Color Lie (super)algebra symmetries in Lévy-Leblond equation

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Based on

- N. Aizawa, Z.K., H. Tanaka and F. Toppan, *$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie symmetries of the Lévy-Leblond equations*, Progress of Theoretical and Experimental Physics (2016) 123A01.

- N. Aizawa, Z.K., H. Tanaka and F. Toppan, *Generalized supersymmetry and Lévy-Leblond equation*, to appear in Proceedings of GROUP31, Springer.

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Symmetries

A **symmetry** of a differential equation is a transformation that keeps its family of solutions invariant.

If Ω is a differential operator,
$\Psi(x)$ is a solution and
Z is a symmetry operator, then

$$\Omega\Psi(x) = 0 \quad \Rightarrow \quad \Omega(Z\Psi(x)) = 0$$

This condition is equivalent to

$$[\Omega, Z]\Psi(x) = 0$$

or

$$\{\Omega, Z\}\Psi(x) = 0$$

Algebras, superalgebras and color Lie (super)algebras

- Lie algebra: G, multiplication: $[,]$
- Lie superalgebra: grading $\mathbf{Z}_2$, $G = G_0 \oplus G_1$, multiplication: $[,]$ and $\{,\}$.
- $\mathbf{Z}_2 \times \mathbf{Z}_2$ Lie (super)algebras:

$$G = G_{(0,0)} \oplus G_{(0,1)} \oplus G_{(1,0)} \oplus G_{(1,1)}$$

Color Lie algebras and superalgebras

Color Lie (super)algebras were introduced in 1978
V. Rittenberg, D. Wyler, Nucl.Phys.**B139** (1978) 189;
V. Rittenberg, D. Wyler, J. Math. Phys.**19** (1978) 385.

Color algebras are graded algebras.
The grading of generators is associated with n-dimensional grading vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. **Examples**:

1. Lie algebras: one grading space.
2. Lie superalgebras: $\mathbf{a} \in Z_2$, $n = 1$, $G = G_0 \oplus G_1$.
3. $\mathbf{Z}_3$ graded (super)algebras, $\mathbf{a} \in Z_2$, $n = 1$, $G = G_0 \oplus G_1 \oplus G_2$.
4. $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded (super)algebras, $\mathbf{a} \in \mathbf{Z}_2 \times \mathbf{Z}_2$, $n = 2$,
   $G = G_{(0,0)} \oplus G_{(0,1)} \oplus G_{(1,0)} \oplus C_{(1,1)}$,
   simple example: (split) quaternions
5. $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2$, $n = N$,
   $G = G_{(0,0,\ldots,0)} \oplus G_{(0,1,\ldots,0)} \oplus \ldots \oplus G_{(1,1,\ldots,1)}$
   example: $\mathbf{Z}_2^3$ graded analogue of the Heisenberg Lie algebra
   (Sigurdsson, Silvestrov, 2003)

$\mathbf{Z}_2 \times \mathbf{Z}_2$ graded Lie (super)algebras

The multiplication $[[,]] : G \times G \to G$ satisfies the following properties:

1. $[[X_{\mathbf{a}}, X_{\mathbf{b}}]] \subset g_{\mathbf{a}+\mathbf{b}}$
2. $[[X_{\mathbf{a}}, X_{\mathbf{b}}]] = -(-1)^{\mathbf{a}\cdot\mathbf{b}}[[X_{\mathbf{b}}, X_{\mathbf{a}}]]$
3. $[[X_{\mathbf{a}}, [[X_{\mathbf{b}}, X_{\mathbf{c}}]]]] = [[[[X_{\mathbf{a}}, X_{\mathbf{b}}]], X_{\mathbf{c}}]] + (-1)^{(\mathbf{a}\cdot\mathbf{b})}[[X_{\mathbf{b}}, [[X_{\mathbf{a}}, X_{\mathbf{c}}]]]]$

where $\mathbf{a}, \mathbf{b} \in \{(0,0), (0,1), (1,0), (1,1)\}$.

