# On current contribution to Fronsdal equations

(arXiv:1706.04605, work in progress)

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# HS fields and Fronsdal equations

- EoM for free massless spin-$s$ field on $\mathcal{M}_d$ [Fronsdal, Phys.Rev.D 18 (1978) 3624]

$$\Box\phi_{A(s)} - D_A D_M \phi^M{}_{A(s-1)} + \frac{1}{2} D_A D_A \phi_{A(s-2)M}{}^M = 0,$$

$$\phi_{A(s-4)MN}{}^{MN} = 0.$$

- Gauge transformations

$$\delta\phi_{A(s)} = D_A \xi_{A(s-1)}.$$

- Linearized gauge-invariant spin-$s$ curvature (*e.g.* Maxwell tensor $F_{AB}$, Weyl tensor $C_{ABCD}$) is of order $s$ in derivatives

$$C_{A(s),B(s)} = D_{A_1}...D_{A_s}\phi_{B(s)} + ...$$

# HS master fields

- Gauge-noninvariant d.o.f. (Fronsdal spin-$s$ fields and their first $s-1$ derivatives): 1-form

$$\omega(Y|x) = \sum_{n,m=0}^{\infty} \omega^{\alpha(n),\dot\beta(m)}(x)\, y_{\alpha_1}...y_{\alpha_n}\bar y_{\dot\beta_1}...\bar y_{\dot\beta_m},$$

$$\delta\omega(Y|x) = \mathcal{D}_{ad}\xi(Y|x).$$

- Gauge-invariant d.o.f. (HS curvatures and their descendants): 0-form

$$C(Y|x) = \sum_{n,m=0}^{\infty} C^{\alpha(n),\dot\beta(m)}(x)\, y_{\alpha_1}...y_{\alpha_n}\bar y_{\dot\beta_1}...\bar y_{\dot\beta_m}$$

- Spin-$s$ submodule is spanned by

$$\omega^{\alpha(n),\dot\beta(m)}(x), \quad n+m = 2(s-1);$$
$$C^{\alpha(n),\dot\beta(m)}(x), \quad |n-m| = 2s.$$

# Nonlinear HS Equations

- Nonlinear higher-spin equations [Vasiliev, Phys.Lett.B 285 (1992) 225]

$$
\begin{aligned}
&\mathrm{d}W + W * \wedge W = -i\theta_\alpha \wedge \theta^\alpha \left(1 + \eta B * \varkappa k\right) - i\bar\theta_{\dot\alpha} \wedge \bar\theta^{\dot\alpha} \left(1 + \bar\eta B * \bar\varkappa \bar k\right),\\
&\mathrm{d}B + W * B - B * W = 0.
\end{aligned}
$$

- Master fields: $W = W\left(Z; Y|K|x|\theta^A, dx^{\underline{m}}\right), \quad B = B\left(Z; Y|K|x\right)$.
- Star-product:

$$
(f * g)(Z, Y) = \int d^4U d^4V e^{iU_A V^A} f(Z+U, Y+U) g(Z-V, Y+V).
$$

- Inner Klein operators:

$$
\varkappa := \exp\left(iz_\alpha y^\alpha\right), \quad \bar\varkappa := \exp\left(i\bar z_{\dot\alpha} \bar y^{\dot\alpha}\right).
$$

$$
\varkappa * \varkappa = 1, \quad \varkappa * f(z^\alpha; y^\alpha) = f(-z^\alpha; -y^\alpha) * \varkappa.
$$

- Outer Klein operators $k$ and $\bar k$:

$$
kk = 1, \quad kf(z^\alpha; y^\alpha; \theta^\alpha) = f(-z^\alpha; -y^\alpha; -\theta^\alpha)k.
$$

