Analysis of the one-loop divergences in $6D$, $\mathcal{N}=(1,0)$ and $\mathcal{N}=(1,1)$ SYM theories

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We consider $6D$, $\mathcal{N}=(1,0)$ and $\mathcal{N}=(1,1)$ supersymmetric Yang–Mills (SYM) theories. Our aim is to study an *off-shell* structure of the one-loop divergences of $6D$, $\mathcal{N}=(1,0)$ and $\mathcal{N}=(1,1)$ SYM theories, in both the gauge multiplet and the hypermultiplet sectors. We consider the $6D$, $\mathcal{N}=(1,0)$ model in which the SYM multiplet interacts with the hypermultiplet in an arbitrary representation of gauge group. We develop the background superfield method in case of $6D$, $\mathcal{N}=(1,0)$ SYM theory with the hypermultiplet matter.

It is well known that both $6D$, $\mathcal{N}=(1,0)$ and $6D$, $\mathcal{N}=(1,1)$ SYM theories at one loop are on-shell finite [**P. S. Howe, K. S. Stelle** (1984); **E. S. Fradkin and A. A. Tseytlin** (1983);**N. Markus and A. Sagnotti** (1984);**G. Bossard, P. S. Howe and K. S. Stelle** (2009); **L. V. Bork, D. I. Kazakov, M. V. Kompaniets, D. M. Tolkachev and D. E. Vlasenko** (2015);**A. Smilga** (2016)]. Our aim is the explicit proof of the absence of one-loop logarithmic divergencies in $6D$, $\mathcal{N}=(1,1)$ SYM theory *off shell*. We demonstrate this by calculating the divergent part of the one-loop effective action in $\mathcal{N}=(1,1)$ SYM theory formulated in terms of $\mathcal{N}=(1,0)$ harmonic gauge and hypermultiplet superfields, both in the adjoint representation of the gauge group.

[1] The talk is based on

I.L. Buchbinder, E.A. Ivanov, K.V. Stepanayantz, B.S.M., JHEP (2017), arXiv:1612.03190.

Plan

- Gauge theory in $6D$, $\mathcal{N} = (1,0)$ harmonic superspace
- The model and background field method
- Divergent part of the one-loop effective action
- Conclusion

We use the notations and conventions from [**G. Bossard, E. Ivanov, A. Smilga,** *JHEP* (2015)] (in $4D$ case see [**A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky, E. S. Sokatchev,** *Harmonic Superspace*, 2001]).
The $6D$, $\mathcal{N}=(1,0)$ harmonic superspace was introduced in [**P.S. Howe, K.S. Stelle, P.C. West,** *Class. Quant. Grav.* 2 (1985); **B.M. Zupnik**, *Sov. J. Nucl. Phys.* 44 (1986)] and in the central basis is parametrized by the coordinates

$$(z,u) = (x^M, \theta_i^a, u^{\pm i}) \quad M = 0,..,5, \quad a = 1,..4, \,. \tag{1}$$

where θ_i^a are Grassmann pseudoreal left-handed spinors and $u^{\pm i}$ $(u_i^- = (u_i^+)^*, u^{+i}u_i^- = 1)$ describe the "harmonic sphere" $SU(2)_R/U(1)$.
The harmonic superspace in the analytic basis is parametrized by

$$(\zeta, u) = (x^M_{(\mathrm{an})}, \theta^{+a}, u^{\pm i})\,, \tag{2}$$

$$x^M_{(\mathrm{an})} = x^M + \frac{i}{2}\theta_k^a \gamma^M_{ab}\theta_l^b u^{+k}u^{-l}, \quad \theta^{\pm a} = u_k^{\pm}\theta^{ak} \tag{3}$$

