Basic quantizations of the complex $D = 4$ Lie algebra $\mathfrak{o}(4;\mathbb{C})$ and of its real forms $\mathfrak{o}(4)$, $\mathfrak{o}(3,1)$, $\mathfrak{o}(2,2)$ and $\mathfrak{o}^{\star}(4)$

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Quantum deformations of $D = 4$ Euclidean, Lorentz, Kleinian and quaternionic $\mathfrak{o}^\star(4)$ symmetries in unified $\mathfrak{o}(4;\mathbb{C})$ setting

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A B S T R A C T

We employ new calculational technique and present complete list of classical r-matrices for $D = 4$ complex homogeneous orthogonal Lie algebra $\mathfrak{o}(4;\mathbb{C})$, the rotational symmetry of four-dimensional complex space–time. Further applying reality conditions we obtain the classical r-matrices for all possible real forms of $\mathfrak{o}(4;\mathbb{C})$: Euclidean $\mathfrak{o}(4)$, Lorentz $\mathfrak{o}(3,1)$, Kleinian $\mathfrak{o}(2,2)$ and quaternionic $\mathfrak{o}^\star(4)$ Lie algebras. For $\mathfrak{o}(3,1)$ we get known four classical $D = 4$ Lorentz r-matrices, but for other real Lie algebras (Euclidean, Kleinian, quaternionic) we provide new results and mention some applications.

1. Introduction

In recent years due to efforts to construct quantum gravity characterized by noncommutative space–time structures at Planckian distances [1–3], the ways in which one deforms the space–time coordinates and space–time symmetries became important. A principal tool for the classification of quantum deformations is provided by the classical r-matrices [4–7].

In this paper we shall consider $D = 4$ orthogonal Lie algebras

important in present studies of gravity models and string theory, in particular for the formulation of quantum-deformed field-theoretic models and related gravity/gauge correspondence. Since introduction in 2002 two-dimensional Yang–Baxter deformed σ-models [10–12] there are available techniques linking classical r-matrices of space–time symmetry algebras with various gravity solutions describing the string theory backgrounds [13–18]. In such framework the classical r-matrices are useful in description of gravity/gauge correspondence for the gauge sector described by non-

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Quantizations of $D = 3$ Lorentz symmetry

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Abstract Using the isomorphism $\mathfrak{o}(3;\mathbb{C}) \simeq \mathfrak{sl}(2;\mathbb{C})$ we develop a new simple algebraic technique for complete classification of quantum deformations (the classical r-matrices) for real forms $\mathfrak{o}(3)$ and $\mathfrak{o}(2,1)$ of the complex Lie algebra $\mathfrak{o}(3;\mathbb{C})$ in terms of real forms of $\mathfrak{sl}(2;\mathbb{C})$: $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$. We prove that the $D = 3$ Lorentz symmetry $\mathfrak{o}(2,1) \simeq \mathfrak{su}(1,1) \simeq \mathfrak{sl}(2;\mathbb{R})$ has three different Hopf-algebraic quantum deformations, which are expressed in the simplest way by two standard $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ q-analogs and by simple Jordanian $\mathfrak{sl}(2;\mathbb{R})$ twist deformation. These quantizations are presented in terms of the quantum Cartan–Weyl generators for the quantized algebras $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ as well as in terms of quantum Cartesian generators for the quantized algebra $\mathfrak{o}(2,1)$. Finally, some applications of the deformed $D = 3$ Lorentz symmetry are mentioned.

bra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, Lie bialgebras $(\mathfrak{g}, \delta)$ play an essential role (see e.g. [4–7]). Here the *co-bracket* δ is a linear skew-symmetric map $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ with the relations consistent with the Lie bracket in $\mathfrak{g}$:

$$\delta([x, y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] - [y \otimes 1 + 1 \otimes y, \delta(x)],$$
$$(\delta \otimes \mathrm{id})\delta(x) + \text{cycle} = 0 \qquad (1.1)$$

for any $x, y \in \mathfrak{g}$. The first relation in (1.1) is a condition of the 1-cocycle and the second one is the co-Jacobi identity (see [4,7]). The Lie bialgebra $(\mathfrak{g}, \delta)$ is a correct infinitesimalization of the quantum Hopf deformation of $U(\mathfrak{g})$ and the operation δ is an infinitesimal part of the difference between a coproduct Δ and an opposite coproduct $\tilde{\Delta}$ in the Hopf algebra, $\delta(x) = h^{-1}(\Delta - \tilde{\Delta}) \mod h$ where h is a deformation parameter. Any two Lie bialgebras $(\mathfrak{g}, \delta)$ and $(\mathfrak{g}, \delta')$ are isomorphic (equivalent) if they are connected by a $\mathfrak{g}$-automorphism φ satisfying the condition

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Addendum

Addendum to “Quantum deformations of $D = 4$ Euclidean, Lorentz, Kleinian and quaternionic $\mathfrak{o}^\star(4)$ symmetries in unified $\mathfrak{o}(4;\mathbb{C})$ setting” [Phys. Lett. B 754 (2016) 176–181]

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ABSTRACT

In our previous paper [1] we obtained a full classification of nonequivalent quasitriangular quantum deformations for the complex $D = 4$ Euclidean Lie symmetry $\mathfrak{o}(4;\mathbb{C})$. The result was presented in the form of a list consisting of three three-parameter, one two-parameter and one one-parameter nonisomorphic classical r-matrices which provide ‘directions’ of the nonequivalent quantizations of $\mathfrak{o}(4;\mathbb{C})$. Applying reality conditions to the complex $\mathfrak{o}(4;\mathbb{C})$ r-matrices we obtained the nonisomorphic classical r-matrices for all possible real forms of $\mathfrak{o}(4;\mathbb{C})$: Euclidean $\mathfrak{o}(4)$, Lorentz $\mathfrak{o}(3,1)$, Kleinian $\mathfrak{o}(2,2)$ and quaternionic $\mathfrak{o}^\star(4)$ Lie algebras. In the case of $\mathfrak{o}(4)$ and $\mathfrak{o}(3,1)$ real symmetries these r-matrices give the full classifications of the inequivalent quasitriangular quantum deformations, however for $\mathfrak{o}(2,2)$ and $\mathfrak{o}^\star(4)$ the classifications are not full. In this paper we complete these classifications by adding three new three-parameter $\mathfrak{o}(2,2)$-real r-matrices and one new three-parameter $\mathfrak{o}^\star(4)$-real r-matrix. All nonisomorphic classical r-matrices for all real forms of $\mathfrak{o}(4;\mathbb{C})$ are presented in the explicit form what is convenient for providing the quantizations. We will mention also some applications of our results to the deformations of space–time symmetries and string σ-models.

