Exact Results in Explicit Three-Loop Calculations Using Higher Derivatives for N=1 SYM

The Adler D-function for $N = 1$ SQCD regularised by higher covariant derivatives in the three-loop approximation, INR-TH-2017-017

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The Adler D-function[1] is related to the cross-section of annihilation of the electron-positron pair into hadrons (denoted as $R(s)$) through a dispersion relation

$$D(Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{(S+Q^2)^2} = -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi(-Q^2), \qquad (1)$$

where $\Pi(Q^2)$ is the photon polarisation operator.
Recently a formula was proposed[2] and then proved to all loops[3] connecting the Adler D-function and the anomalous dimension

$$D(\alpha_{s0}) = \frac{3}{2} N \sum_{\alpha=1}^{N_f} q_\alpha^2 (1-\gamma(\alpha_{s0})), \qquad (2)$$

where α_{s0} is the bare strong coupling constant.

[1] Adler, S. L. *Phys. Rev. D* **10,** 3714–3728 (1974). The Adler D-function has been calculated to order $O(\alpha_s^4)$ in QCD: Baikov, P. A. *et al. Phys. Rev. Lett.* **101,** 012002 (2008); Baikov, P. A. *et al. Phys. Lett.* **B714,** 62–65 (2012), and to order $O(\alpha_s)$ in $N=1$ SQCD: Kataev, A. L. & Pivovarov, A. A. *JETP Lett.* **38,** 369–371 (1983); Altarelli, G. *et al. Phys. Lett.* **129B,** 456–460 (1983).

[2] Shifman, M. & Stepanyantz, K. *Phys. Rev. Lett.* **114,** 051601 (2015).

[3] Shifman, M. A. & Stepanyantz, K. V. *Phys. Rev. D* **91,** 105008 (2015).

The proof of this formula utilized the regularisation by higher covariant derivatives[4]. Its main features that made this derivation possible include

- retaining both gauge symmetry and supersymmetry;
- mathematical self-consistency;
- factorisation of contributions to the β-function into integrals of total[5] and even double total derivatives[6], which allows one to take the integral explicitly with respect to a loop momentum in such a contribution at n loops and compare it with the contribution to the anomalous dimension at $n-1$ loops.

[4] Slavnov, A. A. *Nucl. Phys. B* **31,** 301–315 (1971); Slavnov, A. A. *Theor. Math. Phys.* **13,** 1064–1066 (1972); Krivoshchekov, V. K. *Theor. Math. Phys.* **36,** 745–752 (1978); West, P. *Nucl. Phys. B* **268,** 113 –124 (1986).

[5] Soloshenko, A. A. & Stepanyantz, K. V. *Theor. Math. Phys.* **140,** 1264–1282 (2004).

[6] Smilga, A. V. & Vainshtein, A. *Nucl. Phys. B* **704,** 445–474 (2005), proof to all loops of both statements for $N=1$ SQED can be found in Stepanyantz, K. *Nucl. Phys. B* **852,** 71 –107 (2011) and Stepanyantz, K. V. *JHEP* **08,** 096 (2014) respectively.

N=1 SQCD: the Action

Consider a gauge theory with a gauge group $G \times U(1)$, with G an arbitrary simple group. Let chiral matter superfields ϕ_α, $\alpha = 1 \ldots N_f$, lie each in a representation R of the group G and have charges q_α with respect to U(1) and chiral matter superfields $\tilde{\phi}_\alpha$ lie in the complex conjugate representation $\overline{R}$ and have charges $-q_\alpha$:

$$S = \frac{1}{2g_0^2} \mathrm{tr}\ \mathrm{Re} \int d^4x d^2\theta W^\alpha W_\alpha + \frac{1}{4e_0^2} \mathrm{Re} \int d^4x d^2\theta \mathbf{W}^\alpha \mathbf{W}_\alpha + \\ + \frac{1}{4} \sum_{\alpha=1}^{N_f} \Bigg[\int d^4x d^4\theta \left(\phi_\alpha^\dagger e^{2V + 2q_\alpha \mathbf{V}} \phi_\alpha + \tilde{\phi}_\alpha^\dagger e^{-2V^t - 2q_\alpha \mathbf{V}} \tilde{\phi}_\alpha \right) + \\ + \frac{1}{2} \left(\int d^4x d^2\theta m_{0\alpha} \tilde{\phi}_\alpha^t \phi_\alpha + h.c. \right) \Bigg]. \quad (3)$$