# Perturbation theory: linear order

- $AdS_4$ vacuum:

$$B = 0, \quad W_0 = \Omega_0^{AB} Y_A Y_B + Z^A \theta_A .$$

$$\Omega_0^{AB} Y_A Y_B = -\frac{i}{4}\left(\omega_L^{\alpha\beta} y_\alpha y_\beta + \bar{\omega}_L^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + 2h^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}\right).$$

- Linear order

$$\begin{aligned} \mathcal{D}_{ad}\omega\left(Y|K|x\right) &= L\left(C\right), \\ \mathcal{D}_{tw}C\left(Y|K|x\right) &= 0, \end{aligned}$$

where

$$L\left(C\right) := -\frac{i\eta}{4} h_\beta{}^{\dot{\alpha}} h^{\beta\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} C\left(0, \bar{y}|K|x\right) k - \frac{i\bar{\eta}}{4} h^\alpha{}_{\dot{\beta}} h^{\alpha\dot{\beta}} \partial_\alpha \partial_\alpha C\left(y, 0|K|x\right) \bar{k},$$

$$\mathcal{D}_{ad} f := D^L f + h^{\alpha\dot{\beta}}\left(y_\alpha \bar{\partial}_{\dot{\beta}} + \bar{y}_{\dot{\beta}} \partial_\alpha\right) f,$$

$$\mathcal{D}_{tw} f := D^L f - i h^{\alpha\dot{\beta}}\left(y_\alpha \bar{y}_{\dot{\beta}} - \partial_\alpha \bar{\partial}_{\dot{\beta}}\right) f,$$

$$D^L f := \mathrm{d}f + \left(\omega_L^{\alpha\beta} y_\alpha \partial_\beta + \bar{\omega}_L^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}}\right) f.$$

# Perturbation theory: quadratic corrections

- Quadratic order in canonical form [Gelfond, Vasiliev, arXiv:1706.03718]:

$$\begin{aligned} \mathcal{D}_{ad}\omega + [\omega,\omega]_* &= L(C) + Q(C,\omega) + \Gamma(J), \\ \mathcal{D}_{tw}C + [\omega, C]_* &= -\mathcal{H}_\eta(J) - \mathcal{H}_{\bar{\eta}}(J) + \mathcal{D}_{tw}\mathcal{B}(J), \end{aligned}$$

$$J = C(Y|x)\, C(Y|x).$$

- $[\omega,\omega]_*$contribution is fixed by HS algebra; $Q(C,\omega)$ is gauge-noinvariant contribution to the same sector; both are *a priori* local due to restrictive degree of $\omega$.
- $\Gamma(J)$ is gauge-invariant current contribution which now has local form; it is this sector that we will study and rewrite in terms of Fronsdal fields.
- For HS cubic vertices the Minkowski limit exists (as opposed to the full nonlinear theory), which we will make use of.
- Currents in question are conformal [Gelfond, Vasiliev, arXiv:1510.03488] so we should keep track only of traceless terms.

# Restoring Fronsdal fields

First we need to express unfolded HS fields via (traceless) Fronsdal ones.

- We expand the fields of the theory as

$$\omega = \sum_{n,m=0}^{\infty} \omega_{n,m}, \quad C = \sum_{n,m=0}^{\infty} C_{n,m},$$

  where $f_{n,m} = f_{\alpha(n),\dot{\alpha}(m)} \left(y^{\alpha}\right)^{n} \left(\bar{y}^{\dot{\alpha}}\right)^{m}$.
- All 1-forms on hand can be expanded in terms of vierbein

$$D^{L} = h^{\alpha\dot{\beta}} D_{\alpha\dot{\beta}}, \quad \omega_{n,m} = h^{\alpha\dot{\beta}} \omega_{n,m|\alpha\dot{\beta}}.$$