1. Basic quantizations of $\mathfrak{sl}(2;\mathbb{C}) \simeq \mathfrak{o}(3;\mathbb{C})$ and of its real forms $\mathfrak{o}(3) \simeq \mathfrak{su}(2)$ and $\mathfrak{o}(2,1) \simeq \mathfrak{sl}(2;\mathbb{R}) \simeq \mathfrak{su}(1,1)$

1.1 Classical r-matrices of $\mathfrak{sl}(2;\mathbb{C})$ and of its real forms: $\mathfrak{su}(2), \mathfrak{su}(1,1), \mathfrak{sl}(2;\mathbb{R})$

Any classical skew-symmetric r-matrix of arbitrary complex or real Lie algebra $\mathfrak{g}$, $r \in \mathfrak{g} \wedge \mathfrak{g}$, satisfy CYBE or mCYBE:

$$[[r,\, r]] \;=\; \tilde{\Omega} \qquad r = r_{(1)} \wedge r_{(2)}. \tag{1}$$

Here $[[\cdot,\cdot]]$ is the Schouten bracket which for any skew-symmetric two-tensors $r_1 = x \wedge y$ and $r_2 = u \wedge v$ $(x, y, u, v \in \mathfrak{g})$ is given by

$$\begin{aligned} [[x \wedge y,\, u \wedge v]] \;&:=\; x \wedge \big([y,u] \wedge v + u \wedge [y,v]\big) \\ &\quad -y \wedge \big([x,u] \wedge v + u \wedge [x,v]\big) \\ &=\; [[u \wedge v,\, x \wedge y]] \end{aligned} \tag{2}$$

and $\tilde{\Omega}$ is the $\mathfrak{g}$-invariant element, $\tilde{\Omega} \in (\overset{3}{\wedge}\, \mathfrak{g})_{\mathfrak{g}}$ (i.e. $[\Delta^2(x),\tilde{\Omega}] = 0$ for $\forall x \in \mathfrak{g}$) which in the case of $\mathfrak{g} = \mathfrak{sl}(2;\mathbb{C})$ takes unique form

$$\tilde{\Omega} \;=\; \gamma\, \Omega(\mathfrak{sl}(2) \;=\; \gamma\, E_+ \wedge H \wedge E_- \qquad \gamma \in \mathbb{C} \tag{3}$$

Cartan-Weyl basis $(E_{\pm}, H)$:

$$[H, E_{\pm}] = \pm H \qquad [E_{+}, E_{-}] = 2H$$

One can show that *any two-tensor of* $\mathfrak{sl}(2;\mathbb{C}) \wedge \mathfrak{sl}(2;\mathbb{C})$ *is a classical* $\mathfrak{sl}(2;\mathbb{C})$ r*-matrix*. Indeed, let

$$r \ := \ \beta_{+} r_{+} + \beta_{0} r_{0} + \beta_{-} r_{-} \quad (\beta_{+}, \beta_{0}, \beta_{-} \in \mathbb{C}) \tag{4}$$

be arbitrary element of $\mathfrak{sl}(2;\mathbb{C}) \wedge \mathfrak{sl}(2;\mathbb{C})$, where

$$r_{+} \ := \ E_{+} \wedge H, \quad r_{0} \ := \ E_{+} \wedge E_{-}, \quad r_{-} \ := \ H \wedge E_{-} \tag{5}$$

are basis elements of $\mathfrak{sl}(2;\mathbb{C}) \wedge \mathfrak{sl}(2;\mathbb{C})$. All basis elements (5) are classical r-matrices, and one can calculate that

$$[[r, r]] = -4(\beta_{0}^{2} + \beta_{+}\beta_{-})E_{+} \wedge H \wedge E_{-} \ \equiv \ \gamma\Omega. \tag{6}$$

i) If the coefficients of (4) satisfy the condition $\gamma := \beta_{0}^{2} + \beta_{+}\beta_{-} = 0$ then it satisfies the *homogeneous CYBE (CYBE)*
ii) If $\gamma := \beta_{0}^{2} + \beta_{+}\beta_{-} \neq 0$ then it satisfies the *non-homogeneous CYBE (mCYBE)*.

One can show that the general form (4) can be reduced to *one of two monomial skew-symmetric two-tensors* by using $\mathfrak{sl}(2;\mathbb{C})$-automorphisms.
i) If the parameters satisfy the condition $\gamma^2 = \beta_0^2 + \beta_+\beta_- = 0$ then one can check the following relationship:

$$\beta_+ E_+ \wedge H + \beta_0 E_+ \wedge E_- + \beta_- H \wedge E_- \; = \; \beta\, \varphi_0(E_+) \wedge \varphi_0(H), \tag{7}$$

where φ is the explicite $\mathfrak{sl}(2;\mathbb{C})$-automorphism and $\beta = \kappa\beta_+ - \beta_- \quad (\kappa = \pm 1)$.
If $\gamma = 0$ we obtain *Jordanian $\mathfrak{sl}(2;\mathbb{C})$ r-matrix*