The inner product is defined by

a) $\mathbf{a} \cdot \mathbf{b} = a_1 b_2 - a_2 b_1$ for color Lie algebras
b) $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$ for color Lie superalgebras

Commutation relations

- For $\mathbf{Z}_2 \times \mathbf{Z}_2$ color Lie algebras

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	$[,]$	$[,]$	$[,]$	$[,]$
(0,1)	$[,]$	$[,]$	$\{,\}$	$\{,\}$
(1,0)	$[,]$	$\{,\}$	$[,]$	$\{,\}$
(1,1)	$[,]$	$\{,\}$	$\{,\}$	$[,]$

- For $\mathbf{Z}_2 \times \mathbf{Z}_2$ color Lie superalgebras

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	$[,]$	$[,]$	$[,]$	$[,]$
(0,1)	$[,]$	$\{,\}$	$[,]$	$\{,\}$
(1,0)	$[,]$	$[,]$	$\{,\}$	$\{,\}$
(1,1)	$[,]$	$\{,\}$	$\{,\}$	$[,]$

Lèvy-Leblond Equation (LLE)

was introduced in 1967 in J. M. Lèvi-Leblond, *Nonrelativistic particles and wave equations*, Comm. Math. Phys., **6** (1967) 286.

The author studied nonrelativistic particles and their properties, as described by Galilei invariant wave equations.
LLE, being a nonrelativistic analog of the Dirac equation, is a square root of the Schroedinger equation.

Here we consider a generalized LLE in sense that

- it is a square root of *d*-dimensional
- Schroedinger (free or with potential) or heat equations.

$$\Omega\Psi(\vec{x}, t) = 0$$

with

$$\Omega^2 = -\lambda\partial_t + \partial_x^2 \quad or \quad \Omega^2 = -i\lambda\partial_t + \partial_x^2 + V(x)$$

LLE general construction

We use irreducible real matrix representations of Clifford algebras $Cl(p, q)$:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\eta_{ij}$$

where η_{ij} is a diagonal matrix with p positive and q negative entries.

For any $d = p + q$ dimensions the Clifford representations can be constructed using 2×2 "building blocks":

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the X, Y and A are $Cl(2, 1)$ Clifford algebra representations.

Generalized LLE related with heat and Schroedinger equations, different cases

- for (1,1) dimensional heat equation

$$\Omega\Psi(x,t) = 0, \qquad \Omega = \frac{1}{2}(AA + AY)\partial_t + \frac{1}{2}(AA - AY)\lambda + AX\partial_x$$
$$\Omega^2 = II(-\lambda\partial_t + \partial_x^2)$$

$AA = A \otimes A$, $AY = A \otimes Y$ is a tensor product,

so AA, AY and AX are (4×4) matrices

and $\Psi(x,t)$ is a 4-component real spinor

$\lambda \in \mathbf{R} - \{0\}$ is an arbitrary real parameter

- free Schroedinger equation

$$\Omega = \frac{1}{2}(AA + AY) \cdot AI\partial_t + \frac{1}{2}(AA - AY)\lambda + AX\partial_x$$
$$\Omega^2 = AI\partial_t - II\partial_x^2$$

$(A \cdot A)$ and $(A \cdot Y)$ denote the usual matrix multiplication.

- Schroedinger equation with potential

$$\begin{aligned} \Omega &= \frac{1}{2}(AAA + AAY) \cdot AII\partial_t + \frac{1}{2}(AAA - AAY)\lambda \\ &+ AYI\partial_x + AAXf(x) \\ \Omega^2 &= AII\lambda\partial_t + III\left(-\partial_x^2 + f^2(x)\right) + IIX\partial_x f(x) \end{aligned}$$

This equation needs 8×8-component Gamma matrices and $\Psi(x, t)$ are 8-component real spinors

Symmetry operator Σ for $[\Omega, \Sigma] = \Phi(x,t)\Omega$

For the 1-dimensional LLE

$$\Omega = \gamma_+\partial_t + \gamma_-\lambda + \gamma_3\partial_x$$

where $\gamma_\pm = \frac{1}{2}(Y \pm A)$ and $\gamma_3 = X$

and $\Phi(x,t)$ is a matrix valued function.

