Conformally flat mechanics of many $\mathcal{N} = 4$ superparticles

Nikolay Kozyrev

BLTP JINR, Dubna, Russia

in collaboration with S.Krivonos, O. Lechtenfeld, A. Nerssesian, A. Sutulin

Introduction

The supersymmetric extension of Calogero-like systems has long story of study. One of known properties of these systems is their relation to the WDVV equation, which in this case acts as a necessary condition of existence of $\mathcal{N} = 4$ supersymmetry. The solution of WDVV also enters the equation which determines acceptable potentials.

The study of particle mechanics near black hole horizons in odd dimension produces the closely related systems, but with nontrivial metric in the bosonic sector. With purpose of obtaining $\mathcal{N} = 4$ supersymmetric analogues of these systems, we modify the supercharges of usual $\mathcal{N} = 4$ many particles mechanics by a functional factor to produce nontrivial conformally flat metric in the Hamiltonian and consider more general three-fermion term. The conditions demanded by the closure of supersymmetry algebra are the modified WDVV equation on curved space and covariantized equations on the potential. The solutions of these equations are considered in two simplifying cases.

Usual mechanics

Usual $\mathcal{N}=4$ many particles mechanics with potential, as was discussed by Galajinsky, Lechtenfeld and Polovnikov (2007), can be described by following pair of supercharges

$$Q^a = p_i\,\psi_i^a + \mathrm{i}W_i\,\psi_i^a + \mathrm{i}F_{ijk}\,\psi_i^b\,\psi_{bj}\,\bar\psi_k^a,$$
$$\overline{Q}_a = p_i\,\bar\psi_{ia} - \mathrm{i}W_i\,\bar\psi_{ai} + \mathrm{i}F_{ijk}\,\bar\psi_{ib}\,\bar\psi_j^b\,\psi_{ka}.$$

Here, the basic variables are assumed to have Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}, \quad \{\psi_i^a, \bar\psi_{bj}\} = \frac{\mathrm{i}}{2}\delta_b^a\,\delta_{ij}.$$

The quantities W_i and F_{ijk} depend on x_m, and tensor F_{ijk} is considered completely symmetric. For supercharges Q^a, $\overline{Q}_a$ to form $\mathcal{N}=4$, $d=1$ superalgebra

$$\left\{Q^a, Q^b\right\} = 0,\ \left\{\overline{Q}_a, \overline{Q}_b\right\} = 0, \qquad \left\{Q^a, \overline{Q}_b\right\} = \frac{\mathrm{i}}{2}\,\delta_b^a\,H$$

the following equations should be satisfied

$$\partial_i W_j - \partial_j W_i = 0 \quad \Rightarrow \quad W_i = \partial_i W, \quad \partial_m F_{ijk} - \partial_k F_{ijm} = 0 \quad \Rightarrow \quad F_{ijk} = \partial_i\partial_j\partial_k F,$$
$$\partial_i\partial_j W + \partial_i\partial_j\partial_k F\,\partial_k W = 0, \quad \partial_i\partial_j\partial_n F\,\partial_k\partial_m\partial_n F - \partial_i\partial_k\partial_n F\,\partial_j\partial_m\partial_n F = 0. \quad (WDVV)$$

The Hamiltonian of this system shows that W produces the specific potential term:

$$H_W = p_i p_i + \partial_i W \partial_i W + \text{fermions}.$$

Potential from harmonics

Let us note that it is possible to introduce the potential in different way (Krivonos and Lechtenfeld, 2012). One may take the harmonics u^a, $\bar{u}_b$ with brackets $\{u^a, \bar{u}_b\} = -\mathrm{i}\,\delta^a_b$ and build from them quantities J^{ab} that form $su(2)$ algebra,

$$J^{ab} = \frac{\mathrm{i}}{2}\left(u^a\bar{u}^b + u^b\bar{u}^a\right) \quad \Rightarrow \quad \left\{J^{ab}, J^{cd}\right\} = -\epsilon^{ac}J^{bd} - \epsilon^{bd}J^{ac}.$$

Then one may modify the supercharges as

$$Q^a = p_i\,\psi^a_i + U_i\,J^{ac}\,\psi_{ci} + \mathrm{i}F_{ijk}\,\psi^b_i\,\psi_{bj}\,\bar{\psi}^a_k.$$