$$r_J = \beta\, E_+ \wedge H \tag{8}$$

ii) In the case of $\gamma \equiv \beta_0^2 + \beta_+\beta_- \neq 0$ we have

$$\beta_+ E_+ \wedge H + \beta_0 E_+ \wedge E_- + \beta_- H \wedge E_- \; = \; 2\sqrt{\gamma}\, \varphi_\gamma(E_+) \wedge \varphi_\gamma(E_-), \tag{9}$$

where φ is another explicite $\mathfrak{sl}(2;\mathbb{C})$-automorphism.
Thus, if the general classical r-matrix (4) satisfies the non-homogeneous CYBE ($\gamma \neq 0$) then it can be reduced to *standard Drinfeld-Jimbo* form by using the $\mathfrak{sl}(2;\mathbb{C})$ automorphisms

$$\widehat{r}_{\mathrm{st}} = \gamma\, E_+ \wedge E_- \tag{10}$$

1.2 Complex quantized Lie algebras and universal R-matrix.

Quantization: Lie bialgebra structure $\longrightarrow$ quantum Lie algebras $U_\xi(\hat{g})$
(classical r-matrices) $\longrightarrow$ (quasi-triangular Hopf algebras)

Universal R-matrix $R(\xi) \in U_\xi(\hat{g}) \otimes U_\xi(\hat{g})$ defines flip operation $\tau : (a \otimes b)^\tau = b \otimes a$ on the coproduct Δ_ξ

$$\Delta_\xi^\tau = R(\xi) \circ \Delta_\xi \circ R^{-1}(\xi) \tag{11}$$

and satisfies the quasitriangularity condition

$$(\Delta_\xi \otimes id)R(\xi) = R_{12}(\xi)R_{23}(\xi) \qquad (id \otimes \Delta_\xi)R(\xi) = R_{13}(\xi)R_{12}(\xi) \tag{12}$$

and

$$R(\xi) = 1 \otimes 1 + \xi\,\tilde{r} + O(\xi^2) \qquad \tilde{r} = r_{(1)} \oplus r_{(2)} \tag{13}$$

If antisymmetric r satisfies mCYBE, $\tilde{r}$ is *not antisymmetric.*

If $\tilde{r}$ satisfies *classical YBE* after quantization one obtains *quantum YBE*

$$R_{12}(\xi)R_{13}(\xi)R_{23}(\xi) = R_{23}(\xi)R_{13}(\xi)R_{12}(\xi) \tag{14}$$

If r is antisymmetric ($r_{as} = -r^{\tau}_{as}$) and satisfies CYBE (triangular r-matrix) it defines the twist quantization ($F \in U_{\circ}(\hat{g}) \otimes U_{\circ}(\hat{g})$)

$$R(\xi) = F^{\tau}(\xi) \circ F^{-1}(\xi) \qquad \Delta_{\xi} = F(\xi)\Delta_{\circ}(\xi)F^{-1}(\xi) \tag{15}$$

If r is antisymmetric and satisfies mCYBE one should add to r a symmetric part r_s, which is $\hat{g}$-invariant ($[\Delta(x), r_s] = 0$ for $\forall\, x \in \hat{g}$), with $r^{BD} = r + r_s$ satisfying CYBE (Belavin-Drinfeld form of r-matrix). The resulting universal R-matrix satisfies *quantum YBE*.

Therefore

- – for classification purposes we use $r_{as} = -r^{\tau}_{as}$ satisfying CYBE or mCYBE
- – for quantization of mCYBE case we modify $r_{as} \to r_{BD} = r_{as} + r_s$ ($r^{\tau}_{BD} \neq -r_{BD}$), then CYBE is valid and we obtain $R(\xi)$ satisfying quantum YBE.

1.3 Real quantized Lie algebras - general remarks

$\circledast$-Hopf algebraic structure on complex quantized enveloping algebra $U_q(\mathfrak{g})$ is represented by $\circledast$-involution $(a^\circledast)^\circledast = a$ satisfying conditions for coproducts and antipodes

$$\Delta_q(a^\circledast) \;=\; (\Delta_q(a))^\circledast, \qquad S_q((S_q(a^\circledast))^\circledast) \;=\; a \quad (\forall a \in U_q(\mathfrak{g}) \;. \tag{16}$$

where involution acts in standard way on tensor products

$$(a \otimes b)^\circledast = a^\circledast \otimes b^\circledast \tag{17}$$

One imposes two distinct reality constraints on the universal R-matrices

a) when $r^{\circledast \otimes \circledast} = r^\tau \Leftrightarrow R^{\circledast \otimes \circledast} = R^\tau$ (R is called τ-real);

b) when $r^{\circledast \otimes \circledast} = -r \;\; \Leftrightarrow \;\; R^{\circledast \otimes \circledast} = R^{-1}$ (R is called unitary)

For **triangular** deformation $r^\tau = -r$ both conditions for universal R-matrix *coincide*.

1.4 Standard $\mathfrak{sl}(2;\mathbb{C})$ quantization

One gets quasitriangular Hopf algebra $U_q(\mathfrak{sl}(2;\mathbb{C}))$ $(q = \exp\frac{1}{2}\gamma)$

$$q^H E_\pm = q^{\pm 1} E_\pm\, q^H \;, \qquad [E_+,\, E_-] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}} = \frac{\sinh(\gamma H)}{\sinh(\frac{1}{2}\gamma)} \,, \tag{18}$$

$$\Delta_q(q^{\pm H}) = q^{\pm H} \otimes q^{\pm H} \;, \qquad \Delta_q(E_\pm) \;=\; E_\pm \otimes q^H + q^{-H} \otimes E_\pm \,, \tag{19}$$

$$S_q(q^{\pm H}) = q^{\mp H} \;, \qquad S_q(E_\pm) \;=\; -q^{\pm 1} E_\pm \;, \tag{20}$$

$$R(\gamma) = \exp_{q^{-2}}\Big((q - q^{-1}) E_+\, q^{-H} \otimes q^H E_-\Big) q^{2H \otimes H} \;, \tag{21}$$

corresponding to the non-skew-symmetric Belavin-Drinfeld r-matrix

$$r_{BD} \;=\; \gamma\big(E_+ \otimes E_- + H \otimes H\big) \tag{22}$$

1.5 (Jordanian) nonstandard $\mathfrak{sl}(2;\mathbb{C})$ quantization

From Jordanian r-matrix one derives Jordanian twist

$$F_J = \exp(H \otimes \sigma) \qquad \sigma = ln(1 + \beta E_+) \tag{23}$$

which satisfies 2-cocycle condition defines $R_J = F_J^\tau F_J^{-1}$ as well as deformed coproducts and antipodes, e.g.

