates, generating function
- Constants of motion
- Concluding remarks

Conformal mechanics

$$\{H, D\} = 2H, \qquad \{K, D\} = -2K, \qquad \{H, K\} = D$$

Most known example

$$\omega = d\mathbf{p}\wedge d\mathbf{r}, \quad \mathcal{H} = \frac{\mathbf{p}^2}{2} + V(\mathbf{r}), \quad \text{where} \quad \mathbf{r}\cdot\nabla V(\mathbf{r}) = -2V(\mathbf{r}).$$

$$D = \mathbf{p}\cdot\mathbf{r}, \qquad K = \frac{\mathbf{r}^2}{2}$$

The phase space of conformal mechanics is noncompact one.

Canonical basis and its "angular part"

$$SO(1.2)\ \text{Casimir}: \qquad \mathcal{I} = 2KH - \frac{1}{2}D^2$$

Introduce radial coordinate and momentum

$$\mathcal{D} \equiv p_r r, \quad \mathcal{K} \equiv \frac{r^2}{2} \ : \ \mathcal{H} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2},$$

with $\{p_r, \mathcal{I}\} = \{r, \mathcal{I}\}, \{H, \mathcal{I}\} = 0$. One can choose the appropriate "angular coordinates",

$$u^A = (p_\alpha, \varphi^\alpha) \ : \quad \{p_r, u^A\} = \{r, u^A\} = 0, \quad \{p_\beta, \varphi^\alpha\} = \delta^\alpha_\beta \ ,$$

separating the "radial" and "angular" parts of the system. Thus, we can extract the compact ("angular") part of conformal mechanics:

$$(\mathcal{I}(u),\ dp_\alpha \wedge d\varphi^\alpha)$$

Relativistic conformal mechanics: Particle near extreme black hole throat

- Near-horizon limits of extreme black holes metrics are conformal invariant.
- The motion of particle near extreme black hole throat is described by relativistic conformal mechanics

$$H = r\left(\sqrt{(rp_r)^2 + L(p_a, \varphi^a)} - q(p_a)\right),$$

$$D = rp_r, \quad K = \tfrac{1}{r}\left(\sqrt{(rp_r)^2 + L(p_a)} + q(p_a, \varphi^a)\right)$$

- The angular part of this system is given by the expression

$$\mathcal{I} = \frac{1}{2}\left(L(\varphi^a, p_a) - q^2(p_a)\right), \qquad \omega = dp_a \wedge d\varphi^a$$

It contains whole information on the dynamics of particle near extremal BH.

Example: Reissner-Nordström BH

$$\mathcal{I} = p_\theta^2 + \frac{(p_\varphi + ep\cos\theta)^2}{\sin^2\theta} + (mM)^2 - (eq)^2,$$

Spherical Landau problem (particle on the sphere in the presence of constant magnetic field generated by Dirac monopole), shifted on the constant $\mathcal{I}_0 = (mM)^2 - (eq)^2$.

Equivalently, spherical top.

Respective 3d system corresponds to the 3d charge-monopole system with additional potential $V = \frac{\mathcal{I}_0}{R^2}$

Example: Kerr BH

$$\mathcal{I} = \frac{p_\theta^2}{2} + \frac{1}{2}\left[\left(\frac{1+\cos^2\theta}{2\sin\theta}\right)^2 - 1\right] p_\varphi^2 + \frac{(mr_0)^2}{4}(1+\cos^2\theta)$$

Integrable, but not exactly solvable system.

- Critical point $p_\varphi^2 = 2(mr_0)^2$: one-dimensional Higgs oscillator:

$$\mathcal{I} = \frac{p_\theta^2}{2} + (mr_0)^2\cot^2\theta - (mr_0)^2$$

$5d$ Myers-Perry BH

Myers-Perry BH

$$\mathcal{I} = \frac{1}{4}p_\theta^2 + \frac{\rho_0^4}{ab(a+b)^2}\left(\frac{p_\phi^2}{\sin^2\theta} + \frac{p_\psi^2}{\cos^2\theta} - \right.$$

$$\left. - \frac{1}{\rho_0^2}(bp_\phi + ap_\psi)^2 - (p_\phi + p_\psi)^2 + m^2\rho_0^2, \right.$$

where $\rho_0^2 = ab + a^2\cos^2\theta + b^2\sin^2\theta$

Integrable, but not exactly solvable system for $a \neq b$.

