The correct coset space construction for the conformal group

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Outline

1. Motivation and current status
2. Mathematical preliminaries: Induced representations
3. The correct coset space for the conformal group
4. Reproducing properties of CFT
5. Connection with the inverse Higgs phenomenon

arXiv:1705.04568

Current status and motivation

- Construction of conformally–invariant theories:
    - Biconformal space?
- Spontaneous breakdown of the conformal group:
    - Redundant Nambu–Goldstone fields
    - Inverse Higgs phenomenon?
- The role of discrete symmetries?

Induced representations

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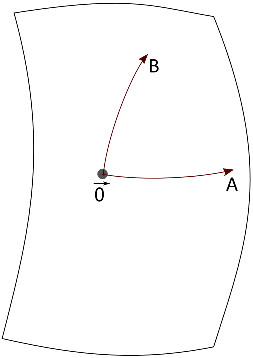
  G – symmetry group,
  H – stability group of the origin,
  $A = G/H$ – homogeneous space.

  Goal: $\psi(x)$ living on A.

- The coordinates on A:

  Isomorphism: $g_H \in G/H \leftrightarrow A$
  $g_H = e^{iP_\mu x^\mu}: \quad \vec{0} \quad \to A\,, B\,, \ldots$



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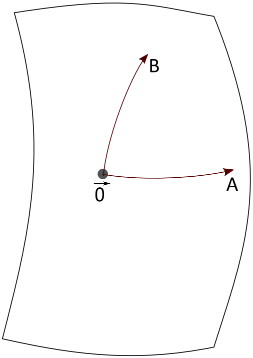

Coset space endows A with an atlas!

The correct coset space

- The conformal group:

$$\{e^{iP_\mu x^\mu},\ e^{iL_{\mu\nu}\omega^{\mu\nu}},\ e^{iD\sigma},\ I\},$$
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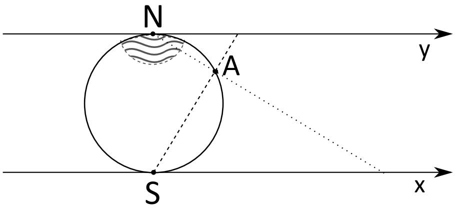

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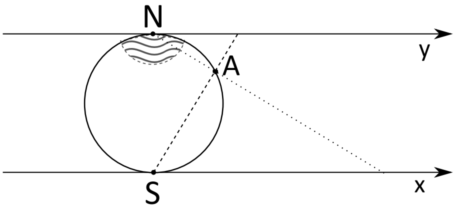

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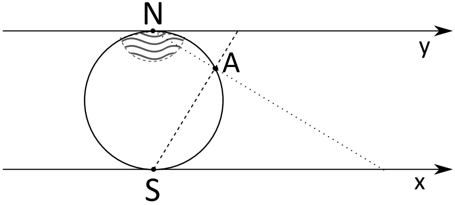

$$H = SO(1,d)\ \times D \times I\ ,$$

$$g_H = e^{iP_\mu x^\mu} e^{iK_\nu y^\nu(x)}$$

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- Mathematical explanation:
  - $g_H^{-1} dg_H = i\omega_P^\mu P_\mu + i\omega_K^\nu K_\nu + i\omega_D D + i\omega_L^{\mu\nu} L_{\mu\nu}$
  - Transformation under the Inversion:

    $\omega_P^\mu \to \omega_K^\mu\,, \quad \omega_K^\nu \to \omega_P^\nu\,, \quad \omega_D \to -\omega_D\,, \quad \omega_L^{\mu\nu} \to \omega_L^{\mu\nu}\,.$
  - $\omega_P^\mu = dx^\mu\,,\ \tilde{\omega}_K^\nu = dy^\nu \;\; \Rightarrow \;\; dy^\nu$ must be the pullback of dx^μ.