$$\Delta_J = F_J \circ \Delta_o \circ F_J^{-1} \tag{24}$$

and

$$\begin{aligned}
\Delta_J(H) &= H \otimes e^{-\sigma} &&+ \quad 1 \otimes H \\
\Delta_J(E_+) &= E_+ \otimes e^{-\sigma} &&+ \quad 1 \otimes E_+ \\
\Delta_J(E_-) &= E_- \otimes e_\sigma &&+ \quad 1 \otimes E_- + 2\beta H \otimes H e^{-\sigma} \\
& &&- \quad \beta^2 H(H-1) \otimes E_+ e^{-2\sigma}
\end{aligned} \tag{25}$$

The algebra sector remains classical and

$$R(\beta) = F_J^\tau \circ F_J^{-1}$$

1.6 Real quantizations of $\mathfrak{sl}(2;\mathbb{C})$;

The complex Lie algebra $\mathfrak{o}(3;\mathbb{C}) \simeq \mathfrak{sl}(2;\mathbb{C})$ admits only two real forms: $\mathfrak{o}(3) \simeq \mathfrak{su}(2)$ and $\mathfrak{o}(2,1) \simeq \mathfrak{su}(1,1) \simeq \mathfrak{sl}(2)$.
For q-deformed standard quantizations there are *three real forms* which are the following ($q = \frac{1}{2}\gamma$)

$$\begin{array}{lllll} i) & H^{*}{=}H, & E_{\pm}^{*} = E_{\mp}, & q \in \mathbb{R} \Leftrightarrow \gamma \in \mathbb{R} & \text{for } \mathfrak{o}_q(3){\simeq}\mathfrak{su}_q(2), \\ ii) & H^{*}{=}H, & E_{\pm}^{*} = -E_{\mp}, & q \in \mathbb{R} \Leftrightarrow \gamma \in \mathbb{R} & \text{for } \mathfrak{o}_q(2,1){\simeq}\mathfrak{su}_q(1,1), \\ iii) & H^{\star}{=}{-}H, & E_{\pm}^{\star} = -E_{\pm}, & |q| = 1 \Leftrightarrow \gamma \in \imath\mathbb{R} & \text{for } \mathfrak{o}'_q(2,1){\simeq}\mathfrak{sl}_q(2), \end{array} \tag{26}$$

In i)–iii) $(H, E_{\pm})$ describes q-deformed CW basis. The last two (non-compact) real forms are isomorphic in the classical limit $\gamma \mapsto 0$.
For the first two cases the corresponding universal R-matrix is τ-real, in the last case it is unitary.
For ξ-deformed Jordanian quantization one gets only *one real form* $\mathfrak{sl}(2)$ with $\xi \in \mathbb{R}$

Real bialgebras for $\mathfrak{su}(2) \simeq \mathfrak{o}(3)$ and $\mathfrak{su}(1,1) \simeq \mathfrak{sl}(2;\mathbb{R}) \simeq \mathfrak{o}(2,1)$ are given by real classical r-matrices.

i) Compact real form $\mathfrak{su}(2) \simeq \mathfrak{o}(3)$

There is *only one real $\mathfrak{su}(2)$ bialgebra* (up to $\mathfrak{su}(2)$ automorphisms)

$$r_{\rm st} = \alpha\, E_+ \wedge E_- \qquad [[r_{\rm st}, r_{\rm st}]] = \alpha^2 \Omega \qquad \alpha - \text{real} \tag{27}$$

ii) Noncompact real form $\mathfrak{su}(1,1) \simeq \mathfrak{o}(2,1)$

There are *three real $\mathfrak{su}(1,1)$ bialgebras* (up to $\mathfrak{su}(1,1)$ automorphism)

$$\begin{aligned} r_{\rm st} &= \alpha E_+ \wedge E_- & [[r_{\rm st}, r_{\rm st}]] &= \alpha^2 \Omega \\ \tilde{r}_{\rm st} &= \alpha (E_+ + E_-) \wedge H & [[\tilde{r}_{\rm st}, \tilde{r}_{\rm st}]] &= -\alpha^2 \Omega \\ r_{\rm qJ} &= \tfrac{\alpha}{2} (i(E_+ - E_-) \wedge H + E_+ \wedge E_-) & [[r_{\rm qJ}, r_{\rm qJ}]] &= 0 \end{aligned} \tag{28}$$

One can map $\mathfrak{su}(1,1)$ basis $(E_\pm, H) \;\rightarrow\; \mathfrak{sl}(2;\mathbb{R})$ basis (E'_+, H')

$$H = -\frac{i}{2}(E'_+ - E'_-) \qquad E_\pm = \mp i H' + \frac{1}{2}(E'_+ + E'_-) \tag{29}$$

One gets the following *three real $\mathfrak{sl}(2;\mathbb{R})$ bialgebras*, isomorphic to $\mathfrak{su}(1,1)$ bialgebras

$$\begin{aligned} r'_{\rm st} &= i\alpha(E'_+ + E'_-)\wedge H' \qquad & [[r'_{\rm st}, r'_{\rm st}]] &= \alpha^2\Omega \\ \tilde{r}'_{\rm st} &= i\alpha E'_+ \wedge E'_- \qquad & [[\tilde{r}'_{\rm st}, \tilde{r}'_{\rm st}]] &= -\alpha^2\Omega \\ r'_{\rm qJ} &= i\alpha E'_+ \wedge H' \qquad & [[r'_{\rm qJ}, r'_{\rm qJ}]] &= 0 \end{aligned} \tag{30}$$