Superintegrable and exactly solvable system for $a = b$

$d = 2N + 1$ EMPBH with equal rotation parameters

$$\mathcal{I} = \sum_{a,b=1}^{N-1} (\delta_{ab} - x_a x_b) p_a p_b + \sum_{i=1}^{N} \frac{p_{\phi_i}^2}{x_i^2} - \frac{(N+1)}{N} \left(\sum_{i=1}^{N} p_{\phi_i} \right)^2,$$

where $\sum_{i=1}^{N} x_i^2 = 1$.

- Superintegrable system!
- Fixing $p_{\varphi_i} = g_i$ we get singular spherical (Higgs) oscillator
- Separation of variables in spherical coordinates

$d = 2N + 2$ EMPBH with equal rotation parameters

$$\mathcal{I} = \sum_{i,j=1}^{N-1} ((2N-3)\rho_0^2\delta_{ij} - x_i x_j) p_i p_j +$$
$$\sum_{i,j=1}^{N-1} \left(\frac{(2N-3)\rho_0^2}{x_i^2}\delta_{ij} - \frac{(2N-3)^2\rho_0^2}{2(N-1)} - \frac{2}{N-1} \right) p_{\phi_i} p_{\phi_j} + m^2\rho_0^2,$$

where $\rho_0^2 = \frac{2(N-1)}{2N-3} - \sum_{i=1}^{N-1} x_i^2$.

- Almost superintegrable system (lacks of one constant of motion
- Fixing $p_{\varphi_i} = g_i$ leads to $(N-1)$-dimensional system with configuration space different from sphere
- Separation of variables in spherical coordinates

Near-horizon $d = (2N+1)$ Extreme Myers-Perry BH

$$\frac{ds^2}{r_H^2} = A(x)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{i=1}^{N} dx_i dx_i + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij} x_i x_j D\varphi^i D\varphi^j ,$$

where r_H is the horizon radius,

$$A(x) = \frac{\sum_{i=1}^{N} x_i^2/m_i^2}{1 + 4\sum_{i<j}(m_i m_j)^{-1}}, \qquad k^i = 2\frac{\sqrt{m_i - 1}/m_i}{1 + 4\sum_{l<n}(m_l m_n)^{-1}}$$

$$\tilde{\gamma}_{ij} = \delta_{ij} + \frac{1}{\sum_l x_l^2/m_l^2} \frac{\sqrt{m_i - 1}x_i}{m_i} \frac{\sqrt{m_j - 1}x_j}{m_j}, \qquad D\varphi^i \equiv d\varphi^i + k^i r d\tau$$

and

$$\sum_{i=1}^{N} \frac{x_i^2}{m_i} = 1, \qquad \sum_{i=1}^{N} \frac{1}{m_i} = 1, \qquad m_i \geq 1, \quad 0 < x_i \leq \sqrt{m_i}$$

Angular Hamiltonian (initial coordinates)

$$\mathcal{I} = \frac{\sum_{i=1}^{N} x_i^2/m_i^2}{1 + 4\sum_{i<j}(m_i m_j)^{-1}} \left[\sum_{a,b=1}^{N-1} h^{ab} p_a p_b + \sum_{i=1}^{N} \frac{p_{\varphi_i}^2}{x_i^2} + g_0 \right] - \mathcal{I}_0,$$

where

$$g_0 = -\left(\sum_{i=1}^{N} \frac{\sqrt{m_i - 1} p_{\varphi_i}}{m_i} \right)^2 + m_0^2 r_H^2, \quad \mathcal{I}_0 = 4\left(\sum_i \frac{\sqrt{m_i - 1} p_{\varphi_i}}{m_i^2} \right)^2$$

and

$$h^{ab} = \delta^{ab} - \frac{1}{\sum_{i=1}^{N} x_i^2/m_i^2} \frac{x_a}{m_a} \frac{x_b}{m_b}$$

h^{ab} is the inverse metric of $(N-1)$-ellipsoid, with $\sum_{i=1}^{N} \frac{x_i^2}{m_i} = 1.$

Some comments

- The metric of configuration space of the system with fixed p_{φ_i} is $(N-1)$-dimensional ellipsoid multiplied by some conformal factor.
- Hence, fixing its energy surface we will get some dual system on ellipsoid which could be interpreted as an analog of singular oscillator.
- With $m_i = 1/N$ the system (singular spherical(Higgs) oscillator) admits separation of variables in spherical coordinates.
- Could we separate the variables in ellipsoidal coordinates in the opposite case of $m_i \neq m_j$? YES!