- Then linearized equations go over

$$\begin{aligned}
D^{\beta}{}_{\dot{\alpha}} \omega_{n,\bar{n}|\beta\dot{\alpha}} &= -y^{\beta} \bar{\partial}_{\dot{\alpha}} \omega_{n-1,\bar{n}+1|\beta\dot{\alpha}} - \partial^{\beta} \bar{y}_{\dot{\alpha}} \omega_{n+1,\bar{n}-1|\beta\dot{\alpha}} + \frac{i}{2} \eta \delta_{n,0} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} C_{0,\bar{n}+2} k, \\
D_{\alpha}{}^{\dot{\beta}} \omega_{n,\bar{n}|\alpha\dot{\beta}} &= -y_{\alpha} \bar{\partial}^{\dot{\beta}} \omega_{n-1,\bar{n}+1|\alpha\dot{\beta}} - \partial_{\alpha} \bar{y}^{\dot{\beta}} \omega_{n+1,\bar{n}-1|\beta\dot{\alpha}} + \frac{i}{2} \bar{\eta} \delta_{\bar{n},0} \partial_{\alpha} \partial_{\alpha} C_{n+2,0} \bar{k}, \\
D_{\alpha\dot{\beta}} C_{n,m} &= i y_{\alpha} \bar{y}_{\dot{\beta}} C_{n-1,m-1} - \partial_{\alpha} \bar{\partial}_{\dot{\beta}} C_{n+1,m+1}.
\end{aligned}$$

# Restoring Fronsdal fields

- We explore traceless component of the fields $\phi_{n,m} := \omega_{n,m|\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}}$ (thus, Fronsdal spin-$s$ corresponds to $\phi_{s,s}$)

$$\begin{aligned}
\bar{y}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} \partial^\alpha \phi_{n,m} &= n \cdot m \phi_{n-1,m+1} + \frac{i}{2} \eta \delta_{n,1} m (m+1) C_{0,m+1} k, \\
y^\alpha D_{\alpha\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} \phi_{n,m} &= n \cdot m \phi_{n+1,m-1} + \frac{i}{2} \bar{\eta} \delta_{m,1} n (n+1) C_{m+1,0} \bar{k}, \\
y^\alpha D_{\alpha\dot{\alpha}} \bar{y}^{\dot{\alpha}} C_{n,m} &= -i (n+1)(m+1) C_{n+1,m+1}.
\end{aligned}$$

- From this one finds

$$\begin{aligned}
C_{2s+p,p} &= \frac{2 \cdot i^{p+1}}{\bar{\eta} s (2s+p)! p!} \left( y^\beta D_{\beta\dot{\beta}} \bar{y}^{\dot{\beta}} \right)^p \left( y^\alpha D_{\alpha\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} \right)^s \phi_{s,s} \bar{k}, \\
C_{p,2s+p} &= \frac{2 \cdot i^{p+1}}{\eta s (2s+p)! p!} \left( y^\beta D_{\beta\dot{\beta}} \bar{y}^{\dot{\beta}} \right)^p \left( \bar{y}^{\dot{\alpha}} D_{\alpha\dot{\alpha}} \partial^\alpha \right)^s \phi_{s,s} k.
\end{aligned}$$

# HS current contribution

- Current sector ($s \geq s_1 + s_2$):

$$D_{\alpha\dot{\beta}}\omega_{s-2,s|\alpha}{}^{\dot{\beta}} = -\bar{y}_{\dot{\beta}}\partial_{\alpha}\omega_{s-1,s-1|\alpha}{}^{\dot{\beta}} - y_{\alpha}\bar{\partial}_{\dot{\beta}}\omega_{s-3,s+1|\alpha}{}^{\dot{\beta}} + \partial_{\alpha}\partial_{\alpha}\mathcal{J}_{s,s}$$
$$D_{\beta\dot{\alpha}}\omega_{s,s-2|}{}^{\beta}{}_{\dot{\alpha}} = -y_{\beta}\bar{\partial}_{\dot{\alpha}}\omega_{s-1,s-1|}{}^{\beta}{}_{\dot{\alpha}} - \bar{y}_{\dot{\alpha}}\partial_{\beta}\omega_{s+1,s-3|}{}^{\beta}{}_{\dot{\alpha}} + \bar{\partial}_{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}}\mathcal{J}_{s,s}$$

