Superfield generating equation of field-antifield formalism

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I.A. Batalin, PML, Eur. Phys. J. C77 (2017) 121 (arXiv:1702.00570[hep-th])

I.A. Batalin, PML, Phys. Lett. B758 (2016) 54 (arXiv:1604.01888[hep-th])

I.A. Batalin, PML, Int. J. Mod. Phys. A31 (2016) 1650054 (arXiv:1603.01825[hep-th])

Contents

BV formalism

The field-antifield (BV) formalism [Batalin, Vilkovisky (1981), (1983)] is known as the most powerful method for covariant (Lagrangian) quantization of gauge-field theories of the general kind, with general open gauge algebra, both irreducible or any-stage reducible. The BV formalism is characterized by the following features:

Configuration space

The configuration space Φ^α for irreducible theories

$$\Phi^\alpha = (A^i,\ B^{\alpha_0},\ C^{\alpha_0},\ \bar{C}^{\alpha_0}), \quad \varepsilon(\Phi^\alpha) = \varepsilon_\alpha.$$

For reducible theories the configuration space Φ^α contains main chains of the ghost $C_s^{\alpha_s}$, antighost $\bar{C}_s^{\alpha_s}$ and auxiliary $B_s^{\alpha_s}$ fields as well as pyramids of the ghost for ghost $C_{s(n_s)}^{\alpha_s}$ and auxiliary $B_{s(n_s)}^{\alpha_s}$ fields

$$\Phi^\alpha = \Big(A^i; B_s^{\alpha_s}, C_s^{\alpha_s}, \bar{C}_s^{\alpha_s}, s = 0, ..., L; B_{s(n_s)}^{\alpha_s}, C_{s(n_s)}^{\alpha_s}, s = 1, ..., L, n_s = 1, ..., s\Big)$$

Antifields

To each field Φ^α (of the total configuration space) one introduces corresponding antifield Φ^*_α

$$\Phi^*_\alpha = \Big(A^*_i, B^*_{s\alpha_s}, C^*_{s\alpha_s}, \bar{C}^*_{s\alpha_s}, s=0,...,L; B^*_{s(n_s)\alpha_s}, C^*_{s(n_s)\alpha_s}, s=1,...,L, n_s=1,..., s\Big)$$

The statistics of Φ^*_α is opposite to the statistics of the fields Φ^α

$$\varepsilon(\Phi^*_\alpha) = \varepsilon_\alpha + 1.$$

The total space of BV formalism

$$Z^A = (\Phi^\alpha; \Phi^*_\alpha), \quad \varepsilon(Z^A) = \varepsilon_A,$$

Quantum master equation

The main ingredient of the field-antifield formalism is the quantum master equation for quantnm action $W = W(Z)$ formulated in terms of the nilpotent odd Laplacian Δ also known as the delta-operator,

$$\Delta \exp\left\{\frac{i}{\hbar}W\right\} = 0, \quad \Delta^2 = 0, \quad \varepsilon(\Delta) = 1,$$

or, equivalently,

$$\frac{1}{2}(W, W) = i\hbar\Delta W.$$

Antibracket (F, G) can be defined by the Witten's formula [Witten (1991)]

$$\Delta[F \cdot G] = (\Delta F) \cdot G + F \cdot (\Delta G)(-1)^{\varepsilon(F)} + (F, G)(-1)^{\varepsilon(F)}.$$

Natural arbitrariness

The nilpotency of the delta-operator $\Delta^2 = 0$ causes the natural arbitrariness for the quantum master action W. That quantum arbitrariness is realized in the form of the so-called anticanonical master transformations, infinitesimal [Batalin, Vilkovisky (1981), (1983)] or finite [Batalin, PML,Tyutin (2015)]. These master transformations are described with the help of a generator $F = F(\Phi, \Phi^*) = F(Z)$ being the odd function,

$$\exp\left\{\frac{i}{\hbar}W'\right\} = \exp\{[\Delta, F]\}\exp\left\{\frac{i}{\hbar}W\right\}$$

BRST symmetry

The BV formalism respects the BRST symmetry discovered for the first time in Yang-Mills theories [Becchi,Rouet,Stora (1974),Tyutin (1975)]. The BRST symmetry is characterized by constant anticommuting parameter θ. Recent developments in study of the BRST symmetry are related to finite BRST transformations [PML,Lechtenfeld (2013); Batalin,PML,Tyutin (2014)].