The term with higher powers of covariant derivatives

$$S_\Lambda = \frac{1}{2g_0^2}\mathrm{tr}\mathrm{Re}\int d^4x d^2\theta W^\alpha \left[R\left(-\frac{\overline{\nabla}^2\nabla^2}{16\Lambda^2}\right) - 1\right] W_\alpha. \qquad (4)$$

where we introduce covariant derivatives in the chiral representation

$$\overline{\nabla}_{\dot\alpha} = \overline{D}_{\dot\alpha} \qquad \nabla_\alpha = e^{-2V} D_\alpha e^{2V} \qquad (5)$$

and choose the regulator to be $R(x) = 1 + x^n$.
To regularise one-loop divergences we introduce Pauli-Villars fields for matter loops

$$S_{PV} = \sum_{\alpha=1}^{N_f}\left[\frac{1}{4}\int d^4x d^4\theta \left(\Phi_\alpha^\dagger e^{2V+2q_\alpha \mathbf{V}}\Phi_\alpha + \tilde\Phi_\alpha^\dagger e^{-2V^t-2q_\alpha\mathbf{V}}\tilde\Phi_\alpha\right) + \right.$$
$$\left. +\frac{1}{2}\left(\int d^4x d^2\theta M \tilde\Phi_\alpha^t \Phi_\alpha + h.c.\right)\right] \qquad (6)$$

and for loops of the non-Abelian gauge superfield and the Faddeev-Popov ghosts

$$S_\varphi = \frac{1}{2}\mathrm{tr}\int d^4x d^4\theta \varphi_1^\dagger \left[e^{2V} R\left(-\frac{\overline{\nabla}^2\nabla^2}{16\Lambda^2}\right)\right]_{adj}\varphi_1 + \frac{1}{2}\mathrm{tr}\int d^4x d^4\theta \varphi_2^\dagger \left[e^{2V}\right]_{adj}\varphi_2 +$$
$$+\frac{1}{2}\mathrm{tr}\int d^4x d^4\theta \varphi_3^\dagger\left[e^{2V}\right]_{adj}\varphi_3 + \frac{1}{2}\left(\mathrm{tr}\int d^4x d^2\theta M_\varphi(\varphi_1^2+\varphi_2^2+\varphi_3^2) + h.c.\right). \qquad (7)$$

The gauge-fixing term:

$$S_{gf} = -\frac{1}{16g^2}\mathrm{tr}\int d^4x d^4\theta \overline{D}^2 V R\left(\frac{\partial^2}{\Lambda^2}\right) D^2 V, \tag{8}$$

Accordingly, the action for Faddeev-Popov ghosts:

$$S_{FP} = \frac{1}{g_0^2}\int d^4x d^4\theta (\overline{c} + \overline{c}^\dagger)\left(\left(\frac{V}{1-e^{2V}}\right)_{adj} c^\dagger - \left(\frac{V}{1-e^{-2V}}\right)_{adj} c\right) \tag{9}$$

Due to the explicit gauge invariance with respect to $U(1)$ corrections to the two-point Green function of the Abelian gauge superfield are transversal and the corresponding part of the effective action has the form:

$$\Delta\Gamma^{(2)}_{\mathbf{V}} = -\frac{1}{16\pi}\int \frac{d^4p}{(2\pi)^4} d^4\theta \mathbf{V}(-p,\theta)\partial^2\Pi_{1/2}\mathbf{V}(p,\theta)\left(d^{-1}\left(\alpha_0,\alpha_{s0},\frac{\Lambda}{p}\right)-\alpha_0^{-1}\right),$$

where $\alpha_0 = e_0^2/4\pi$ and $\alpha_{s0} = g_0^2/4\pi$ are bare charges corresponding to $U(1)$ and G respectively and $\Pi_{1/2} = -D^\alpha \overline{D}^2 D_\alpha/8\partial^2$ is the transversal projection operator.

The same applies to the non-Abelian gauge superfield but this time due to BRST-invariance and the Slavnov-Taylor identities.

Quantum corrections to the two-point Green function of the chiral matter superfields enter the effective action as

$$\Gamma^{(2)}_\phi = \frac{1}{4}\sum_{\alpha=1}^{N_f}\int \frac{d^4p}{(2\pi)^4} d^4\theta \left(\phi^{\dagger i}_\alpha(-p,\theta) G_i{}^j\left(\alpha_{s0},\Lambda/p\right)\phi_j(p,\theta)+\right.$$

$$\left. +\tilde{\phi}^\dagger_{\alpha j}(-p,\theta) G^j{}_i\left(\alpha_{s0},\Lambda/p\right)\tilde{\phi}^{\alpha i}(p,\theta)\right).$$