Important conclusion: we are able to quantize all three noncompact real r-matrices if we choose

$$r_{\rm st} = \alpha E_+ \wedge E_- \qquad \tilde{r}'_{\rm st}, = i\alpha E'_+ \wedge E'_- \qquad \tilde{r}'_{\rm qJ} = i\alpha E'_+ \wedge H' \tag{31}$$

i.e. *we should quantize one in $\mathfrak{su}(1,1)$ basis and two in $\mathfrak{sl}(2;\mathbb{R})$ basis*. In this way *both $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ bialgebras are needed* for quantization of real noncompact classical r-matrices $\mathfrak{su}(1,1) \simeq \mathfrak{su}(2;\mathbb{R}) \simeq \mathfrak{o}(2,1)$. Other way of quantization is to use three real $\mathfrak{o}(2,1)$ bialgebras in q-deformed Cartesian $\mathfrak{o}(2,1)$ basis (VN Tolstoy+JL, 2017).

2. Basic quantizations of $\mathfrak{o}(4,\mathbb{C}) = \mathfrak{sl}(2;\mathbb{C}) \oplus \overline{\mathfrak{sl}(2;\mathbb{C})}$ and of its real forms $\mathfrak{o}(4), \mathfrak{o}(3,1), \mathfrak{o}(2,2), \; \mathfrak{o}^*(4)$

2.1 Five basic skew-symmetric $\mathfrak{o}(4;\mathbb{C})$ r-matrices

$$
\begin{aligned}
r_1(\chi) &= \chi(E_+ + \bar{E}_+) \wedge (H + \bar{H}) \;, && \leftarrow CYBE \\
r_2(\chi,\bar{\chi},\xi) &= \chi\, E_+ \wedge H + \bar{\chi}\, \bar{E}_+ \wedge \bar{H} + \xi E_+ \wedge \bar{E}_+ \,, && \leftarrow CYBE \\
r_3(\gamma,\bar{\gamma},\eta) &= \gamma\, E_+ \wedge E_- + \bar{\gamma}\, \bar{E}_+ \wedge \bar{E}_- + \eta\, H \wedge \bar{H} \;, && \leftarrow mCYBE \\
r_4(\gamma,\xi) &= \gamma\left(E_+ \wedge E_- - \bar{E}_+ \wedge \bar{E}_- - 2H \wedge \bar{H}\right) + \xi E_+ \wedge \bar{E}_+, && \leftarrow mCYBE \\
r_5(\gamma,\bar{\chi},\rho) &= \gamma\, E_+ \wedge E_- + \bar{\chi}\, \bar{E}_+ \wedge \bar{H} + \rho H \wedge \bar{E}_+ \;, && \leftarrow mCYBE
\end{aligned}
\tag{32}
$$

$$(E_\pm, H) \subset \mathfrak{sl}(2;\mathbb{C}); (\bar{E}_\pm, \bar{H}) \in \mathfrak{sl}(2,\mathbb{C})$$

$$r_A \in \mathfrak{sl}(2,\mathbb{C}) \wedge \mathfrak{sl}(2;\mathbb{C}) + \underset{\uparrow}{\mathfrak{sl}(2;\mathbb{C})} \wedge \underset{\uparrow}{\overline{\mathfrak{sl}(2;\mathbb{C})}} + \overline{\mathfrak{sl}(2,\mathbb{C})} \wedge \overline{\mathfrak{sl}(2;\mathbb{C})}$$

Abelian twist generator

2.2 Reality condition for $\mathfrak{o}(4;\mathbb{C})$ Lie algebra

The decomposition

$$\mathfrak{o}(4;\mathbb{C}) = \mathfrak{sl}(2;\mathbb{C}) \oplus \bar{\mathfrak{sl}}(2;\mathbb{C}) \tag{33}$$

we use to obtain real forms of $\mathfrak{o}(4;\mathbb{C})$.
All possible *seven real forms* of $\mathfrak{o}(4;\mathbb{C})$ are the following:

$$\begin{array}{llllll}
1) & H^*{=}H, & E_\pm^*{=}E_\mp; & \bar{H}^*{=}\bar{H}, & \bar{E}_\pm^*{=}\bar{E}_\mp & \text{for } \mathfrak{o}(4){=}\mathfrak{su}(2) \oplus \overline{\mathfrak{su}(2)}, \\
2) & H^*{=}{-}\bar{H}, & E_\pm^*{=}{-}\bar{E}_\pm; & \bar{H}^*{=}{-}H, & \bar{E}_\pm^*{=}{-}E_\pm & \text{for } \mathfrak{o}(3,1){=}\mathfrak{sl}(2;\mathbb{C}){\oplus}(\overline{\mathfrak{sl}(2;\mathbb{C})})^* \\
3) & H^*{=}{-}H, & E_\pm^*{=}{-}E_\pm, & \bar{H}^*{=}{-}\bar{H}, & \bar{E}_\pm^*{=}{-}\bar{E}_\pm & \text{for } \mathfrak{o}(2,2){=}\mathfrak{sl}(2;\mathbb{R}){\oplus}\overline{\mathfrak{sl}(2;\mathbb{R})} \\
4) & H^*{=}{-}H, & E_\pm^*{=}{-}E_\pm; & \bar{H}^*{=}\bar{H}, & \bar{E}_\pm^*{=}\bar{E}_\mp & \text{for } \mathfrak{o}^*(4){=}\mathfrak{sl}(2;\mathbb{R}){\oplus}\overline{\mathfrak{su}(2)} \\
3') & H^*{=}H, & E_\pm^*{=}{-}E_\mp, & \bar{H}^*{=}{-}\bar{H}, & \bar{E}_\pm^*{=}{-}\bar{E}_\mp, & \text{for } \mathfrak{o}(2,2){=}\mathfrak{su}(1,1){\oplus}\mathfrak{sl}(2;\mathbb{R}), \\
3'') & H^*{=}{-}\bar{H}, & E_\pm^*{=}{-}\bar{E}_\mp, & \bar{H}^*{=}{-}H, & \bar{E}_\pm^*{=}{-}E_\mp, & \text{for } \mathfrak{o}(2,2){=}\mathfrak{su}(1,1){\oplus}\mathfrak{su}(1,1), \\
4') & H^*{=}{-}H, & E_\pm^*{=}{-}E_\mp, & \bar{H}^*{=}H, & \bar{E}_\pm^*{=}E_\mp, & \text{for } \mathfrak{o}^*(4){=}\mathfrak{su}(1,1){\oplus}\mathfrak{su}(2).
\end{array} \tag{34}$$