So we have three basic quantities specific for the BV formalism

$$\Delta, \quad F = F(Z), \quad \theta,$$

$$\Delta^2 = 0, \quad \varepsilon(\Delta) = \varepsilon(F) = \varepsilon(\theta) = 1$$

The Sp(2) covariant quantization [Batalin, PML,Tyutin (1990),(1991)] is based on the principle of extended BRST symmetry (BRST-antiBRST).

Configuration space

Original formulation includes the following set of variables

$$(\Phi^{\alpha}; \Phi^{*}_{\alpha a}, \Phi^{**}_{\alpha}), \quad \varepsilon(\Phi^{\alpha}) = \varepsilon(\Phi^{**}_{\alpha}) = \varepsilon_{\alpha}, \quad \varepsilon(\Phi^{*}_{\alpha}) = \varepsilon_{\alpha} + 1$$

Later it was proposed the symmetric formulation of Sp(2) covariant formalism [Batalin, Marnelius (1995)]

$$Z^{A} = (\Phi^{\alpha}, \Phi^{\alpha a}; \Phi^{*}_{\alpha a}, \Phi^{**}_{\alpha}), \quad \varepsilon(\Phi^{*}_{\alpha}) = \varepsilon_{\alpha} + 1, \quad \varepsilon(Z^{A}) = \varepsilon_{A}.$$

Extended quantum master equations

The main ingredient of the Sp(2) formalism is the extended quantum master equations formulated in terms of the nilpotent anticommuting Sp(2)-doublet operators Δ^a_+, $[\Delta^a_+, \Delta^b_+] = 0$,

$$\Delta^a_+ \exp\left\{\frac{i}{\hbar}W\right\} = 0, \quad \Delta^a_+ = \Delta^a + \frac{i}{\hbar}V^a, \quad \varepsilon(\Delta^a_+) = 1,$$

or, equivalently,

$$\frac{1}{2}(W,W)^a = i\hbar\Delta^a_+ W.$$

Extended antibrackets $(F,G)^a$ can be defined by the formulas

$$\Delta^a_+[F\cdot G] = (\Delta^a_+ F)\cdot G + F\cdot(\Delta^a_+ G)(-1)^{\varepsilon(F)} + (F,G)^a(-1)^{\varepsilon(F)},$$

$$\Delta^a[F\cdot G] = (\Delta^a F)\cdot G + F\cdot(\Delta^a G)(-1)^{\varepsilon(F)} + (F,G)^a(-1)^{\varepsilon(F)}.$$

Natural arbitrariness

The Sp(2) nilpotency of the delta-operators $[\Delta^a_+, \Delta^b_+] = 0$, causes the natural arbitrariness for the quantum master action W. That quantum arbitrariness is realized in the form of the so-called extended anticanonical master transformations, infinitesimal [Batalin, PML,Tyutin (1990), (1991)] or finite [Batalin, Bering, PML (2016)]. These master transformations can be described with the help of a Boson function $B = B(\Phi)$,

$$\exp\left\{\frac{i}{\hbar}W'\right\} = \exp\{[\Delta^a_+, \varepsilon_{ab}[\Delta^b_+, B]]\}\exp\left\{\frac{i}{\hbar}W\right\}$$

Extended BRST symmetry

The Sp(2) formalism respects the extended BRST symmetry which is characterized by constant anticommuting parameters θ_a.
Recent results in study of the extended BRST symmetry within the Sp(2) covariant quantization are connected with finite extended BRST transformations [Batalin, Bering,PML, Tyutin (2014)].

So we have the following basic quantities specific for the Sp(2) formalism

$$\Delta^a_+, \quad B = B(\Phi), \quad \theta_a,$$

$$[\Delta^a_+, \Delta^b_+] = 0, \quad \varepsilon(\Delta^a_+) = \varepsilon(\theta_a) = 1, \quad \varepsilon(B) = 0$$

Phase space :

$$Z^A =: (\Phi^\alpha; \Phi^*_\alpha), \quad \varepsilon(\Phi^\alpha) = \varepsilon(\Phi^*_\alpha) + 1, \quad \varepsilon(Z^A) = \varepsilon_A$$

$$P_A =: -i\hbar \frac{\partial}{\partial Z^A} (-1)^{\varepsilon_A} \quad \Rightarrow \quad [Z^A, P_B] = i\hbar \delta^A_B,$$

$$[Z^A, Z^B] = [P_A, P_B] = 0.$$

Superfield :

$$\Psi =: \Psi(Z; t, \theta), \quad \varepsilon(t) = 0, \quad \varepsilon(\theta) = 1.$$