We then introduce renormalised charges $\alpha(\alpha_0, \alpha_{s0}, \Lambda/\mu)$, $\alpha_s(\alpha_{s0}, \Lambda/\mu)$, or equivalently, renormalisation constants

$$Z_\alpha \equiv \alpha/\alpha_0, \qquad Z_{\alpha_s} \equiv \alpha_s/\alpha_{s0}; \tag{10}$$

and field strength renormalisation constants for the chiral matter superfields:

$$\phi_{\alpha i} = (\sqrt{Z})_i{}^j (\phi_R)_{\alpha j}; \quad \tilde{\phi}^i_\alpha = (\sqrt{Z})^i{}_j (\tilde{\phi}_R)^{\alpha j}, \tag{11}$$

where R stands for *renormalised.*

At one loop the strong coupling α_s is renormalised as follows:

$$\frac{1}{\alpha_{s0}} - \frac{1}{\alpha_s} = \frac{1}{2\pi}\left[3C_2\left(\ln\frac{\Lambda}{\mu} + b_{11}\right) - 2N_f T(R)\left(\ln\frac{\Lambda}{\mu} + b_{12}\right)\right] + O(\alpha_s), \tag{12}$$

where b_{11} and b_{12} are finite constants defined through a particular renormalisation scheme.

We define our RG functions in terms of the bare coupling constant as follows. The Adler D-function:

$$D(\alpha_{s0}) = -\frac{3\pi}{2}\frac{d}{d\ln\Lambda}\alpha_0^{-1}\left(\alpha,\alpha_s,\Lambda/\mu\right)\Big|_{\alpha,\alpha_s=\text{const}},$$

which is equivalent to

$$D(\alpha_{s0}) =$$
$$= \frac{3\pi}{2}\ \frac{d}{d\ln\Lambda}\left[d^{-1}(\alpha_0(\alpha,\alpha_s,\Lambda/\mu),\alpha_{0s}(\alpha_s,\Lambda/\mu),\Lambda/p) - \alpha_0^{-1}(\alpha,\alpha_s,\Lambda/\mu)\right]\Big|_{\substack{\alpha,\alpha_s=\text{const}\\ p=0}},$$

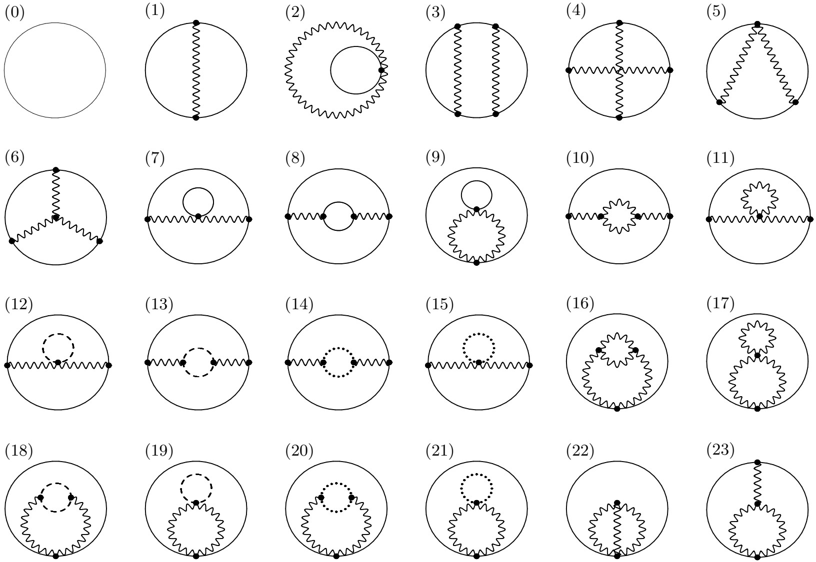
and the anomalous dimension:

$$\gamma_i{}^j(\alpha_{s0}) = -\frac{d\ln Z_i{}^j}{d\ln\Lambda}\Big|_{\alpha_s=\text{const}} = \frac{d\ln G_i{}^j(\alpha_{s0}(\alpha_s,\Lambda/\mu),\Lambda/q)}{d\ln\Lambda}\Big|_{\substack{\alpha_s=\text{const}\\ q=0}}.$$

One of the main goals of this work is to verify at three loops the relation

$$D(\alpha_{s0}) = \frac{3}{2}\sum_{\alpha=1}^{N_f} q_\alpha^2\left(\dim(R) - \text{tr}\gamma(\alpha_{s0})\right).$$