The general form of a symmetry operator is

$$\Sigma = f(x,t)\partial_t + g(x,t)\partial_x + h(x,t)$$

where $f(x,t)$, $g(x,t)$ and $h(x,t)$ are matrix valued functions in γ's basis

$$\begin{aligned} f(x,t) &= \phi_+(x,t)\gamma_+ + \phi_-(x,t)\gamma_- + \phi_3(x,t)\gamma_3 + \phi_0(x,t)\mathbf{1} \\ g(x,t) &= \alpha_+(x,t)\gamma_+ + \alpha_-(x,t)\gamma_- + \alpha_3(x,t)\gamma_3 + \alpha_0(x,t)\mathbf{1} \\ h(x,t) &= \beta_+(x,t)\gamma_+ + \beta_-(x,t)\gamma_- + \beta_3(x,t)\gamma_3 + \beta_0(x,t)\mathbf{1} \end{aligned}$$

The following symmetry operators are found

$$
\begin{aligned}
\Sigma_1 = H &= \mathbf{1}\partial_t, \\
\Sigma_2 = D &= -\left(\mathbf{1}(t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{2}) + \frac{1}{4}\gamma_3\right) \\
\Sigma_3 = K &= -\left[\mathbf{1}\left(t^2\partial_t + tx\partial_x - \frac{\lambda}{4}x^2 + t\right) - \frac{1}{2}\gamma_+ x + \frac{1}{2}\gamma_3 t\right] \\
\Sigma_4 = P_+ &= \mathbf{1}\partial_x \\
\Sigma_5 = P_- &= \mathbf{1}\left(t\partial_x - \frac{\lambda}{2}x\right) - \frac{1}{2}\gamma_+, \\
\Sigma_6 = C &= \mathbf{1}, \\
\Sigma_7 = \Omega_{z(x,t)} &= z(x,t)\Omega = z(x,t)(\gamma_+\partial_t + \gamma_-\lambda + \gamma_3\partial_x)
\end{aligned}
$$

The operators $\{H, D, K, P_+, P_-, C\}$ span one-dimensional Schroedinger algebra $sch(1)$.

The operator $\Omega_{z(x,t)}$ depends on an arbitrary function $z(x,t)$.

Symmetry operators Λ for $\{\Omega, \Lambda\} = \Phi(x,t)\Omega$

$$
\begin{aligned}
\Lambda_1 &= \gamma_+\partial_t - \lambda\gamma_-, \\
\Lambda_2 &= \gamma_+\left(t\partial_t + \frac{1}{2}\right) - \gamma_-\lambda t + \frac{\lambda}{2}x\mathbf{1}_2, \\
\Lambda_3 &= \gamma_3\partial_t - 2\gamma_-\partial_x, \\
\Lambda_4 &= \gamma_3\left(t\partial_t + \frac{1}{4}\right) + \gamma_-\left(-2t\partial_x + \frac{\lambda}{2}\right) - \gamma_+\frac{x}{2} + \frac{1}{2}\mathbf{1}_2, \\
\Lambda_5 &= \gamma_3\left(\frac{1}{2}t^2\partial_t + \frac{t}{4}\right) + \gamma_-\left(-t^2\partial_x + \frac{\lambda}{2}tx\right) + \gamma_+\left(-\frac{1}{2}tx\partial_t - \frac{x}{4}\right) \\
&+ \left(\frac{t}{2} - \frac{\lambda x^2}{8}\right)\mathbf{1}_2, \\
\Lambda_{\omega(x,t)} &= \omega(x,t)\left(\gamma_+\partial_x - \frac{\lambda}{2}(\mathbf{1}_2 + \gamma_3)\right).
\end{aligned}
$$

The last symmetry operator depends on an arbitrary real function $\omega(x,t)$.

Scroedinger algebra $sch(2)$

from the operators with from $[\Omega, \Sigma] = \Phi(x, t)\Omega$

$$\begin{aligned}
[D, H] &= H, \\
[D, K] &= -K, \\
[H, K] &= 2D, \\
[D, P_\pm] &= \pm\frac{1}{2}P_\pm, \\
[H, P_-] &= P_+, \\
[K, P_+] &= P_-, \\
[P_+, P_-] &= -\frac{\lambda}{2}C.
\end{aligned}$$

The operator D defines a grading $[D, X] = \xi X$.