Superfield dynamics :

Superfield Schroedinger equation,

$$(i\hbar D - Q)\Psi = 0,$$

Covariant super-time derivative,

$$D =: \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}, \quad \varepsilon(D) = 1, \quad D^2 = \frac{1}{2}[D, D] = \frac{\partial}{\partial t},$$

Super-charge

$$Q =: \Delta - F, \quad \varepsilon(Q) = \varepsilon(\Delta) = \varepsilon(F) = 1,$$

Superfield generating equation

The odd Laplacian,

$$\Delta =: \frac{1}{2} P_A E^{AB} P_B (-1)^{\varepsilon_B}, \quad E^{AB} = \text{const},$$

Antisymplectic structure, E^{AB},

$$\varepsilon(E^{AB}) =: \varepsilon_A + \varepsilon_B + 1,$$

$$E^{AB} = -E^{BA}(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)},$$

The nilpotency of Δ

$$\Delta^2 = \frac{1}{2}[\Delta, \Delta] = 0.$$

Super-potential

$$F =: F(Z) \quad \Rightarrow \quad [F, F] = 0,$$

Decomposition of superfield

$$\Psi(Z; t, \theta) = \exp\{\theta(i\hbar)^{-1}Q\}\Psi_0(Z; t),$$

Schroedinger equation,

$$(i\hbar\partial_t - H)\Psi_0 = 0,$$

Hamiltonian

$$H =: -\frac{1}{2}(i\hbar)^{-1}[Q, Q] = (i\hbar)^{-1}[\Delta, F],$$

Superfield generating equation

We have the following implications

$$[\Delta, H] = 0 \quad \Rightarrow \quad (i\hbar\partial_t - H)\Delta\Psi_0 = 0,$$

$$\Delta\Psi_0|_{t=0} = 0 \quad \Rightarrow \quad \Delta\Psi_0|_{any\ t} = 0.$$

Time evalution

$$\Psi_0(Z;t) = \exp\{(i\hbar)^{-2}[\Delta, F]t\}\Psi_0(Z).$$

Using anzatz

$$\Psi_0(Z) = \exp\Big\{\frac{i}{\hbar}W(Z)\Big\}.$$

we reproduce the master transformations [Batalin, PML,Tyutin (2015)]

$$\exp\Big\{\frac{i}{\hbar}W'(Z)\Big\} = \exp\{(i\hbar)^{-2}[\Delta, F]\}\exp\Big\{\frac{i}{\hbar}W(Z)\Big\}.$$

If $F = F(\Phi)$ this transformation corresponds to anticanonical transformation of W.

Generalization of master transformations

Consider the case when

$$F = F(Z, P)$$

Hamiltonian

$$H =: -\frac{1}{2}(i\hbar)^{-1}[Q, Q] = (i\hbar)^{-1}\big([\Delta, F] - \frac{1}{2}[F, F]\big), \quad [\Delta, H] \neq 0$$

It follows from the definition of H that

$$[Q, H] = 0.$$

We arrive at the implication

$$[\Delta, H] = 0 \;\Rightarrow\; [H, F] = 0,$$

or more explicitly

$$[[\Delta, F], F] = [\Delta, \frac{1}{2}[F, F]] = 0.$$

Due to the Poincare lemma, we have

$$\frac{1}{2}[F,F] = -i\hbar H_S - [\Delta, G],$$

where H_S is a Boson singlet component,

$$[\Delta, H_S] = 0, \quad H_S \neq [\Delta, \text{anything}],$$

G is an arbitrary Fermion operator.
With the above implication we arrive at the final presentation of the Hamiltonian

$$H = H_S + H_\Phi,$$

where the Δ-exact Φ component is defined as

$$H_\Phi =: (i\hbar)^{-1}[\Delta, \Phi], \quad \Phi =: F + G.$$

Generalization of master transformations

Once the Δ operator commutes with the Hamiltonian H, it follows for the zero component $\Psi_0(Z,t)$

$$i\hbar\frac{\partial}{\partial t}\Delta\Psi_0 = H\Delta\Psi_0,$$

$$\Delta\Psi_0|_{t=0} = 0 \;\Rightarrow\; \Delta\Psi_0|_{\mathrm{any}\,t} = 0.$$

The arbitrariness of a solution to the quantum master equation,

$$\Delta\Psi_0 = 0, \quad \varepsilon(\Psi_0) = 0, \quad \Psi_0(Z) =: \exp\Big\{\frac{i}{\hbar}W(Z)\Big\},$$

is measured by the evolution operator,

$$\Psi_0|_{t=0} \;\rightarrow\; \Psi_0|_{any\ t} = \exp\Big\{-\frac{i}{\hbar}H\,t\Big\}\Psi_0|_{t=0}.$$