Diagrams Contributing to the D-Function in the Three-Loop Approximation



The contributions of these graphs to the Adler D-function are given by an integral of a double total derivative of the form:

$$D(\alpha_{s0}) = \frac{3\pi}{2}\cdot 4\pi\sum_{\alpha=1}^{N_f} q_\alpha^2 \int \frac{d^4q}{(2\pi)^4}\frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q^\mu}\frac{d}{d\ln\Lambda}\Big(I_0(q) + \mathrm{tr}\, C(R) I_1(q) +$$
$$+\,\mathrm{tr}\, C(R)^2 I_2(q) + C_2 \mathrm{tr}\, C(R)(I_3(q) + I_4(q)) + N_f T(R) \mathrm{tr}\, C(R) I_5(q)\Big), \quad (13)$$

where

$$I_i = I_i(q, M=0) - I_i(q, M) \quad \text{for} \quad i = 1\ldots 4$$

with $I_i(q,M)$, $i = 1\ldots 4$, $I_0(q)$ and $I_5(q)$ defined as

$$I_0(q) = \frac{1}{4q^2}\ln\left(1+\frac{M^2}{q^2}\right);$$

$$I_1(q,M) = \int \frac{d^4k}{(2\pi)^4}\frac{g_0^2}{R_k k^2(q^2+M^2)((q+k)^2+M^2)};$$

$$I_2(q,M) = \int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{g_0^4}{R_k k^2 R_l l^2 (q^2+M^2)((q+k)^2+M^2)((q+l)^2+M^2)}\times$$
$$\times\Big(\frac{2(q^2-M^2)}{q^2+M^2}+\frac{(2q+k+l)^2+2M^2}{(q+k+l)^2+M^2}-4\Big);$$

$$I_3(q,M) = \int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{g_0^4}{R_k k^2 R_l l^2 (q^2+M^2)((q+k)^2+M^2)((q+l)^2+M^2)}\times$$
$$\times\Big(\frac{(2q+k+l)^2+2M^2}{2((q+k+l)^2+M^2)}+1-\frac{2((q+k)_\mu(q+l)^\mu+M^2)}{(l-k)^2}-\frac{4}{(l-k)^2 R_{l-k}}\frac{R_l-R_k}{l^2-k^2}\times$$
$$\times\Big(q^2(q+k)_\mu l^\mu + l^2(q+k)_\mu q^\mu + M^2(q+k+l)_\mu l^\mu\Big)\Big);$$

$$I_4(q,M) = -\int \frac{d^4k}{(2\pi)^4}\frac{2g_0^4 f(k)}{R_k^2 k^2 (q^2+M^2)((q+k)^2+M^2)};$$

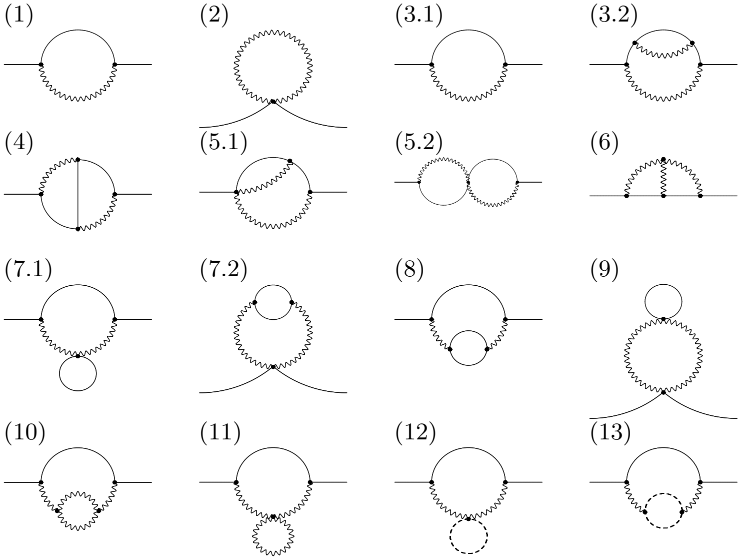
$$I_5(q) = -\int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{4g_0^4}{k^2 R_k^2}\Big(\frac{1}{q^2(q+k)^2}-\frac{1}{(q^2+M^2)((q+k)^2+M^2)}\Big)\times$$
$$\times\Big(\frac{1}{q^2(q+l)^2}-\frac{1}{(q^2+M^2)((q+l)^2+M^2)}\Big).$$