N-dimensional ellipsoidal coordinates

$$x_i^2 = (m_i - \lambda_i) \prod_{j=1, j\neq i}^{N} \frac{m_i - \lambda_j}{m_i - m_j} : \qquad \lambda_N = 0 \quad \Rightarrow \quad \sum_{i=1}^{N} \frac{x_i^2}{m_i} = 1.$$

Here $\lambda_N < m_N < \ldots < \lambda_2 < m_2 < \lambda_1 < m_1$.

Useful identities coming from $(N-1)$-th order Lagrange polynomials

$$\frac{1}{\prod_{a=1}^{N-1} \lambda_a} = \sum_{a=1}^{N-1} \frac{\lambda_a^{-1}}{\prod_{b=1;b\neq a}^{N-1} (\lambda_b - \lambda_a)}, \quad \sum_{i=1}^{N} \frac{\lambda_i^{a-1}}{\prod_{j=1;i\neq j}^{N} (\lambda_i - \lambda_j)} = \delta_{a,N-1}.$$

Angular mechanics: Ellipsoidal coordinates

$$\tilde{\mathcal{I}} = \left(\prod_{c=1}^{N-1}\lambda_c\right)\left[-\sum_a \frac{4\prod_{i=1}^{N}(m_i-\lambda_a)\pi_a^2}{\lambda_a\prod_{b=1,a\neq b}^{N-1}(\lambda_b-\lambda_a)} + \sum_{i=1}^{N}\frac{g_i^2}{\prod_{a=1}^{N-1}(m_i-\lambda_a)} + g_0\right]$$

where

$$g_i^2 = \frac{p_{\varphi_i}^2}{m_i}\prod_{j=1,j\neq i}^{N}(m_i - m_j), \qquad \tilde{\mathcal{I}} \equiv (\mathcal{I}+\mathcal{I}_0)\,(1+4\sum_{i<j}(m_i m_j)^{-1})\prod_{i=1}^{N} m_i$$

In these variables we can find the general solution of the Hamilton-Jacobi equation $\tilde{\mathcal{I}}(\lambda_a, \pi_a = \partial S/\partial\lambda_a) = \mathcal{E}$.

Energy Surface

$$\tilde{\mathcal{I}}(\lambda_a, \pi_a) = \mathcal{E} \quad \Leftrightarrow \quad \sum_{a=1}^{N-1} \frac{R_a - \mathcal{E}}{\lambda_a \prod_{b=1, a\neq b}^{N-1} (\lambda_b - \lambda_a)} = 0$$

where

$$R_a = -4 \prod_{i=1}^{N} (m_i - \lambda_a) \pi_a^2 + (-1)^N \sum_{i=1}^{N} \frac{g_i^2 \lambda_a}{m_i - \lambda_a} + g_0 (-\lambda_a)^{N-1}.$$

Equivalently,

$$\sum_{a=1}^{N-1} \frac{R_a - \mathcal{E} - \sum_{\alpha=1}^{N-2} \nu_\alpha \lambda_a^\alpha}{\lambda_a \prod_{b=1, a\neq b}^{N-1} (\lambda_b - \lambda_a)} = 0,$$

where ν_α are arbitrary constants.
This representation allows to obtain the (general) solution of Hamilton-Jacobi equation S_{gen} depending on $N-1$ constants $\nu_\alpha, \mathcal{E}$!

Hamilton-Jacobi equation. General Solution

$$S_{\text{gen}} = \sum_{a=1}^{N-1} S(\lambda_a), \quad R\left(\lambda_a, \frac{dS(\lambda_a)}{d\lambda_a}\right) - \sum_{b=1}^{N-1} \nu_b \lambda_a^{b-1} = 0,$$

where $\nu_1 = \mathcal{E}$.

This yields the general solution

$$S_{\text{gen}} = \sum_{a=1}^{N-1} S(\lambda_a, \nu_\alpha, \mathcal{E}), \qquad S(\lambda, \nu_\alpha, \mathcal{E}) =$$

$$= \frac{1}{2} \int \frac{d\lambda}{\sqrt{\prod_{i=1}^{N} (m_i - \lambda)}} \sqrt{(-1)^N \left[\sum_{i=1}^{N} \frac{g_i^2 \lambda}{m_i - \lambda} + g_0 \lambda^{N-1}\right] - \sum_{b=1}^{N-1} \nu_b \lambda^{b-1}} \, .$$