The $6D, \mathcal{N} = (1,0)$ gauge theory can be describe in terms of the hermitian analytic gauge connection $V^{++}(\zeta, u) = \widetilde{V^{++}(\zeta, u)}$,

$$V^{++} = (V^{++})^A T^A, \qquad (T^A)^+ = T^A, \tag{4}$$

where T^A are the generators of the gauge group

$$[T^A, T^B] = i f^{ABC} T^C, \ \mathrm{tr}\,(T^A T^B) = T(R)\delta^{AB}, \ (T^A)_m{}^l (T^A)_l{}^n = C(R)_m{}^n.$$

For the adjoint representation the generators are written as $(T^C_{\mathsf{Adj}})_A{}^B = i f^{ACB}$ and

$$T(\mathsf{Adj}) = C_2, \qquad C(\mathsf{Adj})_m{}^n = C_2 \delta^n_m, \tag{6}$$

where C_2 is the second Casimir operator.
In the fundamental representation $T^A_{\mathsf{f}} \equiv t^A$ are normalized by the condition $\mathsf{tr}(t^A t^B) = \frac{1}{2}\delta^{AB}$.

The component expansion of $V^{++}(\zeta, u)$ in the Wess-Zumino gauge

$$V^{++}_{WZ} = \theta^{+\alpha}\theta^{+\beta}A_{\alpha\beta}(x_A) + (\theta^+)^3_\alpha\lambda^{-\alpha}(x_A) + 3(\theta^+)^4F^{--}(x_A)\ , \qquad (7)$$

involves the physical fields $A_{\alpha\beta} = (\gamma^m)_{\alpha\beta}A_m$ and $\lambda^{i\,\alpha}$ and the auxiliary ones which are collected in F^{--}.

We introduce the non-analytic harmonic connection $V^{--}(z, u) = (V^{--})^A T^A$ as a solution of the zero-curvature condition

$$D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0. \qquad (8)$$

The condition (8) can be solve for V^{--} in the form

$$V^{--}(z, u) = \sum_{n=1}^{\infty}(-i)^{n+1}\int du_1 \dots du_n \frac{V^{++}(z, u_1)\dots V^{++}(z, u_n)}{(u^+u_1^+)(u_1^+u_2^+)\dots(u_n^+u^+)}\,. \qquad (9)$$

With the help of the connection V^{--} we construct the spinor and the vector superfield connections and determine the superfield strength $W^{+a} = -\frac{1}{6}\varepsilon^{abcd}D^+_bD^+_cD^+_dV^{--}$.

The model

The superfield action of $6D$, $\mathcal{N}=(1,0)$ SYM interacting with a hypermultiplet has the form

$$S_0[V^{++},q^+] = \frac{1}{f^2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\mathrm{tr}\int d^{14}z\,du_1\dots du_n\frac{V^{++}(z,u_1)\dots V^{++}(z,u_n)}{(u_1^+u_2^+)\dots(u_n^+u_1^+)} - \int d\zeta^{(-4)}du\,\tilde{q}^{+\,m}(\nabla^{++})_m{}^n q_n^+\,, \quad (10)$$

where f is a dimensionful coupling constant ($[f]=-1$) and $(\nabla^{++})_m{}^n = D^{++}\delta_m{}^n + i(V^{++})^C(T^C)_m{}^n$. The hypermultiplet $q^+(x,\theta)$ has a short expansion $q^+(z)=f^+(x)+\theta^{\alpha+}\psi_\alpha(x)+\dots$, with a doublet of massless scalars f^i and the spinor ψ_α fields.
The action (10) is invariant under the gauge transformation

$$V^{++\prime} = -ie^{i\lambda}D^{++}e^{-i\lambda} + e^{i\lambda}V^{++}e^{-i\lambda}\,, \qquad q^{+\prime} = e^{i\lambda}q^+\,, \quad (11)$$

where $\lambda=\lambda(\zeta,u)$ is an analytic gauge parameter.