$$
\begin{aligned}
f(k) = -\int \frac{d^4 l}{(2\pi)^4}\Bigg[&\frac{3}{2}\left(\frac{1}{l^2(l+k)^2} - \frac{1}{(l^2+M_\varphi^2)((l+k)^2+M_\varphi^2)}\right) - \frac{R_l - R_k}{R_l l^2}\left(\frac{1}{(l+k)^2} - \right.\\
&\left.-\frac{1}{l^2-k^2}\right) - \frac{2}{(l+k)^2-l^2}\left(\frac{R_{l+k}-R_l}{(l+k)^2-l^2} - \frac{R_l'}{\Lambda^2}\right)\left(\frac{1}{R_l} - \frac{R_l l^2}{R_l^2 l^2 + M_\varphi^2}\right) + \\
&+\left(\frac{R_{l+k}-R_l}{(l+k)^2-l^2}\right)^2\left(\frac{1}{R_l R_{l+k}} - \frac{(l+k)^2(l^2 R_l R_{l+k} - M_\varphi^2)}{(l^2 R_l^2 + M_\varphi^2)((l+k)^2 R_{l+k}^2 + M_\varphi^2)}\right) + \frac{R_{l+k}-R_l}{(l+k)^2-l^2}\times\\
&\times\frac{2\,M_\varphi^2 R_l}{((l+k)^2 R_{l+k}^2 + M_\varphi^2)(l^2 R_l^2 + M_\varphi^2)} + \frac{1}{2(l^2+M_\varphi^2)((l+k)^2+M_\varphi^2)} - \frac{R_l^2 R_{l+k}^2}{2(l^2 R_l^2 + M_\varphi^2)}\times\\
&\times\frac{1}{(l+k)^2 R_{l+k}^2 + M_\varphi^2} + \frac{R_{l+k}-R_k}{(l+k)^2-k^2}\,\frac{2\,R_k k^2}{l^2(l+k)^2 R_l R_{l+k}} + \frac{(l,k)R_k}{l^2 R_l (l+k)^2 R_{l+k}}\,\frac{R_{l+k}-R_l}{(l+k)^2-l^2} - \\
&-\frac{R_{l+k}-R_k}{(l+k)^2-k^2}\,\frac{R_{l+k}-R_l}{(l+k)^2-l^2}\,\frac{2\,(l,k)}{l^2 R_l R_{l+k}} + \left(\frac{R_l-R_k}{l^2-k^2}\right)^2\frac{2\,k^2}{(l+k)^2 R_l R_{l+k}} + \\
&+\frac{k^2(l,l+k)}{l^2(l+k)^2 R_l R_{l+k}}\,\frac{R_l-R_k}{l^2-k^2}\,\frac{R_{l+k}-R_k}{(l+k)^2-k^2} - \frac{2}{(l+k)^2-k^2}\left(\frac{R_{l+k}-R_k}{(l+k)^2-k^2} - \frac{R_k'}{\Lambda^2}\right)\frac{k^2}{l^2 R_l} + \\
&+\frac{2\,(l,k)}{l^2 R_l}\left(\frac{R_l}{(l^2-(l+k)^2)\,(l^2-k^2)} + \frac{R_{l+k}}{((l+k)^2-l^2)\,((l+k)^2-k^2)} + \right.\\
&\left.+\frac{R_k}{(k^2-l^2)\,(k^2-(l+k)^2)}\right)\Bigg].
\end{aligned}
$$

After performing integration with respect to q we obtain

$$D(\alpha_{s0}) = \frac{3}{2}\sum_{\alpha=1}^{N_f} q_\alpha^2 \Bigg[\dim\,(R) + \frac{d}{d\ln\Lambda}\Bigg\{\ \mathrm{tr}\,C(R)\int\frac{d^4k}{(2\pi)^4}\frac{2g_0^2}{k^4R_k}\times$$
$$\times\left(1-\frac{C_2 g_0^2 f(k)}{R_k}\right) - N_f T(R)\ \mathrm{tr}\,C(R)\int\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{4g_0^4}{k^4R_k^2}\left(\frac{1}{l^2(l+k)^2}-\right.$$
$$\left.-\frac{1}{(l^2+M^2)((l+k)^2+M^2)}\right) + \mathrm{tr}\,C(R)^2\int\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{2g_0^4}{k^2R_kl^2R_l}\left(\frac{1}{l^2k^2}-\right.$$
$$\left.\left.\left.-\frac{2}{l^2(l+k)^2}\right)\right\}\right] + O(\alpha_{s0}^3). \quad (14)$$

Diagrams Contributing to the Anomalous Dimension in the Two-Loop Approximation



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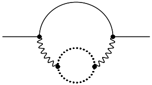

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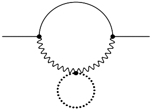

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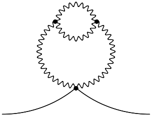

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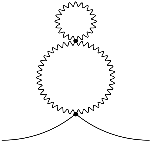