where

$$\mathcal{J}_{s,s} = i\frac{(s-2)!}{8\,(2s)!}\sum_{k,m=0}^{s}\frac{(m+k)!\,(2s-m-k)!}{(s-k)!k!\,(s-m!)\,m!}\cdot$$
$$\cdot\left(y^{\alpha}\partial^{1}_{\alpha}\right)^{m}\left(-y^{\beta}\partial^{2}_{\beta}\right)^{s-m}\left(\bar{y}^{\dot{\alpha}}\bar{\partial}^{1}_{\dot{\alpha}}\right)^{s-k}\left(-\bar{y}^{\dot{\beta}}\bar{\partial}^{2}_{\dot{\beta}}\right)^{k}\cdot$$
$$\cdot\left\{\sum_{n=0}^{s}\frac{i^{n}}{(s+n-1)!}\left(\left(\partial^{1}_{\gamma}\partial^{2\gamma}\right)^{n} + \left(\bar{\partial}^{1}_{\dot{\gamma}}\bar{\partial}^{2\dot{\gamma}}\right)^{n}\right)C\left(Y^{1}|K|x\right)C\left(Y^{2}|K|x\right)\right\}\Big|_{Y^{1}=Y^{2}=0}.$$

- In terms of Fronsdal fields this yields

$$\Box\phi_{s,s} + \ldots = -s^{2}\,(s-1)\,\mathcal{J}_{s,s} + \ldots$$

# Structure of the current

- Operator $\frac{1}{2}\left(y^{\alpha}\partial_{\alpha}-\bar{y}^{\dot\alpha}\bar\partial_{\dot\alpha}\right)$ provides a helicity-sign expansion

$$C = C_+ + C_- + C_0, \qquad \begin{array}{l} C_{n,m}\in C_+,\ \text{if } n>m,\\ C_{n,m}\in C_0,\ \text{if } n=m,\\ C_{n,m}\in C_-,\ \text{if } n<m. \end{array}$$

- $\mathcal{J}_{s,s}$ consists of

$$\mathcal{J}_{s,s}=\sum_{s_1,s_2}\left(\mathcal{J}^{H}_{s-s_1-s_2}+\mathcal{J}^{L}_{s-s_1-s_2}\right)$$

where $\mathcal{J}^{H}_{s-s_1-s_2}$ is built of codirectional helicities and have $(s+s_1+s_2)$ derivatives, while $\mathcal{J}^{L}_{s-s_1-s_2}$ is built of opposite helicities and have $(s+|s_1-s_2|)$ derivatives (that matches [Metsaev, hep-th/0512342]).

# Spinor identities

- We treat derivatives as commuting, working in the flat limit,

$$\left[\partial_{\alpha\dot\alpha}, \partial_{\beta\dot\beta}\right] = 0.$$

- We keep track only of transverse traceless contribution of the second order in fields, so we lop off

$$\phi^{\alpha\ldots\dot\alpha\ldots}{}_{\alpha\ldots\dot\alpha\ldots} = 0,$$

$$\partial_{\alpha\dot\alpha}\phi^{\alpha\ldots\dot\alpha\ldots} = 0,$$

$$\ldots(\Box\phi)\ldots\phi = 0,$$

$$\Box(\ldots\phi\ldots\phi) = 0.$$

- All that together with two-componentness of spinors ($f_{\alpha\beta} - f_{\beta\alpha} = \frac{1}{2}\epsilon_{\alpha\beta}f_\gamma{}^\gamma$) allows to put long spinor expressions in necessary form, for instance,

$$\partial_{\alpha\dot\beta}\phi_\gamma{}^{\dot\beta\ldots} = \partial_{\gamma\dot\beta}\phi_\alpha{}^{\dot\beta\ldots}, \quad \partial_{\alpha\dot\beta}\partial_\beta{}^{\dot\beta\ldots} = 0, \quad \partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta} = \partial_{\beta\dot\alpha}\partial_{\alpha\dot\beta}$$

$$\partial_{\gamma\dot\gamma}\partial_{\alpha\dot\beta}\phi_\beta{}^{\dot\beta\ldots} = \partial_{\beta\dot\gamma}\partial_{\alpha\dot\beta}\phi_\gamma{}^{\dot\beta\ldots}$$