According to the general analysis performed in [**G. Bossard, E. Ivanov, A. Smilga**, *JHEP* (2015)] the logarithmic divergences in the one-loop approximation can be written as

$$\Gamma^{(1)}_{\text{ln}} = \int d\zeta^{(-4)}\, du \Big[c_1 (F^{++A})^2 + i c_2 F^{++A} (\widetilde{q}^{+})^m (T^A)_m{}^n (q^+)_n + c_3 \Big((\widetilde{q}^{+})^m (q^+)_m \Big)^2 \tag{12}$$

where c_1, c_2, and c_3 are numerical real coefficients. Here F^{++} is an analytic superfield, $F^{++} = (D^+)^4 V^{--}$, where $(D^+)^4 = -\frac{1}{24}\varepsilon^{abcd} D^+_a D^+_b D^+_c D^+_d$.

Using superficial degree of divergences [**I.L.Buchbinder, E.A. Ivanov, K.V. Stepanayantz, B.S.M**, *Phys.Lett. B (2016)*, arXiv:1609.00975] we show that the last term in (12) does not contribute to divergent part of effective action.

Our aim is to obtain the coefficients c_1 and c_2 by direct calculation in the one-loop approximation.

Following the background field method in superspace [**S. J. Gates, M. T. Grisaru, M. Rocek, W. Siegel,** *Superspace*, 1983] we split the fields V^{++}, q^{+} into the sum of the 'background' fields V^{++}, Q^{+} and the 'quantum' ones v^{++}, q^{+} and then we expand the Lagrangian in a power series in quantum fields.

$$V^{++} \to V^{++} + f v^{++}, \qquad q^{+} \to Q^{+} + q^{+} . \tag{13}$$

In order to construct the effective action for the model under consideration, we follow the Faddeev-Popov ansatz. In the full analogy with $4D$, $\mathcal{N} = 2$ case [**I.L. Buchbinder, E.I. Buchbinder, S.V. Kuzenko, B.A. Ovrut**, Phys. Lett. B (1998)], we use the gauge fixing function in the form

$$\mathcal{F}_{\tau}^{(+4)} = D^{++} v_{\tau}^{++} = e^{-ib} (\nabla^{++} v^{++}) e^{ib} = e^{-ib} \mathcal{F}^{(+4)} e^{ib} \,, \tag{14}$$

where $b(z)$ is a background-dependent gauge bridge superfield.

According to (14), the gauge-fixing part of the quantum field action has the form

$$S_{GF}[v^{++},V^{++}] = -\frac{1}{2}\mathrm{tr}\int d^{14}z du_1 du_2 \frac{v_\tau^{++}(1)v_\tau^{++}(2)}{(u_1^+u_2^+)^2} + \frac{1}{4}\mathrm{tr}\int d^{14}z du\, v_\tau^{++}(D^{--})^2 \qquad (15)$$

The action (15) depends on the background field V^{++} through the background gauge bridge b, $v_\tau^{++} = e^{-ib}v^{++}e^{ib}$. The final expression for the effective action is

$$e^{i\Gamma[V^{++},Q^+]} = \mathrm{Det}^{1/2}\,\widehat{\Box}\int \mathcal{D}v^{++}\,\mathcal{D}q^+\,\mathcal{D}\mathbf{b}\,\mathcal{D}\mathbf{c}\,\mathcal{D}\varphi\; e^{iS_{quant}[v^{++},q^+,\mathbf{b},\mathbf{c},\varphi,V^{++},Q^+]}. \qquad (16$$

Here, the quantum action S_{quant} has the structure

$$\begin{aligned} S_{quant} \quad = \quad & S_0[V^{++}+fv^{++},Q^++q^+] + S_{GF}[v^{++},V^{++}] \\ & +S_{FP}[\mathbf{b},\mathbf{c},v^{++},V^{++}] + S_{NK}[\varphi,V^{++}], \end{aligned} \qquad (17)$$

where S_0 is a classical action of the theory.