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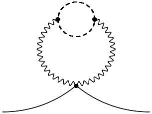

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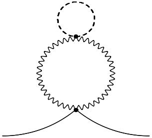

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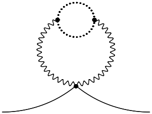

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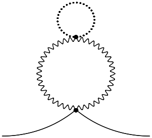

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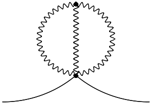

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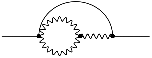

Evaluating these diagrams we obtain:

$$
\begin{aligned}
\gamma(\alpha_{s0})_i{}^j = -C(R)_i{}^j \frac{d}{d\ln\Lambda}\int \frac{d^4k}{(2\pi)^4}\frac{2g_0^2}{k^4R_k} - \\
-(C(R)^2)_i{}^j \frac{d}{d\ln\Lambda}\int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{2g_0^4}{k^2R_kl^2R_l}\left(\frac{1}{l^2k^2}-\frac{2}{l^2(l+k)^2}\right)+ \\
+N_fT(R)C(R)_i{}^j \frac{d}{d\ln\Lambda}\int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{4g_0^4}{k^4R_k^2}\left(\frac{1}{l^2(l+k)^2}-\right. \\
\left.-\frac{1}{(l^2+M^2)((l+k)^2+M^2)}\right)+C_2C(R)_i{}^j \frac{d}{d\ln\Lambda}\int \frac{d^4k}{(2\pi)^4}\frac{4g_0^4}{k^4R_k^2}f(k)+O(g_0^6),
\end{aligned}
\tag{15}
$$

The integrals are the same as those entering the Adler D-function. In fact, one can easily see that

$$D(\alpha_{s0}) = \frac{3}{2}\sum_{\alpha=1}^{N_f} q_\alpha^2 \left(\dim(R) - \mathrm{tr}\gamma(\alpha_{s0})\right) + O(\alpha_{s0}^3).$$

In the case of the simplest regulator possible $R(x) = 1 + x$ the integrals can be evaluated producing a complete analytical expression for $\gamma(\alpha_{s0})_i{}^j$ (and consequently for $D(\alpha_{s0})$) in powers of α_{s0}:

$$\gamma(\alpha_{s0})_i{}^j = -C(R)_i{}^j \frac{\alpha_{s0}}{\pi} - C(R)_i{}^j C_2 \frac{3\alpha_{s0}^2}{2\pi^2} (\ln a_\varphi + 1) + \\ + N_f T(R) C(R)_i{}^j \frac{\alpha_{s0}^2}{\pi^2} (\ln a + 1) + (C(R)^2)_i{}^j \frac{\alpha_{s0}^2}{2\pi^2} + O(\alpha_{s0}^3); \quad (16)$$

$$D(\alpha_{s0}) = \frac{3}{2} \sum_{\alpha=1}^{N_f} q_\alpha^2 \Big(\dim(R) + \mathrm{tr}\, C(R) \frac{\alpha_{s0}}{\pi} + \\ + C_2 \mathrm{tr}\, C(R) \frac{3\alpha_{s0}^2}{2\pi^2} (\ln a_\varphi + 1) - N_f T(R) \mathrm{tr}\, C(R) \frac{\alpha_{s0}^2}{\pi^2} (\ln a + 1) - \\ - \mathrm{tr}\, C(R)^2 \frac{\alpha_{s0}^2}{2\pi^2} \Big) + O(\alpha_{s0}^3), \quad (17)$$

where $a_\varphi \equiv M_\varphi/\Lambda$ and $a \equiv M/\Lambda$ are arbitrary constants.

RG functions defined in terms of the bare coupling constant do not depend on the subtraction scheme (although they do depend on the regularisation)while RG functions defined in terms of renormalised couplings are scheme-dependent[7].

To fix the scheme in which the NSVZ-like relation between D and γ holds one has simply to impose a certain boundary condition on the renormalisation constants.

Let us introduce for convenience a new variable

$$x \equiv \ln \Lambda/\mu.$$

Then we require that for some arbitrary but fixed value x_0 of x

$$Z(\alpha_s, x_0)_i{}^j = \delta_i^j; \qquad Z_\alpha(\alpha, \alpha_s, x_0) = 1; \qquad Z_{\alpha_s}(\alpha_s, x_0) = 1. \tag{18}$$

[7] Kataev, A. L. & Stepanyantz, K. V. *Nucl. Phys. B* **875,** 459–482 (2013).