$\{D, H, K\}$ generate a $sl(2)$ algebra with D as a Cartan element.

$\{P_\pm, C\}$ realize a $h(1)$ Heisenberg algebra.

Super-Schroedinger algebra

The odd operators $Q_\pm$ with $Q_+^2 = -H$ and $Q_-^2 = K$ are constructed with the operators from Λ sector.

$$\begin{aligned} Q_+ &= \frac{1}{\sqrt{\lambda}}\Lambda_1 = \frac{1}{\sqrt{\lambda}}(\gamma_+\partial_t - \lambda\gamma_-), \\ Q_- &= \frac{1}{\sqrt{\lambda}}(\Lambda_2 + \Lambda_{\omega(x,t)=x}) \\ &= \frac{1}{\sqrt{\lambda}}\left[\gamma_+\left(t\partial_t + x\partial_x + \frac{1}{2}\right) - \gamma_-\lambda t - \gamma_3\frac{\lambda x}{2}\right]. \end{aligned}$$

The commutators are usual for superconformal algebra. The new odd operator X from $[P_\pm, Q_\mp] = \pm X$ is expressed in terms of Λ‘s as

$$X = \frac{1}{\sqrt{\lambda}}\Lambda_{\omega(x,t)=1} + \frac{\sqrt{\lambda}}{2}C = \frac{1}{\sqrt{\lambda}}\left(\gamma_+\partial_x - \gamma_3\frac{\lambda}{2}\right)$$

The operators $\{H, D, K, P_\pm, Q_\pm, X, C\}$ define one-dimensional super-Schroedinger algebra of the symmetries of the Lèvi-Leblond equation.

Three $osp(1|2)$ superalgebras

1. **from the $P_\pm$ operators**
   Introducing the notation $P_{\pm\frac{1}{2}} \equiv P_\pm$ and defining new operators
   $$P_{\pm 1} = \{P_{\pm\frac{1}{2}}, P_{\pm\frac{1}{2}}\}, \quad P_0 = \{P_{+\frac{1}{2}}, P_{-\frac{1}{2}}\}$$
   The operators $\{P_{\pm\frac{1}{2}}, P_{\pm 1}, P_0\}$ define the first $osp(1|2)$.
2. **from the Lèvy-Leblond operator Ω**
   $\Omega_{\frac{1}{2}} \equiv \Omega_{z(x,t)=1}$ has a half-integer grading $[D, \Omega] = \frac{1}{2}\Omega$.
   From $[K, \Omega] = t\Omega = \Omega_{z(t,x)=t}$ we have $[D, t\Omega] = -\frac{1}{2}t\Omega$.
   Defining $\Omega_{-\frac{1}{2}} \equiv t\Omega$ and constructing $\Omega_{\pm 1}, \Omega_0$ by anticommutators we obtain the second $osp(1|2)$ structure with $\{\Omega_{\pm\frac{1}{2}}, \Omega_{\pm 1}, \Omega_0\}$.
3. **from the supersymmetry operators $Q_\pm$**
   $Q_{\pm\frac{1}{2}} \equiv Q_\pm$. The generators $\{Q_{\pm\frac{1}{2}}, H, K, D\}$ define the third $osp(1|2)$ structure.

$\mathbf{Z}_2 \times \mathbf{Z}_2$ color superalgebra symmetries

With the two pairs of $Q_{\pm\frac{1}{2}}, P_{\pm\frac{1}{2}}$ operators we can define $\mathbf{Z}_2 \times \mathbf{Z}_2$ grading with

$$\begin{aligned} G_{00} &= \{H, D, K, P_{\pm 1}, P_0\}, \\ G_{01} &= \{P_{\pm\frac{1}{2}}\}, \\ G_{10} &= \{Q_{\pm\frac{1}{2}}, X_{\pm\frac{1}{2}}\}, \\ G_{11} &= \{X\} \end{aligned}$$