Quantum master transformations

$$\exp\left\{\frac{i}{\hbar}W'(Z)\right\} = \exp\{(i\hbar)^{-2}[\Delta, \Phi]\}\exp\left\{\frac{i}{\hbar}W(Z)\right\}$$

where

$$\Phi = \Phi(Z, P), \quad \varepsilon(\Phi) = 1$$

is an arbitrary Fermion operator and we put $H_S = 0$.

$Sp(2)$ superfield construction

Phase space:

$$Z^A =: (\Phi^\alpha, \Phi^{\alpha a}; \Phi^*_{\alpha a}, \Phi^{**}_\alpha),$$

$$\varepsilon(\Phi^\alpha) = \varepsilon(\Phi^{**}_\alpha) = \varepsilon(\Phi^*_{\alpha a}) + 1 = \varepsilon(\Phi^{\alpha a}) + 1, \quad \varepsilon(Z^A) = \varepsilon_A$$

$$P_A =: -i\hbar\partial_A(-1)^{\varepsilon_A} \quad \Rightarrow \quad [Z^A, P_B] = i\hbar\delta^A_B,$$

$$[Z^A, Z^B] = 0, \quad [P_A, P_B] = 0.$$

The Sp(2) superfield Schroedinger equation,

$$(i\hbar D^a - Q^a)\Psi = 0, \quad \Psi = \Psi(Z; t, \theta_a)$$

The Sp(2) covariant super-time derivatives,

$$D^a =: \frac{\partial}{\partial\theta_a} + g^{ab}\theta_b\frac{\partial}{\partial t},$$

$$[D^a, D^b] = 2g^{ab}\partial_t, \quad g^{ab} = g^{ba} = \text{const},$$

The Sp(2) super-charges,

$$Q^a =: \Delta^a_+ - F^a,$$

$$\Delta^a_+ =: \Delta^a + \frac{i}{\hbar}V^a,$$

$$\Delta^a =: \frac{1}{2}P_A E^{ABa} P_B (-1)^{\varepsilon_B}, \quad V^a =: -i\hbar\varepsilon^{ab}\Phi^*_{\alpha b}P^{\alpha}_{**}(-1)^{\varepsilon_\alpha},$$

$Sp(2)$ superfield construction

Sp(2) antisymplectic structure, E^{ABa},

$$\varepsilon(E^{ABa}) =: \varepsilon_A + \varepsilon_B + 1,$$

$$E^{ABa} = -E^{BAa}(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)},$$

Sp(2) nilpotency

$$[\Delta^a_+, \Delta^b_+] = 0,$$

Sp(2) super-potential

$$F^a =: g^{ab}\varepsilon_{bc}(i\hbar)^{-1}[\Delta_+^c, B],$$

$$B = B(\Phi) \quad \Rightarrow \quad (B,B)^a = 0 \quad \Rightarrow \quad [F^a, F^b] = 0.$$

Decomposition of superfield,

$$\Psi(Z;t,\theta_a) = \exp\{\theta_a(i\hbar)^{-1}Q^a\}\Psi_0(Z;t),$$

Schroedinger equation

$$(i\hbar\partial_t - H)\Psi_0 = 0,$$

$$H =: -\frac{1}{4}g_{ab}(i\hbar)^{-1}[Q^a, Q^b] = \frac{1}{2}(i\hbar)^{-2}[\Delta_+^a, \varepsilon_{ab}[\Delta_+^b, B]].$$

The $\Delta^a{}_+$ operators commute with the Hamiltonian H, and therefore for the zero component $\Psi_0(Z,t)$ we have

$$i\hbar\frac{\partial}{\partial t}\Delta^a_+\Psi_0 = H\Delta^a_+\Psi_0,$$

$$\Delta^a_+\Psi_0|_{t=0} = 0 \quad \Rightarrow \quad \Delta^a_+\Psi_0|_{any\ t} = 0.$$