Proof of existence. Suppose we are employing a scheme in which these conditions do not hold, then

$$\alpha_0(\alpha,\alpha_s,x_0) = a(\alpha,\alpha_s); \quad \alpha_{s0}(\alpha_s,x_0) = b(\alpha_s); \quad Z(\alpha_s,x_0)_i{}^j = g(\alpha_s)_i{}^j.$$

Let us define new renormilised charges and renormalisation constants:

$$\alpha'(\alpha,\alpha_s) = a(\alpha,\alpha_s); \quad \alpha'_s(\alpha_s) = b(\alpha_s);$$

$$Z'(\alpha'_s,x)_i{}^j = g^{-1}(\alpha_s(\alpha'_s))_i{}^k Z(\alpha_s(\alpha'_s),x)_k{}^j.$$

The boundary conditions obviously hold. Let us now demonstrate that given that the NSVZ-like relation is valid for the RG functions defined in terms of the bare coupling constants then it is true for the RG functions defined in terms of the renormalised coupling constants.

Validity of the NSVZ-like relation. The definition of the Adler D-function and the anomalous dimension in terms of the renormalised couplings is as follows:

$$\tilde{D}(\alpha_s) = -\frac{3\pi}{2}\frac{d}{d\ln\mu}\alpha^{-1}(\alpha_0,\alpha_{s0},\Lambda/\mu)\Big|_{\alpha_{s0},\alpha_0=\text{const}};$$

$$\tilde{\gamma}(\alpha_s)_i{}^j = \frac{d\ln Z(\alpha_s,\Lambda/\mu)_i{}^j}{d\ln\mu}\Big|_{\alpha_{s0}=\text{const}}.$$

$$\tilde{\gamma}(\alpha_s)_i{}^j = -\frac{d\ln Z(\alpha_s,x)_i{}^j}{dx}\Big|_{\alpha_s=\text{const}} - \frac{\partial\ln Z(\alpha_s,x)_i{}^j}{\partial\alpha_s}\frac{d\alpha_s(\alpha_{s0},x)}{dx}$$

Set $x = x_0$; due to the boundary conditions $\alpha_s = \alpha_{s0}$ and also

$$\frac{\partial\ln Z(\alpha_s,x_0)_i{}^j}{\partial\alpha_s} = 0,$$

which in turn means that

$$\tilde{\gamma}(\alpha_{s0})_i{}^j = \gamma(\alpha_{s0})_i{}^j$$

that is to say the two functions are identically equivalent.

The same can be performed for the Adler D-function:

$$\tilde{D}(\alpha_s) = \frac{3\pi}{2\alpha} \frac{d}{d\ln\mu} \ln Z_\alpha(\alpha_s, \alpha, \Lambda/\mu)\Big|_{\alpha_0, \alpha_{s0} = \text{const}}$$

$$\tilde{D}(\alpha_s) = -\frac{3\pi}{2\alpha} \left(\frac{d \ln Z_\alpha(\alpha, \alpha_s, x)}{dx}\Big|_{\alpha, \alpha_s = \text{const}} + \right.$$
$$+ \frac{\partial \ln Z_\alpha(\alpha, \alpha_s, x)}{\partial \alpha} \frac{d\alpha(\alpha_0, \alpha_{s0}, x)}{dx} +$$
$$\left. \frac{\partial \ln Z_\alpha(\alpha, \alpha_s, x)}{\partial \alpha_s} \frac{d\alpha_s(\alpha_0, \alpha_0, x)}{dx} \right)$$

When we set $x = x_0$, the last two terms vanish due to the boundary conditions and we obtain

$$\tilde{D}(\alpha_{s0}) = D(\alpha_{s0}).$$

Now since we have $D(\alpha_{s0})$ and $\gamma(\alpha_{s0})$ satisfy the NSVZ-like relation, then we have $\tilde{D}(\alpha_s)$ and $\tilde{\gamma}(\alpha_s)$ satisfy the same relation since the latter are equal to the former.

In the three-loop approximation with the use of the one-loop running of α_s we obtain

$$\ln Z_i{}^j = C(R)_i{}^j \frac{\alpha_s}{\pi}\left(\ln\frac{\Lambda}{\mu} + g_1\right) + C_2 C(R)_i{}^j \frac{3\alpha_s^2}{4\pi^2}\left(-\ln^2\frac{\Lambda}{\mu} + \right.$$
$$\left. +2\ln\frac{\Lambda}{\mu}(\ln a_\varphi + 1 - b_{11}) + g_{21}\right) - N_f T(R) C(R)_i{}^j \frac{\alpha_s^2}{2\pi^2}\left(-\ln^2\frac{\Lambda}{\mu} + \right.$$
$$\left. +2\ln\frac{\Lambda}{\mu}(\ln a + 1 - b_{12}) + g_{22}\right) - (C(R)^2)_i{}^j \frac{\alpha_s^2}{2\pi^2}\left(\ln\frac{\Lambda}{\mu} + g_{23}\right) + O(\alpha_s^3),$$