As a result of superalgebra relations, $U_i = \partial_i U$ and

$$U_i = \partial_i U,\ \partial_i\partial_j U - \partial_i U\,\partial_j U - F_{ijk}\partial_k U = 0.$$

The contribution of J^{ab} to the bosonic Hamiltonian is proportional to the constant of motion $J^{ab}J_{ab}$

$$H_U = p_i p_i + \frac{1}{2}J^{ab}J_{ab}\partial_i U\partial_i U + \text{fermions}.$$

The equations on functions U and W can be mapped on each other by

$$W \;\Leftrightarrow\; e^{-U}, \qquad F \;\Leftrightarrow\; -F.$$

Let us now modify these supercharges to produce the conformally flat metric in the Hamiltonian $p_i p_i \rightarrow f^2 p_i p_i$.

Modified supercharges

It is possible to obtain the $f^2 p_i p_i$ term in the Hamiltonian just by modifying the first term in the supercharge as $p_i \psi_i^a \to f p_i \psi_i^a$. However, to satisfy algebraic relations the three-fermion term also should be modified:

$$Q_W^a = f\, p_i\, \psi_i^a + \mathrm{i}\, W_i\, \psi_i^a + \mathrm{i}\, F_{ijk}\, \psi_i^b\, \psi_{bj}\, \bar{\psi}_k^a + \mathrm{i}\, G_{ijk}\, \psi_i^a\, \psi_j^b\, \bar{\psi}_{kb} \quad \text{(potential)},$$
$$Q_U^a = f\, p_i\, \psi_i^a + U_i\, J^{ac}\, \psi_{ci} + \mathrm{i}\, F_{ijk}\, \psi_i^c\, \psi_{cj}\, \bar{\psi}_k^a + \mathrm{i}\, G_{ijk}\, \psi_i^a\, \psi_j^c\, \bar{\psi}_{ck} \quad \text{(harmonics)}.$$

Now F_{ijk} is symmetric only in i, j, and $G_{ijk} = -G_{jik}$. They could be combined into most general term $H_{ijk}\, \psi_i^a\, \psi_j^b\, \bar{\psi}_{kb}$ (H_{ijk} has no symmetries), but it is convenient to keep them separated.

The equations on G_{ijk} and F_{ijk}, that do not contain W and U are common for both systems. They directly express G_{ijk} and antisymmetric part of F_{ijk} in terms of $\partial_i f$,

$$G_{ijk} = \delta_{ik} \partial_j f - \delta_{jk} \partial_i f,\; F_{ijk} - F_{ikj} = \frac{3}{2} G_{jki}.$$

This tells us that only completely symmetric part of F_{ijk} remains undetermined. It is related to F_{ijk} by

$$F_{ijk}^S \equiv \frac{1}{3} \left(F_{ijk} + F_{ikj} + F_{jki} \right) = F_{ijk} + \frac{1}{2} G_{kji} + \frac{1}{2} G_{kij}.$$

Generalized WDVV

Upon introduction of F^S_{ijk}, the condition on F splits into two, due to symmetry/antisymmetry in some indices:

$$f\left(\partial_i F^S_{jkl} - \partial_j F^S_{ikl}\right) + \partial_j f\, F^S_{ikl} - \partial_i f\, F^S_{jkl} + \partial_m f\, \left(\delta_{jl} F^S_{ikm} + \delta_{jk} F^S_{ilm} - \delta_{il} F^S_{jkm} - \delta_{ik} F^S_{jlm}\right) = 0,$$

$$f\left(\partial_i G_{lkj} - \partial_j G_{lki}\right) + F^S_{ikm} F^S_{jlm} - F^S_{jkm} F^S_{ilm} + \partial_m f\, \partial_m f\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right) = 0.$$

However, most nice and compact form of these equations can be obtained if one rescales $F^S_{ijk} = f^3 \widetilde{F}_{ijk}$ and introduces the geometric objects on the space with metric $g_{ij} = \frac{1}{f^2}\delta_{ij}$:

$$\Gamma^k_{ij} = \frac{1}{2}(g^{-1})^{km}\left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right) = -\frac{1}{f}\left(\partial_j f\, \delta^k_i + \partial_i f\, \delta^k_j - \delta^{km}\partial_m f\, \delta_{ij}\right),$$

$$R_{imjk} = \frac{1}{f^3}\left(\delta_{ij}\partial^2_{km} f - \delta_{ik}\partial^2_{jm} f + \delta_{mk}\partial^2_{ij} f - \delta_{mj}\partial^2_{ik} f\right) + \frac{1}{f^4}\left(\delta_{ik}\delta_{jm} - \delta_{ij}\delta_{km}\right)\delta^{ps}\partial_p f \partial_s f,$$

$$\nabla_m \widetilde{F}_{ijk} = \partial_m \widetilde{F}_{ijk} - \widetilde{F}_{ijn}\,\Gamma^n_{km} - \widetilde{F}_{ink}\,\Gamma^n_{jm} - \widetilde{F}_{njk}\,\Gamma^n_{im}.$$