Real form $\mathfrak{o}(3,1)$ describes the realification of complex $\mathfrak{sl}(2;\mathbb{C})$ algebra, with reality conditions not preserving the left/right decomposition (33)

Our aims:

- - list all antisymmetric $\mathfrak{o}(4;\mathbb{C})$ complex r-matrices with complex parameters.
- - impose *seven reality conditions, expressed as reality conditions for the parameters.*

Remark: In order to quantize the solutions r_3, r_4, r_5 of mCYBE one should introduce their *nonsymmetric BD* forms, which satisfy CYBE, by replacement of two expressions with added extra term

$$E_+ \wedge E_- \to E_+ \oplus E_- + H \otimes H \qquad \bar{E}_+ \wedge \bar{E}_- \to \bar{E}_+ \otimes \bar{E}_- + \bar{H} \oplus \bar{H} \quad (35)$$

2.3 Explicite quantizations of basic five $\mathfrak{o}(4;\mathbb{C})$ r-matrices:

A) Jordanian twist quantization of $\mathfrak{o}(4;\mathbb{C})$

$$
\begin{aligned}
r_1(\chi) &= \chi(E_+ + \bar{E}_+) \wedge (H + \bar{H}) \\
F_1(\chi) &= \exp\left((H+\bar{H}) \otimes \sigma\right), \qquad \sigma = \ln(1 + \chi(E_+ + \bar{E}_+))
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\Delta_1(E_{k+}) &= F_1(\chi)\Delta^{(0)}(E_k)F_0^{-1} = E_{k+} \otimes e^{\sigma} + 1 \otimes E_{k+} \\
\Delta_1(H_k) &= H_k \otimes 1 + 1 \otimes H_k - \chi(H+\bar{H}) \otimes E_{k+}e^{-\sigma} \\
\Delta_1(E_{k-}) &= E_{k-} \otimes e^{-\sigma} + 1 \otimes E_{k-} + 2\chi(H+\bar{H}) \otimes H_k e^{-\sigma} \\
&\quad -\chi^2(H+\bar{H})(H+\bar{H}-1) \otimes E_{k+}e^{-2\sigma}
\end{aligned} \tag{37}
$$

The quantum R-matrix takes the form $(R = F_1^{21}F_1^{-1})$

$$
R_1(\chi) = \exp\left(\sigma \otimes (H+\bar{H})\right) \exp\left(-(H+\bar{H}) \otimes \sigma\right) \tag{38}
$$

Two real quantizations: $\mathfrak{o}(2,2) \simeq \mathfrak{sl}(2;\mathbb{R}) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})}$ $\quad \chi \in i\mathbb{R}$

$\mathfrak{o}(3,1) \simeq \mathfrak{sl}(2;\mathbb{C}) \oplus \overline{\mathfrak{sl}(2;\mathbb{C})^*}$ $\quad \chi \in i\mathbb{R}$

B) Product of two Jordanian quantizations intertwined by twist

$$r_2(\chi, \bar{\chi}, \xi) = \chi\, E_+ \wedge H + \bar{\chi}\, \bar{E}_+ \wedge \bar{H} + \xi E_+ \wedge \bar{E}_+$$

$$F_2(\chi, \bar{\chi}, \xi) = F_A(\chi, \bar{\chi}, \xi) F_{J,0}(\chi) F_{J,1}(\bar{\chi})$$

$$F_{J,k}(\chi_k) = \exp\left(H_k \otimes \Sigma_k\right) \quad F_A(\chi, \bar{\chi}, \xi) = \exp\left(\tfrac{\xi}{\chi\bar{\chi}} \Sigma \wedge \bar{\Sigma}\right) \qquad k = 0, 1$$

$$\text{where} \quad \Sigma_k = ln(1 + \chi_k E_{k+}); \quad H = H_0, \bar{H} = H_1, E_\pm = E_{0\pm}, \bar{E}_\pm = E_{1\pm} \tag{39}$$

$$R_2 = \exp\left(\tfrac{-\xi}{\chi\bar{\chi}} \Sigma \wedge \bar{\Sigma}\right) \exp\left(\Sigma \otimes H\right) \exp\left(-H \otimes \Sigma\right) \cdot \\ \exp\left(\bar{\Sigma} \otimes \bar{H}\right) \exp\left(-\bar{H} \otimes \bar{\Sigma}\right) \exp\left(\tfrac{-\xi}{\chi\bar{\chi}} \Sigma \wedge \bar{\Sigma}\right). \tag{40}$$

Two real quantizations:

- $\mathfrak{o}(2,2) = \mathfrak{sl}(2;\mathbb{R}) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})} \quad \chi, \bar{\chi}, \xi \in i\mathbb{R}$
- $\mathfrak{o}(3,1) = \mathfrak{sl}(2;\mathbb{C}) \oplus \overline{\mathfrak{sl}(2;\mathbb{C})}^* \quad \chi = -\bar{\chi}^* \ (\chi \in \mathbb{C}), \quad \xi \in \mathbb{R}$