The corresponding ghosts action are

$$S_{FP}[\mathbf{b},\mathbf{c},v^{++},V^{++}] = \mathrm{tr}\int d\zeta^{(-4)}du\ \mathbf{b}\nabla^{++}(\nabla^{++}\mathbf{c}+i[v^{++},\mathbf{c}]). \quad (18)$$

and

$$S_{NK}[\phi,V^{++}] = \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}du\,\varphi(\nabla^{++})^2\varphi\,. \quad (19)$$

Like in $4D$ case, φ is the Nielsen-Kallosh ghost. As a result, we see that the $6D$, $\mathcal{N}=(1,0)$ SYM theory, in the close analogy with $4D$, $\mathcal{N}=2$ SYM, in the background field approach is described by the three ghosts: two fermionic ghosts $\mathbf{b}$ and $\mathbf{c}$ together with the single bosonic ghost φ.

In the one-loop approximation, the first quantum correction to the classical action, $\Gamma^{(1)}[V^{++},\ Q^{+}]$, is given by the quadratic action S_2:

$$
\begin{aligned}
S_2 \quad = \quad & \frac{1}{2}\int d\zeta^{(-4)}du\, v^{++\,A}\,\widehat{\Box}^{AB}\, v^{++\,B} + \int d\zeta^{(-4)}du\,\mathbf{b}^A(\nabla^{++})^{2\,AB}\mathbf{c}^B \\
& +\frac{1}{2}\int d\zeta^{(-4)}du\,\varphi^A(\nabla^{++})^{2\,AB}\varphi^B - \int d\zeta^{(-4)}du\,\tilde{q}^{+\,m}(\nabla^{++})_m{}^n q^+_n \qquad (20) \\
& -\int d\zeta^{(-4)}du\Big\{\widetilde{Q}^{+\,m} if(v^{++})^C(T^C)_m{}^n q^+_n + \tilde{q}^{+\,m} if(v^{++})^C(T^C)_m{}^n Q^+_n\Big\},
\end{aligned}
$$

The operator $\widehat{\Box}= \frac{1}{2}(D^+)^4(\nabla^{--})^2$ transforms the analytic superfields v^{++} into analytic superfields and has the following structure

$$
\begin{aligned}
\widehat{\Box}^{AB} \quad = \quad & \frac{1}{2}(D^+)^4\Big\{(D^{--})^2\delta^{AB} - 2f^{ACB}(V^{--})^C D^{--} - f^{ACB}(D^{--}V^{--})^C \\
& +f^{ACE}f^{EDB}(V^{--})^C(V^{--})^D\Big\}. \qquad (21)
\end{aligned}
$$

For the next step we consider the special change of hypermultiplet variables [**I.L. Buchbinder, N.G. Pletnev,** *JHEP* 0704; **S. M. Kuzenko, S. J. Tyler,** *JHEP* 0705] in the one-loop effective action

$$q^+_n(1) = h^+_n(1) - f \int d\zeta_2^{(-4)} du_2 \, G_{(1,1)}(1|2)_n{}^p \, iv^{++\,C}(2) \, (T^C)_p{}^l \, Q^+_l(2) \,, \qquad (22)$$

with h^+_n are new independent variable in the path integral describing the hypermultiplet. Here $G^{(1,1)}(\zeta_1, u_1|\zeta_2, u_2)$ is the hyper-multiplet Green function which satisfies the equation

$$\begin{aligned} (\nabla^{++}_1)_m{}^p G_{\lambda\,(1,1)}(1|2)_p{}^n &= \delta^n_m \delta^{(3,1)}_A(1|2) \,. \\ G_{\tau\,(1,1)}(1|2)_m{}^n &= (\widehat{\Box}_1^{-1})_m{}^n (\nabla^+_1)^4 (\nabla^+_2)^4 \frac{\delta^{14}(z_1 - z_2)}{(u^+_1 u^+_2)^3} \,. \end{aligned} \qquad (23)$$

where $\delta^{(3,1)}_A(1|2)$ is the covariantly analytic delta-function and $\widehat{\Box}$ is the covariantly analytic d'Alembertian

$$\begin{aligned} \widehat{\Box}_m{}^n = \frac{1}{2}(D^+)^4 \Big\{ & (D^{--})^2 \delta^n_m + 2i(V^{--})^C (T^C)_m{}^n D^{--} + i(D^{--}V^{--})^C (T^C)_{m}{}^{n} \\ & -(V^{--})^C (V^{--})^D (T^C T^D)_m{}^n \Big\} \,. \end{aligned} \qquad (2$$