Then two equations can be written as

$$\nabla_i \widetilde{F}_{jkl} - \nabla_j \widetilde{F}_{ikl} = 0 \quad \text{and} \quad \widetilde{F}_{ikp} g^{pq} \widetilde{F}_{jmq} - \widetilde{F}_{jkp} g^{pq} \widetilde{F}_{imq} = -R_{ijkm}.$$

They are not contradicting due to Bianchi identities.

Potentials and Hamiltonians

Analysis of terms with W_i and U_i show that they can be expressed in terms of prepotential almost as before,

$$f^2\left[\partial_i\left(\frac{W_j}{f}\right)-\partial_j\left(\frac{W_i}{f}\right)\right]=0 \quad \Rightarrow \quad W_i = f\,\partial_i W$$

(the same for U). Differential equations on U and W can be written in terms of geometric objects as

$$\nabla_i\nabla_j W + \widetilde{F}_{ijk}\, g^{km}\,\nabla_m W = 0, \qquad \nabla_i\nabla_j U - \nabla_i U\,\nabla_j U - \widetilde{F}_{ijk}\, g^{km}\,\nabla_m U = 0.$$

Relation $U \leftrightarrow -\log W$, $\widetilde{F}_{ijk} \leftrightarrow -\widetilde{F}_{ijk}$ is also true.

In the case without harmonics, the Hamiltonian can be written as

$$H_W = f^2\tilde{p}_i\tilde{p}_i + f^2\partial_i W\partial_i W + 4f^2\nabla_i\partial_j W\psi_i^c\bar{\psi}_{cj} - 4f^4\left[\nabla_m\widetilde{F}_{ijk} + R_{imjk}\right]\psi_i^c\bar{\psi}_{cm}\,\psi_j^d\bar{\psi}_{dk}.$$

The Hamiltonian with harmonics is

$$H_U = f^2\tilde{p}_i\tilde{p}_i + \frac{f^2}{2}J_{cd}J^{cd}\partial_i U\partial_i U - 4\mathrm{i}f^2\nabla_i\partial_j U J_{cd}\psi_i^c\bar{\psi}_j^d - 4f^4\left[\nabla_m\widetilde{F}_{ijk} + R_{imjk}\right]\psi_i^c\bar{\psi}_{cm}\,\psi_j^d\bar{\psi}_{dk}.$$

Here, $\tilde{p}_k = p_k - \frac{2\mathrm{i}}{f}G_{ijk}\psi_i^c\,\bar{\psi}_{cj}$.

Equations on R=const spaces

Solving the equations on $\widetilde{F}_{ijk}$ and potentials directly is difficult, especially as dimension increases. However, there are two cases, that allow significant simplifications: the constant curvature metrics and the two-particle situation. In the first case, one may bring situation closer to the flat one, there it is possible to define the $\widetilde{F}_{ijk}$ in terms of single prepotential. Indeed, the substitution

$$\begin{aligned}\widetilde{F}_{ijk} &= \frac{1}{3}\left(\nabla_i\nabla_j\nabla_k F + \nabla_j\nabla_i\nabla_k F + \nabla_k\nabla_i\nabla_j F\right) - \\ &\quad -\frac{4R}{3N(N-1)}\left(g_{ij}\nabla_k F + g_{ik}\nabla_j F + g_{jk}\nabla_i F\right) = \\ &= \nabla_i\nabla_j\nabla_k F - \frac{R}{N(N-1)}\left(2g_{jk}\partial_i F + g_{ij}\partial_k F + g_{ik}\partial_j F\right).\end{aligned}$$

identically satisfies the equation

$$\nabla_i \widetilde{F}_{jkm} - \nabla_j \widetilde{F}_{ikm} = 0$$

if R =const, which is valid for spheres and pseudospheres. This allows one to generalize the flat solutions to this spaces. Unfortunately, addition of more general combinations of Ricci tensors does not help to solve this with prepotential on general conformally flat spaces.