C) Twisted pair of left and right q-analogs

$$r_3(\gamma,\bar{\gamma},\eta) = \gamma\ (E_+ \wedge E_-) + \bar{\gamma}\ \left(\bar{E}_+ \wedge \bar{E}_-\right) + \eta\, H \wedge \bar{H}$$

$$r_3^{BD} = \gamma\ (E_+ \otimes E_- + H \otimes H) + \bar{\gamma}\ \left(\bar{E}_+ \otimes \bar{E}_- + \bar{H} \otimes \bar{H}\right) + \frac{\eta}{2}\, H \wedge \bar{H}$$

For $\eta = 0$ the quantization is a product of two independent standard (Drinfeld-Jimbo) deformations ($\eta = 0 :\ \ q = \exp \frac{1}{2}\gamma = q_0$ and $\bar{q} = \exp \frac{1}{2}\bar{\gamma} = q_1 \in \mathbb{C}$).

$$U_{(q,\bar{q})}(\mathfrak{o}(4;\mathbb{C})) \cong U_q(\mathfrak{sl}(2;\mathbb{C})) \otimes U_{\bar{q}}(\mathfrak{sl}(2;\mathbb{C})) \qquad (41)$$

and the universal R-matrix is the product of two factors $R_3^{(0)}, R_3^{(1)}$

$$R_3^{(k)} = \exp_{q_k^{-2}}\Big((q_k - q_k^{-1})E_{k+}\, q_k^{-H_k} \otimes q_k^{H_k} E_{k-}\Big) q_k^{2H_k \otimes H_k} \qquad k = 0,1$$
$$\exp_q(x) := \textstyle\sum_{n\geq 0} \frac{x^n}{(n)_q!}\ , \qquad (n)_q! := (1)_q (2)_q \cdots (n)_q, \quad (n)_q = \frac{1-q^n}{1-q} \qquad (42)$$

The Abelian twist is given by $F_3(\eta) := \tilde{q}^{H\wedge\bar{H}} \qquad \tilde{q} = exp\frac{1}{4}\eta$.
Complete universal R-matrix for $\eta \neq 0$ looks as follows

$$R_3(q_0, q_1, \tilde{q}) = \tilde{q}^{\bar{H}\wedge H} R_3^{(0)}(\eta) R_3^{(1)}(\eta) \tilde{q}^{\bar{H}\wedge H} \ = \ R_3^{(0)}(\eta) R_3^{(1)}(\eta) \tilde{q}^{2\bar{H}\wedge H} \tag{43}$$

where

$$R_3^{(k)}(\eta) = \exp_{q_k^{-2}}\left((q_k - q_k^{-1}) E_{k+} q_k^{-H_k} \tilde{q}^{(-)^{k+1} H_{k+1}} \otimes q_k^{H_k} \tilde{q}^{(-)^{k+1} H_{k+1}} E_{k-}\right) q_k^{2H_k \otimes H_k} \tag{44}$$

Seven real quantizations generated by r_3 with constrained parameters

- $\mathfrak{o}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \qquad \gamma, \bar{\gamma} \in \mathbb{R}, \quad \eta \in i\mathbb{R} \quad$ (unique!)
- $\mathfrak{o}(3,1) = \mathfrak{sl}(2;\mathbb{C}) \oplus (\mathfrak{sl}(2;\mathbb{C}))^+ \qquad \bar{\gamma} = -\gamma^* \in \mathbb{C} \quad (\gamma \text{ complex}), \eta \in i\mathbb{R}$
- $\mathfrak{o}(2,2) = \mathfrak{sl}(2;\mathbb{R}) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})} \qquad \gamma, \bar{\gamma}, \eta \in i\mathbb{R},$
- $\mathfrak{o}(2,2) = \mathfrak{su}(1,1) \oplus \overline{\mathfrak{su}(1,1)} \qquad \gamma, \bar{\gamma} \in \mathbb{R}, \eta \in i\mathbb{R}$
- $\mathfrak{o}(2,2) = \mathfrak{su}(1,1) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})} \qquad \gamma, \eta \in \mathbb{R}, \bar{\gamma} \in i\mathbb{R}$
- $\mathfrak{o}^*(4) = \mathfrak{su}(2) \oplus \overline{\mathfrak{su}(1,1)} \qquad \gamma, \bar{\gamma} \in \mathbb{R}, \eta \in i\mathbb{R},$
- $\mathfrak{o}^*(4) = \mathfrak{su}(2) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})} \qquad \gamma, \eta \in \mathbb{R}, \bar{\gamma} \in i\mathbb{R},$

D) Twisted deformation for Belavin-Drinfeld triple ($q \in C, \bar{q} = q^{-1}$)

$$r_4(\gamma, \xi) = \gamma \left(E_+ \wedge E_- - \bar{E}_+ \wedge \bar{E}_- - 2H \wedge \bar{H} \right) + \xi E_+ \wedge \bar{E}_+$$

$$r_4^{BD} = \gamma \left(E_+ \otimes E_- + H \otimes H - \bar{E}_+ \otimes \bar{E}_- - \bar{H} \otimes \bar{H} - H \wedge \bar{H} \right) + \tfrac{\xi}{2} E_+ \wedge \bar{E}_+$$

Belavin-Drinfeld twist factor:

$$F_4(\xi) := \exp_{q^2} \left(\xi E_+ q^{H+\bar{H}} \otimes q^{H+\bar{H}} \bar{E}_+ \right) \tag{45}$$

The universal R-matrix for this case is the twisted product (now $\bar{\gamma} = -\gamma \Leftrightarrow \bar{q} = q^{-1}$)

$$R_4(\gamma, \xi) = F_4^{\tau}(\xi) R_3^{(0)}(\gamma) R_3^{(1)}(-\gamma) F_4^{-1}(\xi) \tag{46}$$

Two real quantizations generated by r_4:

- $\mathfrak{o}(3,1) = \mathfrak{sl}(2;\mathbb{C}) \oplus (\overline{\mathfrak{sl}(2;\mathbb{C})})^* \quad \gamma, \xi \in \mathbb{R}$
- $\mathfrak{o}(2,2) = \mathfrak{sl}(2;\mathbb{R}) \oplus (\overline{\mathfrak{sl}(2;\mathbb{R})}) \quad \gamma, \xi \in i\mathbb{R}$

E) Left q-analog and right Jordanian quantization intertwined by twist

$$\begin{aligned} \tilde{r}_5(\gamma,\bar{\chi},\rho) &= \gamma\ (E_+ \wedge E_-) + \bar{\chi}\,\bar{E}_+ \wedge \bar{H} + \rho H \wedge \bar{E}_+ \\ \tilde{r}_5(\gamma,\bar{\chi},\rho) &= \gamma\ (E_+ \otimes E_- + H \otimes H) + \bar{\chi}\,\bar{E}_+ \wedge \bar{H} + \rho H \wedge \bar{E}_+ \end{aligned} \tag{47}$$

where first bracket describes standard quantization in first factor, second term generates Jordanian twist F_J in second factor, and last term leads to Abelian twist

$$F_5(\bar{\chi},\rho) = \tilde{q}^{H\wedge\bar{\Sigma}}, \qquad \tilde{q} = \exp\frac{\rho}{4\bar{\chi}} \tag{48}$$

Universal R-matrix for this case takes the form

$$R_5(\gamma,\bar{\chi},\rho) = \tilde{q}^{\bar{\Sigma}\wedge H} R(\gamma) F_J^{\tau}(\bar{\chi}) F_J^{-1}(\bar{\chi}) \tilde{q}^{\bar{\Sigma}\wedge H}. \tag{49}$$

Three real quantizations of $\boldsymbol{r_5}$:

- $\mathfrak{o}(2,2) = \mathfrak{sl}(2;\mathbb{R}) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})}$ $\quad \gamma,\bar{\chi},\rho \in i\mathbb{R}$
- $\mathfrak{o}(2,2) = \mathfrak{su}(1,1) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})}$ $\quad \gamma,\rho \in \mathbb{R} \quad \bar{\chi} \in i\mathbb{R}$
- $\mathfrak{o}^*(4) = \mathfrak{su}(2) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})}$ $\quad \gamma,\rho \in \mathbb{R} \quad \bar{\chi} \in i\mathbb{R}$

Remarks:

1) All coproducts and antipodes are presented in

A.B.+J.L.+V.N.T. arXiv:1708. ...

2) The $\mathfrak{o}(4;\mathbb{C})$ r-matrices (their real forms) can be constructed as follows:

i) Summ of $\mathfrak{sl}(2;\mathbb{C})$ and $\overline{\mathfrak{sl}(2;\mathbb{C})}$ r-matrices r and $\bar{r}$ (or of their real forms) are supplemented by Abelian twist r_t

$$r + \bar{r} + r_t \quad (r_t \in \mathfrak{sl}(2;\mathbb{C}) \wedge \overline{\mathfrak{sl}(2;\mathbb{C})})$$

ii) Using quantization for BD triples (one for $\mathfrak{o}(4;\mathbb{C})$!)

iii) Extending $\mathfrak{sl}(2;\mathbb{C}) \to \mathfrak{sl}(2;\mathbb{C}) \oplus \overline{\mathfrak{sl}(2;\mathbb{C})}$ Jordanian twist by addition:

$$H \wedge E_+ \to (H + \tilde{H}) \wedge (E_+ + \tilde{E}_+)$$

Example: 8 $\mathfrak{o}(2,2)$ matrices $= 6 \times i) + 1 \times ii) + 1 \times iii)$ $(\mathfrak{o}(2,2) \simeq AdS_3)$

Table of real $\mathfrak{o}(4)$, $\mathfrak{o}^*(4)$, $\mathfrak{o}(2,2)$ and $\mathfrak{o}(3,1)$ r-matrices

	r_1	r_2	r_3	r_4	r_5	
$\mathfrak{su}(2) \oplus \overline{\mathfrak{su}(2)}$	0	0	x	0	0	$\mathfrak{o}(4)$
$\mathfrak{su}(2) \oplus \overline{\mathfrak{sl}(2)}$	0	0	x	0	x	$\mathfrak{o}^*(4)$
$\mathfrak{su}(2) \oplus \overline{\mathfrak{su}(1,1)}$	0	0	x	0	0	$\mathfrak{o}^*(4)$
$\mathfrak{su}(1,1) \oplus \overline{\mathfrak{su}(1,1)}$	0	0	x	0	0	$\mathfrak{o}(2,2)$
$\mathfrak{su}(1,1) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})}$	0	0	x	0	x	$\mathfrak{o}(2,2)$
$\mathfrak{sl}(2;\mathbb{R}) \oplus \overline{\mathfrak{sl}(2;\mathbb{R})}$	x	x	x	x	x	$\mathfrak{o}(2,2)$
$\mathfrak{sl}(2;\mathbb{C}) \oplus \overline{\mathfrak{sl}(2;\mathbb{C})}$	x	x	x	x	0	$\mathfrak{o}(3,1)$

↑ Cartan-Weyl basis

↑ (Pseudo)-orthogonal description

Resume of real $\mathfrak{o}(4;\mathbb{C})$ quantizations:

1) one quantization of $\mathfrak{o}(4)$ (r_3)
2) four quantizations of $\mathfrak{o}(3,1)$ (r_1, r_2, r_3, r_4)
3) eight quantizations of $\mathfrak{o}(2,2)$ $(r_1, r_2, r_3(\times 3), r_4, r_5(\times 2))$
4) three quantizations $\mathfrak{o}^*(4)$ $(r_3(\times 2), r_5)$

New results in position 3),4). Possible 16 quantizations of real forms of $\mathfrak{o}(4;\mathbb{C})$!

THANK YOU!