After performing the shift (22), the quadratic part of the action S_2 (20) splits into few terms, each being bilinear in quantum superfields:

$$
\begin{aligned}
S_2 \;=\; & \frac{1}{2}\int d\zeta_1^{(-4)}\, d\zeta_2^{(-4)}\, v_1^{++\,A}\Big\{ \widehat{\square}^{AB}\; \delta_A^{(3,1)}(1|2) - 2f^2 \widetilde{Q}_1^{+\,m}\big(T^A G_{(1,1)} T^B\big)_m{}^n Q^+_{n2}\Big\} v_2^{+
\\
& + \int d\zeta^{(-4)} du\, \mathbf{b}^A (\nabla^{++})^{2\,AB}\mathbf{c}^B + \frac{1}{2}\int d\zeta^{(-4)} du\, \varphi^A (\nabla^{++})^{2\,AB}\varphi^B
\\
& - \int d\zeta^{(-4)} du\, \tilde{h}^{+\,m} (\nabla^{++})_m{}^n h^+_n\,. \qquad (
\end{aligned}
$$

Starting from the action (25) one can construct the one-loop quantum correction $\Gamma^{(1)}[V^{++}, Q^+]$ to the classical action (10), which has the following formal expression

$$
\begin{aligned}
\Gamma[V^{++}, Q] \;=\; & \frac{i}{2}\mathrm{Tr}\ln\Big\{ \widehat{\square}^{AB} - 2f^2 \widetilde{Q}^{+\,m}\big(T^A G_{(1,1)} T^B\big)_m{}^n Q^+_n\Big\} - \frac{i}{2}\mathrm{Tr}\ln \widehat{\square}_{\mathrm{Adj}}
\\
& - i\mathrm{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + \frac{i}{2}\mathrm{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + i\mathrm{Tr}\ln \nabla^{++}_{\mathrm{R}}\,, \qquad (26)
\end{aligned}
$$

where subscripts Adj and R mean that the corresponding operators are taken in the adjoint representation and in R representation for hypermultiplet.

Divergent part of the one-loop effective action

The $(F^{++})^2$ part of the effective action depends only on the background vector multiplet V^{++} and is defined by the last three terms in Eq. (26). More precisely,

$$\begin{aligned}\Gamma^{(1)}_{F^2}[V^{++}] &= -i\mathsf{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + \frac{i}{2}\mathsf{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + i\mathsf{Tr}\ln\nabla^{++}_{\mathrm{R}} \\ &= -i\mathsf{Tr}\ln\nabla^{++}_{\mathrm{Adj}} + i\mathsf{Tr}\ln\nabla^{++}_{\mathrm{R}}\,. \end{aligned} \quad (27)$$

Let us vary the expression (27) with respect to the background gauge multiplet $(V^{++})^A$

$$\delta\Gamma^{(1)}_{F^2}[V^{++}] = i\mathrm{Tr}\, f^{ACB}\,\delta(V^{++})^C\, G^{BA}_{(1,1)} - \mathrm{Tr}\,(T^C)_m{}^n\,\delta(V^{++})^C\,(G_{(1,1)})_n{}^m\,. \quad (28)$$

Here $(G_{(1,1)})_n{}^m$ is the superfield Green function (23) for operator $(\nabla^{++})_n{}^m$ acting on the superfields in the representation R of gauge group to which the hypermultiplet belongs. Also we denoted $G^{BA}_{(1,1)}$ the Green function for the operator $(\nabla^{++})^{BA}$, which acts on superfields in adjoint representation. The Green function $G^{BA}_{(1,1)}$ has the structure similar to (23).