# Currents in tensors

After some playing with spinor indices, it is possible to put $\mathcal{J}_{s,s}$ in the form where vector indices can be easily restored via $\sigma$-matrices. Allowing for $\mathrm{tr}\left(\sigma_a\bar{\sigma}_b\right) = 2\eta_{ab}$, $\left(\sigma_a\bar{\sigma}_b\sigma_c - \sigma_c\bar{\sigma}_b\sigma_a\right) = 2i\epsilon_{abcd}\sigma^d$ this gives

$$\mathcal{J}^H_{s-s_1-s_2,A(s)} = -\frac{(s-2)!\,(s_1-1)!\,(s_2-1)!}{4\cdot(2s)!}\frac{\left(1+(-1)^{s+s_1+s_2}\right)}{(s+s_1+s_2-1)!} i^{s+s_1+s_2+1}2^{s_1+s_2}\left(-1\right)^{s+s_1}$$

$$\sum_{n=0}^{s}(-1)^n\left(\begin{array}{c} s+s_1-s_2 \\ n\end{array}\right)\left(\begin{array}{c} s-s_1+s_2 \\ s-n\end{array}\right)\cdot$$

$$\cdot\Big[\left(\eta^2+\bar{\eta}^2\right)\left(\partial_A\right)^{s+s_1-s_2-n}\left(\partial_B\right)^{s_2}\phi^{C(s_1)}\cdot\left(\partial_A\right)^{-s_1+s_2+n}\left(\partial_C\right)^{s_1}\phi^{B(s_2)}+$$

$$+i\left(\eta^2-\bar{\eta}^2\right)\epsilon_{AFCG}\partial^F\left(\partial_A\right)^{s+s_1-s_2-n-1}\left(\partial_B\right)^{s_2}\phi^{C(s_1)}\cdot\partial^G\left(\partial_A\right)^{-s_1+s_2+n}\left(\partial_C\right)^{s_1-1}\phi^{B(s_2)}\Big];$$

$$\mathcal{J}^L_{s-s_1-s_2,A(s)} = -\frac{(s-2)!\,(s_1-1)!\,(s_2-1)!}{(2s)!}\frac{\left(1+(-1)^{s+s_1+s_2}\right)}{(s+s_1-s_2-1)!} i^{s+s_1-s_2+1}2^{s_1}\left(-1\right)^{s+s_1}$$

$$\sum_{n=0}^{s}(-1)^n\left(\begin{array}{c} s+s_1-s_2 \\ n\end{array}\right)\left(\begin{array}{c} s-s_1+s_2 \\ s-n\end{array}\right)\cdot$$

$$\cdot\left[\left(\partial_A\right)^n\phi^{B(s_1)}\sum_{m=0}^{s_2}(-1)^m\left(\begin{array}{c} s_2 \\ s_2-m\end{array}\right)\left(\partial_A\right)^{s-n-m}\left(\partial_B\right)^{s_1-s_2+m}\phi_{B(s_2-m)A(m)}\right].$$

# Fronsdal equations with currents

- Summation over $n$ can be performed using Vandermonde's identity

$$\sum_{n=0}^{c} \left( \begin{array}{c} a \\ n \end{array} \right) \left( \begin{array}{c} b \\ c-n \end{array} \right) = \left( \begin{array}{c} a+b \\ c \end{array} \right)$$

  and lopping off longitudinal contributions $\Box \phi_{A\ldots} = D_A \left(\ldots\right) + \ldots$