Time involution

$$\Psi_0(Z;t) = \exp\Big\{\frac{1}{2}(i\hbar)^{-3}[\Delta^a_+, \varepsilon_{ab}[\Delta^b_+, B]]t\Big\}\Psi_0(Z),$$

Using anzatz

$$\Psi_0(Z) = \exp\Big\{\frac{i}{\hbar}W(Z)\Big\},$$

we reproduce the extended master transformations [Batalin, Bering, PML, (2016)] in Sp(2) formalism

$$\exp\Big\{\frac{i}{\hbar}W'(Z)\Big\} = \exp\Big\{\frac{1}{2}(i\hbar)^{-3}[\Delta^a_+, \varepsilon_{ab}[\Delta^b_+, B]]\Big\}\exp\Big\{\frac{i}{\hbar}W(Z)\Big\}.$$

Heisenberg equations of motion

Let Γ be the full set of the Schroedinger canonical variable operators,

$$\Gamma =: (Z^A; P_A),$$

and let $\tilde{\Gamma}(t,\theta)$ be the respective superfield Heisenberg canonical variable operators,

$$\tilde{\Gamma}(t,\theta) = U(t,\theta)\,\Gamma\,U^{-1}(t,\theta),$$

The superfield Heisenberg equations of motion have the form,

$$i\hbar D\tilde{\Gamma} = [\tilde{Q},\tilde{\Gamma}], \quad i\hbar D\tilde{Q} = [\tilde{Q},\tilde{Q}].$$

$$U(t,\theta) = \exp\{\theta(i\hbar)^{-1}Q + (i\hbar)^{-2}Q^2 t\},$$

$$\tilde{Q} = Q + 2\theta(i\hbar)^{-1}Q^2.$$

Then it follows

$$(i\hbar)^2\frac{\partial}{\partial t}\tilde{\Gamma} = -\frac{1}{2}[\tilde{\Gamma},[\tilde{Q},\tilde{Q}]] = -\frac{2}{3}(\tilde{\Gamma},\tilde{Q})_{\tilde{Q}}.$$

Here the quantum 2 - antibracket, $(A,B)_Q$, [Batalin, Marnelius (1995), (1996)] is defined by:

$$(A,B)_Q =: \frac{1}{2}([A,[Q,B]] - (A \leftrightarrow B)(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}).$$

In the $Sp(2)$ case, the respective superfield Heisenberg equations of motion have the form,

$$i\hbar D^a \tilde{\Gamma} = [\tilde{Q}^a, \tilde{\Gamma}], \quad i\hbar D^a \tilde{Q}^b = [\tilde{Q}^a, \tilde{Q}^b].$$

It follows from these equations that

$$(i\hbar)^2 \frac{\partial}{\partial t}\tilde{\Gamma} = -\frac{1}{4} g_{ab}[\tilde{\Gamma}, [\tilde{Q}^b, \tilde{Q}^a]] = -\frac{1}{3} g_{ab}(\tilde{\Gamma}, \tilde{Q}^b)^a_{\tilde{Q}},$$

Here the Sp(2) vector valued quantum 2 - antibracket, $(A, B)^a_Q$, [Batalin, Marnelius (1995), (1996)] is defined by:

$$(A, B)^a_Q =: \frac{1}{2}\Big([A, [Q^a, B]] - (A \leftrightarrow B)(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}\Big).$$

Conclusion

- We have proposed the new quantum superfield generating equation for the general field-antifield formalism. The three basic Fermion objects, the super-time covariant derivative D, the odd Laplacian Δ, and the super-potential Fermion F, enter that linear homogeneous generating equation, in a quite symmetric way.
- From the generating equation, we have derived the Schroedinger equation with the Hamiltonian H commuting with the supercharge Q. It follows from the form of $F = F(Z)$ that the Hamiltonian H commutes with the Δ, provided the H commutes with the F, as well. In turn it allows us to discribe natural arbitrareness existing in solutions to the quantum master equation.

Conclusion

- Choosing more general form of the super-potential $F = F(Z, P)$ we have found a new type of transformations (quantum master transformations) respecting the quantum master equation.
- We have also presented an $Sp(2)$ symmetric extension to the main construction, with specific features caused by the principal fact that all basic equations become $Sp(2)$ vector-valued ones.
- We have presented the superfield Heisenberg equations of motion in both schemes of quantization with the help of the quantum antibrackets.

Thank you
for attention!