In the dimension regularization scheme the divergences are associated with the pole terms of the form $\frac{1}{\varepsilon}$, $\varepsilon \to 0$, with $d = 6 - \varepsilon$. Taking into account the expression for the Green functions (23), we obtain

$$\delta\Gamma^{(1)}_{F^2}[V^{++}] = i\int d\zeta_1^{(-4)} du_1 \delta(V_1^{++})^C \Big\{ f^{ACB} G^{BA}(1|2) + i(T^C)_m{}^n G_n{}^m(1|2) \Big\}\Big|^{2=1}_{\mathrm{div}} .$$
$$= -i\int d\zeta_1^{(-4)} du_1 \delta(V_1^{++})^C \int_0^\infty d(is)(is\mu^2)^{\frac{\varepsilon}{2}} \qquad (29)$$
$$\times \Big\{ f^{ACB}(e^{is\widehat{\Box}_1})^{BA} + i(T^C)_m{}^n (e^{is\widehat{\Box}_1})_n{}^m \Big\} (\nabla_1^+)^4 (\nabla_2^+)^4 \frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3}\Big|^{2=1}_{\mathrm{div}} .$$

Here s is the proper-time parameter and μ is an arbitrary regularization parameter of mass dimension.
Like in $4D$ case [**S.M. Kuzenko, I.N. McArthur,** *Phys. Lett. B* 506 (2001)] one can obtain useful identity

$$(\nabla_1^+)^4 (\nabla_2^+)^4 \frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3} = (\nabla_1^+)^4 \Big\{ (u_1^+ u_2^+)(\nabla_1^-)^4 - (u_1^- u_2^+)\Omega_1^{--} + \widehat{\Box}_1 \frac{(u_1^- u_2^+)^2}{(u_1^+ u_2^+)} \Big\} \delta^{14}(z_1 - \qquad (30)$$

where we have introduced the notation

$$\Omega^{--} = i\nabla^{ab}\nabla_a^-\nabla_b^- - W^{-a}\nabla_a^- + \frac{1}{4}(\nabla_a^- W^{-a}) \, . \qquad (31)$$

The divergent contributions arise from the first two contribution in (30). The result is

$$\delta\Gamma^{(1)}_{F^2}[V^{++}] = \frac{(C_2 - T(R))}{3(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du\, \delta V^{++A}\, \widehat{\Box}\, F^{++A}\,. \tag{32}$$

It is ese to see that up to an unessential additive constant the last expression can be obtained from

$$\Gamma^{(1)}_{F^2} = \frac{C_2 - T(R)}{6(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du\, (F^{++A})^2\,. \tag{33}$$

The hypermultiplet-dependent part $\widetilde{Q}^{+}F^{++}Q^{+}$ of the one-loop counterterm comes out from the first term in (26). To calculate this contribution, one expands the logarithm in the first term (26) up to the first order and computes the functional trace,

$$\frac{i}{2}\mathrm{Tr}\ln\Big\{\,\widehat{\Box}^{AB}\;-2f^{2}\widetilde{Q}^{+\,m}\big(T^{A}G_{(1,1)}T^{B}\big)_{m}{}^{n}Q^{+}_{n}\Big\} = \frac{i}{2}\mathrm{Tr}\ln\widehat{\Box} + \frac{i}{2}\mathrm{Tr}\ln\Big\{\delta^{AB} - 2f^{2}(\widehat{\Box}^{-1})^{AC}\widetilde{Q}^{+\,m}\big(T^{C}G_{(1,1)}T^{B}\big)_{m}{}^{n}Q^{+}_{n}\Big\}. \quad (34)$$

We note that, like in $4D$, $\mathcal{N}=2$ SYM theory, the term $\frac{i}{2}\mathrm{Tr}\,\ln\widehat{\Box}$ does not contribute to the divergent part.