- After rescaling fields as

$$\phi_{A(n)} \longrightarrow \frac{2^{-\frac{n}{2}} \left(-i\right)^{n+1}}{\left(n-1\right)!} \phi_{A(n)},$$

  one gets the final answer

$$\Box \phi_{A(s)} + \ldots = \left(1 + \left(-1\right)^{s+s_1+s_2}\right) 2^{\frac{s+s_1+s_2}{2}}$$
$$\left\{ \frac{\left(-1\right)^{s_2} \cos\left(2\varphi\right)}{2 \cdot \Gamma\left(s+s_1+s_2\right)} \left(\partial_A\right)^s \left(\partial_B\right)^{s_2} \phi^{C(s_1)} \cdot \left(\partial_C\right)^{s_1} \phi^{B(s_2)} - \right.$$
$$- \frac{\left(-1\right)^{s_2} \sin\left(2\varphi\right)}{2 \cdot \Gamma\left(s+s_1+s_2\right)} \epsilon_{AECF} \partial^E \left(\partial_A\right)^{s-1} \left(\partial_B\right)^{s_2} \phi^{C(s_1)} \cdot \partial^F \left(\partial_C\right)^{s_1-1} \phi^{B(s_2)} +$$
$$\left. + \frac{\left(-1\right)^s}{\Gamma\left(s+s_1-s_2\right)} \sum_{n=0}^{s_2} k_n \left(\partial_M\right)^{s-n} \phi^{B(s_1)} \cdot \left(\partial_B\right)^{s_1-s_2+n} \phi_{B(s_2-n)M(n)} \right\} + \ldots$$

$$k_n := 2^{-s_2} \left( \begin{array}{c} s_2 \\ n \end{array} \right), \qquad \sum_{n=0}^{s_2} k_n = 1, \qquad \eta = \exp\left(i\varphi\right).$$

# Fronsdal equations with currents

The lower-derivative part of the current is independent of phase angle $\varphi$. For the higher-derivative one there are two distinguished situations:

- parity-invariant models ($\varphi = 0, \frac{\pi}{2}$), where $P$-odd component vanishes

$$\Box\phi_{A(s)} + \ldots = \pm\left(1 + (-1)^{s+s_1+s_2}\right)(-1)^{s_2}\frac{2^{\frac{s+s_1+s_2}{2}-1}}{\Gamma\left(s+s_1+s_2\right)}\cdot$$
$$\cdot\left(\partial_A\right)^s\left(\partial_B\right)^{s_2}\phi^{C(s_1)}\cdot\left(\partial_C\right)^{s_1}\phi^{B(s_2)} + \ldots$$

  (agree with [Metsaev, Mod.Phys.Lett.A 6 (1991) 359], [Sleight, Taronna, arXiv:1603.00022])

- $\varphi = \frac{\pi}{4}$ model, where $P$-even component vanishes:

$$\Box\phi_{A(s)} + \ldots = -\left(1 + (-1)^{s+s_1+s_2}\right)(-1)^{s_2}\frac{2^{\frac{s+s_1+s_2}{2}-1}}{\Gamma\left(s+s_1+s_2\right)}\cdot$$
$$\cdot\epsilon_{AECF}\partial^E\left(\partial_A\right)^{s-1}\left(\partial_B\right)^{s_2}\phi^{C(s_1)}\cdot\partial^F\left(\partial_C\right)^{s_1-1}\phi^{B(s_2)} + \ldots$$

# Conclusions

- Quadratic corrections to Fronsdal equations generated by gauge-invariant HS currents are explicitly obtained;
- For $A$-model results agree with that available in the literature;
- For $\varphi = \frac{\pi}{4}$ model $P$-even higher-derivative contribution vanishes – it would be interesting to study a holographic implication of that;
- To do: theory with fermions, quadratic gauge-nonivariant corrections.