Constructing CFT

$$\omega_P^\mu = dx^\mu \, , \quad \omega_K^\nu = dy^\nu + 2y_\rho dx^\rho y^\nu - y^2 dx^\nu \, , \quad \omega_D = 2y_\rho dx^\rho \, , \quad \omega_L^{\mu\nu} = -2y^\mu dx^\nu \, .$$

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- Direct mixing: $\mathcal{L} = \mathcal{L}(\omega_P, \omega_K, D\psi, \psi)$

$$\frac{\delta\mathcal{L}(\omega_P, \omega_K, D\psi, \psi)}{\delta y^\rho} = 0 \Rightarrow \text{Virial current is a total derivative}$$

Spontaneously broken CG

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- The polar decomposition: $\psi(x) = \gamma(x)\tilde{\psi}(x)$

$$\hat{K}_\mu \varphi(x) = 2x_\mu \Delta\varphi \ \Rightarrow \text{no need for the NGF for SCT}$$

Representations of the conformal algebra

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- Unavoidable problem:

$$x'^{\mu} = \frac{x^{\mu}+b^{\mu}x^{2}}{1+2b_{\mu}x^{\mu}+b^{2}x^{2}}$$

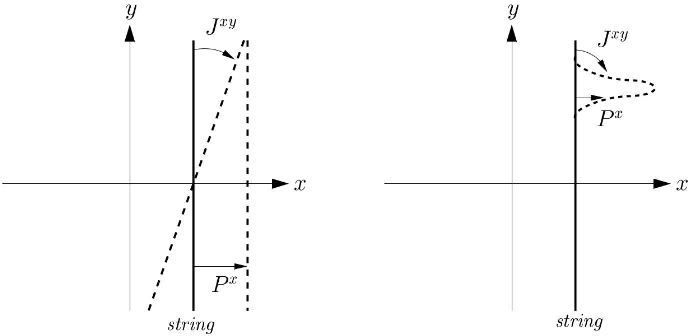
Some of x^{μ} are mapped to the infinity!

Conclusions and final remarks

- The method can be generalized
- The NGF for SCT play a special role
  - Ensure self–consistency of a theory
  - Select Lagrangians that are invariant under discrete symmetries
- Inverse Higgs phenomenon is not applicable to the conformal group

Redundant Nambu–Goldstone fields

- Scalar domain wall (*arXiv:hep-th/0110285*) :



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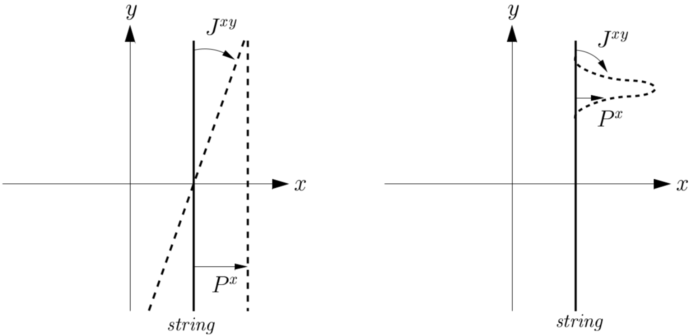

$$\hat{P}_\mu \varphi = \partial_\mu \varphi \,, \quad \hat{J}_{\mu\nu}\varphi(x) = (x_\mu \partial_\nu - x_\nu \partial_\mu)\varphi \,.$$

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- Conformal group:

$$\hat{K}_\mu \Phi(x) = (2x_\mu \Delta - x^\nu \hat{S}_{\mu\nu} - 2i x_\mu x^\nu \partial_\nu + i x^2 \partial_\mu)\Phi(x) \ .$$

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- Inverse Higgs phenomenon: $[P_\mu, K_\nu] \sim \eta_{\mu\nu} D$,

$$\omega_D = 0 \ \Rightarrow \ y_\mu = -\tfrac{1}{2} e^{-\pi} \partial_\mu \pi \ .$$