We decompose the logarithm up to the first order and compute the functional trace

$$\Gamma^{(1)}_{QFQ} = -if^{2}\int d\zeta^{(-4)}du\,\widetilde{Q}^{+\,m}Q^{+}_{n}\,(\widehat{\Box}^{-1})^{AB}\big(T^{B}G_{(1,1)}T^{A}\big)_{m}{}^{n}\Big|^{2=1}_{\mathrm{div}} = -if^{2}\int d\zeta^{(-4)}du\,\widetilde{Q}^{+\,m}Q^{+}_{n} \times(\widehat{\Box}^{-1})^{AB}\big(T^{B}\,\widehat{\Box}^{-1}\,T^{A}\big)_{m}{}^{n}(u^{+}_{1}u^{+}_{2})\,\delta^{6}(x_{1}-x_{2})\Big|_{2=1}. \quad (35)$$

Then we decompose the inverse covariant d'Alembertians (21) and (??) up to the second order and obtain

$$\begin{aligned}
\Gamma^{(1)}_{QFQ} &= -if^2 \int d\zeta^{(-4)} du\, \widetilde{Q}^{+\,m} Q^+_n \left(\frac{\delta^{AB}}{\Box_1} + 2f^{ACB}(F^{++})^C \frac{D_1^{--}}{\Box_1^2} \right) \\
&\quad \times (T^B)_m{}^p \left(\frac{\delta^l_p}{\Box_1} - 2i(F^{++})^C (T^C)_p{}^l \frac{D_1^{--}}{\Box_1^2} \right) (T^A)_l{}^n (u_1^+ u_2^+) \delta^6(x_1 - x_2)\Big|_{2 \\
&= 2if^2 \int d\zeta^{(-4)} du\, \widetilde{Q}^{+\,m} Q^+_n (F^{++})^C \\
&\quad \times \Big\{ f^{ACB}(T^B T^A)_m{}^n - i(T^A T^C T^A)_m{}^n \Big\} \frac{1}{\Box_1^3} \delta^6(x_1 - x_2)\Big|_{2=1} . \qquad (3
\end{aligned}$$

We rewrite the expression within the brackets using commutation relation for the generators we have

$$f^{ACB} T^B T^A - i T^A T^C T^A = 2 f^{ACB} T^B T^A - i T^A T^A T^C \,, \qquad (37)$$

$$f^{ACB} T^B T^A = \frac{i}{2} f^{ACB} f^{BAD} T^D = \frac{i}{2} C_2 T^C \,, \qquad (38)$$

Divergent part of the one-loop effective action

Passing to momentum representation we finally obtain

$$\Gamma^{(1)}_{QFQ}[V^{++},Q^{+}] = -\frac{2if^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)}du \, \widetilde{Q}^{+\,m}(C_2\delta^l_m - C(R)_m{}^l)(F^{++})^A\,(T^A)_l{}^n\,Q^+_n. \quad (39)$$

Summing up the contributions (33) and (39), we finally obtain the total divergent contribution

$$\Gamma^{(1)}_{div}[V^{++},Q^{+}] = \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon}\,\mathrm{tr}\int d\zeta^{(-4)}du\,(F^{++})^2 - \frac{2if^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)}du\,\widetilde{Q}^{+}(C_2 - C(R))F^{++}Q^{+}. \quad (40)$$

Conclusion

We see that the coefficients of the $(F^{++})^2$ and $\widetilde{Q}^+ F^{++} Q^+$ terms in the divergent part of one-loop effective action are proportional to the differences between the second order Casimir operator for the adjoint representation of gauge group and the operators $T(R)$ and $C(R)$ for the hypermultiplet representation R, respectively.
$6D$, $\mathcal{N} = (1,1)$ supersymmetric Yang-Mills theory involves the hypermultiplet in adjoint representation of gauge group where

$$T(\text{Adj}) = C_2, \qquad C(\text{Adj})_m{}^n = C_2 \delta^n_m. \tag{41}$$

As consequence (40) vanishes for this case. Hence, the $6D$, $\mathcal{N} = (1,1)$ SYM theory is one-loop finite *off-shell*.