Constants of motion

$$\sum_{\alpha=1}^{N-1} F_\alpha \lambda_a^{\alpha-1} = R_a$$

Equivalently

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-2} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{N-1} & \lambda_{N-1}^2 & \cdots & \lambda_{N-1}^{N-2} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_{N-1} \end{pmatrix},$$

where

$$F_1 = \tilde{\mathcal{I}}, \quad R_a = -4 \prod_{i=1}^{N} (m_i - \lambda_a)\pi_a^2 + (-1)^N \sum_{i=1}^{N} \frac{g_i^2 \lambda_a}{m_i - \lambda_a} - g_0(-\lambda_a)^{N-1}.$$

Constants of motion: Cartesian coordinates

$$F_{N-1} = \left(\sum_{a=1}^{N-1} p_a x_a\right)^2 - \sum_{a=1}^{N-1} p_a^2 m_a - \sum_{i=1}^{N} \frac{m_i p_{\varphi_i}^2}{x_i^2} + g_0 \sum_{i=1}^{N} x_i^2$$

$$F_{N-2} = \sum_{a,b=1}^{N-1} p_a x_a p_b x_b \sum_{\substack{k=1 \\ k\neq a,b}}^{N} m_k - \sum_{a=1}^{N-1} (p_a x_a)^2 m_a + \sum_{a=1}^{N-1} p_a^2 (m_a^2 - f_1 m_a)$$

$$+ \sum_{i=1}^{N} \frac{p_{\varphi_i}^2}{x_i^2} (m_i^2 - f_1 m_i) + g_0 \sum_{\substack{i,j \\ i\neq j}}^{N} m_i x_j^2, \qquad f_1(x_i, m_j) \equiv \sum_{i}^{N} (-x_i{}^2 + m_i)$$

$\dots$

In the $m_i = 1/N$ limit all these constants of motion result in the Hamiltonian. Hence, we need to find another contraction

Even-dimensional MP black holes

All listed observations held in the case of $2(N+1)$-dimensional Myers-Perry black hole with nonequal non-vanishing rotational parameters upon replacements

$$\sum_{i=1}^{N} \frac{x_i^2}{m_i} = 1 \quad \to \quad \sum_{i=1}^{N+1} \frac{x_i^2}{m_i} = 1, \qquad m_{N+1} = 1$$

$$\sum_{i=1}^{N} \frac{1}{m_i} = 1 \quad \to \quad \sum_{i=1}^{N+1} \frac{x_i^2}{m_i} = \frac{1}{2}$$

$(2N+1)$-dimensional MP black hole with vanishing horizon

In the case of odd-dimensional extreme Myers-Perry black hole with one vanishing rotational parameter the angular mechanics is given by the expression

$$\mathcal{I} = x_N^2 \left[\sum_i^{N-1} \left(p_i^2 + \frac{g_i^2}{x_i^2} - \frac{g_i^2}{m_i x_N^2 + x_i^2} \right) + \frac{m_0^2}{\alpha} \right],$$

where

$$x_N^2 = 1 - \sum_{i=1}^{N-1} \frac{x_i^2}{m_i}, \quad \sum_{i=1}^{N} \frac{1}{m_i} = 1$$

We were unable to separate the variables even when all parameters m_i parameters are equal to each other. But sill hope.....

Supersymmetry

- Let the angular Hamiltonian admits $\mathcal{N}=4$ supersymmetric extension. Then we can immediately construct $D(1.2:\alpha)$ superconformal extension of initial conformal mechanics (T.Hakobyan,O.Lechtenfeld, S.Krivonos, A.N.. 2010).
- Unfortunately, even in "isotropic" case, $m_i = 1/N$ (singular Higgs oscillator), $\mathcal{N}=4$ supersymmetrization in conformally flat coordinates yields strange restriction $\sum_{i=1}^{N} p_{\varphi_i} = 0$ (Kozyrev, Krivonos, Lechtenfeld, A.N., Sutulin). I believe, that it could be avoided in spherical coordinates. But Krivonos disagree with me.

Obvious tasks to be done

- Extension of these results to black rings and black strings(in progress)
- Separation of variables in "intermediate" case when some of m_i parameters coincide. Intuitively it is clear, that in this case they system gets additional constants of motion.
- Semiclassical and canonical quantization. This should clarify, are there special values of rotational parameters when the system gets additional (higher-order) constant(s) of motion.
- Proper formulation of "singular ellipsoidal oscillator" and its investigation.

Once again: main results

- We investigated the Hamiltonian system describing probe particle near horizon of extreme Myers-Perry black hole with nonequal non-vanishing rotational parameters. We found that Hamilton-Jacobi equation admits the separation of variables in ellipsoidal coordinates, and wrote down its general solution.
- We presented the explicit expressions for its Liouville integrals in the ellipsoidal and "Cartesian" coordinates.

Thank you