Solutions on R=const spaces

One may try to generalize WDVV solutions to the spheres by introducing to arbitrary functions, depending on x^2.

$$F = \varphi_1(x^2) + \varphi_2(x^2) F^0,$$

where F_0 is a solution of flat WDVV equation. One example, valid in arbitrary dimension, is a solution by Veselov

$$F^0 = \sum_i K(x_i), \quad K(x) = x^2 \log x.$$

Substituting this ansatz into the equation

$$\widetilde{F}_{ikp} g^{pq} \widetilde{F}_{jmq} - \widetilde{F}_{jkp} g^{pq} \widetilde{F}_{imq} = -R_{ijkm},$$

one may see that terms with logarithms immediately determine

$$\varphi_2 = \frac{C}{\left(1 + \rho x^2\right)^2}$$

and $C = 1/2$ to be able to find potential, satisfying $\nabla_i \nabla_j W + \widetilde{F}_{ijk} g^{km} \nabla_m W = 0$. Then the remaining allow to find the φ_1

$$\varphi_1(x^2)_{C=1/2} = -\frac{1}{4\rho} \mathrm{ArcTanh}(\rho x^2) - x^2 \frac{\log\left(1 - \rho x^2\right)}{2\left(1 + \rho x^2\right)^2}.$$

Potential for Veselov's solution

It is natural to find the complete potential by analogous generalization of flat one,

$$W = \varphi_3(x^2) + \varphi_4(x^2)\, W^0, \quad W^0 = \sum_i a_i \log x_i .$$

Substituting this together with $\widetilde{F}_{ijk}$ into the equation for the potential, we find

$$\nabla_i \nabla_j W + \widetilde{F}_{ijk} g^{km} \nabla_m W = 0 \quad \Rightarrow \varphi_4 = \text{ const} = 1, \varphi_3 = -\log\left(1 - \rho x^2\right) \cdot \sum_i a_i ,$$

$$V = f^2 \sum_i \left(\partial_i W\right)^2 = \left(1 + \rho x^2\right)^2 \sum_i \frac{a_i^2}{x_i^2} + 4\rho \left(\sum_i a_i\right)^2 \frac{\left(1 + \rho x^2\right)^2}{\left(1 - \rho x^2\right)^2} .$$

Just for illustration, let us present the $\widetilde{F}_{ijk}$ in this case

$$\widetilde{F}_{ijk} = -2\rho \frac{\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ik} x_j}{\left(1 + \rho x^2\right)^3} - \frac{4\rho^2 x_i x_j x_k}{\left(1 - \rho x^2\right)\left(1 + \rho x^2\right)^3} + \frac{1}{\left(1 + \rho x^2\right)^2} \sum_m \frac{\Delta_{ijkm}}{x_m} ,$$

where $\Delta_{ijkm} = 1$ if and only if $i = j = k = m$, and 0 else.

Other solutions

One may note that the obtained solution for functions φ_1, φ_2

$$\varphi_2 = \frac{1}{2\left(1+\rho x^2\right)^2}, \quad \varphi_1\left(x^2\right) = -\frac{1}{4\rho}\mathrm{ArcTanh}\left(\rho x^2\right) - x^2\,\frac{\log\left(1-\rho x^2\right)}{2\left(1+\rho x^2\right)^2}$$

is relatively general. For example, the prepotential

$$F = \sigma\varphi_1 + \sigma\varphi_2\, F^B, \quad F^B = 2\beta\sum_i K(x_i) + 2\alpha\sum_{i<j}\left[K(x_i-x_j)+K(x_i+x_j)\right],$$

corresponding to the B root systems, satisfies the WDVV if $\beta = 2\left(N-1\right)\alpha - \frac{1}{2}$, where N =number of particles, and $\sigma = \pm 1$ is overall sign. It turns out that equation for the U potential (related to harmonics)

$$\nabla_i\nabla_j U - \nabla_i U\,\nabla_j U - \widetilde{F}_{ijk}g^{km}\nabla_m U = 0$$

is satisfied if

$$\sigma = -1,\ \alpha = -\frac{2}{N},\ U = -\frac{2}{N}\sum_i \log x_i + 2\log\left(1+\rho x^2\right),$$

$$f^2\sum_i\left(\partial_i U\right)^2 = \frac{4\left(1+\rho x^2\right)^2}{N^2}\sum_i\frac{1}{x_i^2} - \frac{4\rho}{N^2}.$$