The new generators $X_{\pm\frac{1}{2}}$ appear from the anticommutators

$$X_{\pm\frac{1}{2}} = \{X, P_{\pm\frac{1}{2}}\}$$

This operators are symmetry generators satisfying

$$\{\Omega, X\} = \{\Omega, X_{\pm\frac{1}{2}}\} = 0$$

$\mathbf{Z}_2 \times \mathbf{Z}_2$-graded symmetry for $d = 2$ LLE

For the construction are used $\gamma_\mu \in Cl(3,2)$.

$$\Omega\Psi(x_1, x_2, t) = 0, \quad \Omega = \gamma_+\partial_t + \lambda\gamma_- + \gamma_i\partial_i$$

$i = 1, 2$ and $\gamma_\pm = \frac{1}{2}(\gamma_3 \pm \gamma_4)$

The symmetry operators from $[\Sigma, \Omega] = \Phi(x_i, t)\Omega$ are

$$\{H, D, K, P_\pm, J, \bar{X}, C\}$$

where

$$\begin{aligned} J &= \epsilon_{ij}\left(\mathbf{1}x_i\partial_j + \frac{1}{8}[\gamma_i, \gamma_j]\right), \\ \bar{X} &= \epsilon_{ij}\left(\gamma_+\gamma_i\partial_i + \frac{\lambda}{8}[\gamma_i, \gamma_j]\right) \end{aligned}$$

and C is the central charge as before.

From noncommutative sector $\{\Lambda, \Omega\} = \Phi(x_i, t)\Omega$ we have the symmetry operators

$$\begin{aligned} Q_+ &= \gamma_+ \partial_t - \lambda \gamma_-, \\ Q_- &= \gamma_+(t\partial_t + x_i \partial_i + 1) - \frac{1}{2}\lambda x_i \gamma_i - \lambda t \gamma_-, \\ X_i &= \gamma_+ \partial_i - \frac{1}{2}\lambda \gamma_i. \end{aligned}$$

The 2-dimensional super-Schroedinger algebra:
$G_0 = \{H, D, K, J, C\}$ and $G_1 = \{Q_\pm, X_i\}$.

What is the meaning of the "even" symmetry operator $\bar{X}$?

It is related with the X_i by

$$[X_i, X_j] = \lambda \epsilon_{ij} \bar{X}$$

So $\bar{X}$ can enter an algebraic structure if it is connected with X_i by commutators, which is not the case in *ssch*(2).

$\mathbf{Z}_2 \times \mathbf{Z}_2$ structure

The symmetry operators found before can be accomodated in $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded Lie structure with the following grading

$$\begin{aligned} G_{00} &= \{H, D, K, J, \bar{X}, P_{s,ij}\}, \\ G_{01} &= \{P_{\pm i}\}, \\ G_{10} &= \{Q_{\pm}, \hat{X}_{\pm,ij}\} \\ G_{11} &= \{X_i\} \end{aligned}$$

This associating the symmetry generators to grading sectors connects X_i and $\bar{X}$ with commutators, as was needed.

The new (second order) operators $P_{s,ij}$ and $\hat{X}_{\pm,ij}$ are introduced and they also are symmetry operators of LLE.

The symmetry of LLE is $\tilde{G}_{\mathbf{Z}_2\times\mathbf{Z}_2} = G_{\mathbf{Z}_2\times\mathbf{Z}_2} \oplus u(1)$.

Conclusions and open questions

- The LL free heat and free Schroedinger equation possess $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded supersymmetry.
- We expect that for Schroedinger equation with potential the symmetry operators can be associated in $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ graded structure.
- Three independent sets of $osp(1|2)$ are found. They are generated by the pairs $P_{\pm\frac{1}{2}}$, $Q_{\pm\frac{1}{2}}$ and $\Omega_{\pm\frac{1}{2}}$.
- Two of these pairs: (P, Ω) and (P, Q) can be used to introduce finite color superalgebra structures. For the fist one (P, Ω) the sector $G_{(1,1)}$ is empty.
- $\mathbf{Z}_2 \times \mathbf{Z}_2$ superalgebra as a spectrum generating algebra.
- meaning of second order generators
- $\Lambda_{3,4,5}$ symmetries
- consider *Hom* algebras (S. Silvestrov...) as symmetry algebras

THANK YOU FOR THE ATTENTION