More solutions

One may also consider potentials U and W, that depend only on x^2. For the previously mentioned prepotential $F \sim \varphi_1 + \varphi_2 F^B$ they read

$$\begin{aligned} U &= 2\log\left(1+\rho x^2\right) - \log\left(C\left(1+\rho x^2\right)^2 - \rho x^2\right), \quad f^2\left(\partial_i U\right)^2 = \frac{4x^2\left(1-\rho x^2\right)^2}{\left(C\left(1+\rho x^2\right)^2 - \rho x^2\right)^2}, \\ W &= c_1 \frac{\left(1-\rho x^2\right)^2}{\left(1+\rho x^2\right)^2} + c_2, \quad f^2\left(\partial_i W\right)^2 = 64\rho^2 c_1^2 \frac{x^2\left(1-\rho x^2\right)^2}{\left(1+\rho x^2\right)^4}. \end{aligned}$$

As the last comment we note that all previously mentioned potentials make sense both for positive and negative ρ. However, exists a solution that makes sense only for $\rho < 0$.

$$F_{\rho<0} = -\frac{\sqrt{-\rho x^2}\left(1-\rho x^2\right)}{4\rho\left(1+\rho x^2\right)^2} + \frac{\mathrm{ArcTanh}\sqrt{-\rho x^2}}{4\rho}.$$

It depends only on x^2 and vanishes if $\rho \to 0$. Related potential can also be provided

$$W = \frac{\sum_i a_i x_i}{\left(1+\sqrt{-\rho x^2}\right)^2}, \quad \partial_i W \partial_i W = \frac{\sum_i a_i^2}{\left(1+\sqrt{-\rho x^2}\right)^4} + 4\rho \frac{\left(\sum_i a_i x_i\right)^2}{\sqrt{-\rho x^2}\left(1+\sqrt{-\rho x^2}\right)^6}.$$

Equations in two dimensions

Solving equations in two dimensions is relatively easy, as in this dimension one may use the WDVV equation (1 component) and 3 equations on the potentials W or U, to obtain 4 independent F_{ijk} algebraically, and check differential equations on F_{ijk}. We a few examples of solutions for nonspherical metrics can be constructed. Let us, for example, consider $f = \sqrt{1 + \rho x^2}$.

- Logarithmic prepotential

  $$W, U = \sum a_k \log(\alpha_k x_1 + \beta_k x_2), \quad \sum a_k = 0.$$

  works both with and without harmonics. Results in reasonably general Calogero potential. Actually, seems to be a solution for any $f = f\left(x^2\right)$

- The general potential, depending on x^2, is

  $$W = \frac{a}{1 + \rho x^2} + b, \;\; V = f^2 \partial_i W \partial_i W = \frac{4a^2 \rho^2}{\left(1 + \rho x^2\right)^3}.$$

  $U = -\log W$ is a solution with harmonics.

- The potential, suggested by black hole considerations, does not seem to be a solution:

  $$\frac{a_1^2}{x_1^2} + \frac{a_2^2}{x_2^2} - \frac{a_1^2}{1 - x_2^2} - \frac{a_2^2}{1 - x_1^2}.$$

Conclusion

In this work, we considered generalization of the many particle $\mathcal{N}=4$ supersymmetric mechanics by introducing of the conformally flat metric in the configuration space. The equations, that guarantee the existence of $\mathcal{N}=4$ supersymmetry, are the *WDVV* equations, generalized to curved spaces. It was shown, that tensor $\widetilde{F}_{ijk}$, which enters this equation, can be expressed in terms of derivatives of some prepotential on spaces of constant curvature. Using this property of equations, we found some of their solutions on spheres and pseudospheres, which generalize known flat solutions.

Some questions remain open:

- It should be found what kind of flat WDVV solutions can be upgraded to solutions on sphere
- Whether already used method of generalization

$$F^0 \to \varphi_1(x^2) + \varphi_2(x^2) F^0, \quad (W, U)^0 \to \varphi_3(x^2) + \varphi_4(x^2)(W, U)^0$$

  is universal.
- Are there any solutions with no flat analogs?
- It would be desirable to find the solution of the sphere, that vanishes when $\rho \to 0$, or prove its unexistence.