$$\alpha_0^{-1} - \alpha^{-1} = -\frac{1}{\pi}\sum_{\alpha=1}^{N_f} q_\alpha^2 \left(\dim(R)\left(\ln\frac{\Lambda}{\mu} + d_1\right) + \frac{\alpha_s}{\pi}\mathrm{tr}\, C(R)\left(\ln\frac{\Lambda}{\mu} + d_2\right) - \right.$$
$$- \frac{3\alpha_s^2}{2\pi^2} C_2 \mathrm{tr}\, C(R)\left(\frac{1}{2}\ln^2\frac{\Lambda}{\mu} - \ln\frac{\Lambda}{\mu}(\ln a_\varphi + 1 - b_{11}) + d_{31}\right) +$$
$$+ \frac{\alpha_s^2}{\pi^2} N_f T(R) \mathrm{tr}\, C(R)\left(\frac{1}{2}\ln^2\frac{\Lambda}{\mu} - \ln\frac{\Lambda}{\mu}(\ln a + 1 - b_{12}) + d_{32}\right) -$$
$$\left. - \frac{\alpha_s^2}{2\pi^2}\mathrm{tr}\, C(R)^2\left(\ln\frac{\Lambda}{\mu} + d_{33}\right)\right) + O(\alpha_s^3),$$

where the b's, g's and d's are finite constants whose particular values are defined by the renormalisation prescription.

From these equations we can obtain the anomalous dimension and D-function defined in terms of the renormalised couplings:

$$\tilde{\gamma}(\alpha_s)_i{}^j = \left.\frac{d\ln Z_i{}^j}{d\ln\mu}\right|_{\alpha_{s0}=\text{const}} = -C(R)_i{}^j\frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2}\Bigg[-\frac{3}{2}C_2 C(R)_i{}^j\left(\ln a_\varphi + 1 +\right.$$

$$\left. + g_1 - b_{11}\right) + N_f T(R) C(R)_i{}^j\left(\ln a + 1 + g_1 - b_{12}\right) + \frac{1}{2}(C(R)^2)_i{}^j\Bigg] + O(\alpha_s^3).$$

$$\tilde{D}(\alpha_s) = -\left.\frac{3\pi}{2}\frac{d}{d\ln\mu}\alpha^{-1}\right|_{\alpha_0,\alpha_{s0}=\text{const}} = \frac{3}{2}\sum_{\alpha=1}^{N_f} q_\alpha^2\Bigg(\dim(R) + \frac{\alpha_s}{\pi}\text{tr}\, C(R) +$$

$$+\frac{\alpha_s^2}{\pi^2}\Bigg[\frac{3}{2}C_2\text{tr}\, C(R)\left(\ln a_\varphi + 1 + d_2 - b_{11}\right) - N_f T(R)\text{tr}\, C(R)\left(\ln a + 1 + d_2 - b_{12}\right) -$$

$$-\frac{1}{2}\text{tr}(C(R))^2\Bigg]\Bigg) + O(\alpha_s^3).$$

Unlike RG functions defined in terms of the bare couplings these depend on the renormalisation prescription.

Now we will try to satisfy the boundary conditions. Let us choose some value x_0 of $x = \ln \Lambda/\mu$.
One-loop renormalisation of α_s:

$$\alpha_{s0}^{-1} - \alpha_s^{-1} = \frac{1}{2\pi}\left[3C_2\left(\ln\frac{\Lambda}{\mu} + b_{11}\right) - 2N_f T(R)\left(\ln\frac{\Lambda}{\mu} + b_{12}\right)\right] + O(\alpha_s).$$

One can see that to satisfy the boundary conditions for α_s one has to choose

$$b_{11} = b_{12} = -x_0,$$

and for ${Z_i}^j$

$$g_1 = -x_0,$$

and for α

$$d_1 = d_2 = -x_0.$$

But in this case

$$g_1 = b_{11} = b_{12}; \quad d_2 = b_{11} = b_{12},$$

which ensures identical equality between D, γ and $\tilde{D}$, $\tilde{\gamma}$ respectively and the validity of the NSVZ-like relation for the latter.

In thus chosen subtraction scheme (the NSVZ-like scheme) the Adler D-function defined in terms of the renormalised charges has the form

$$\tilde{D}(\alpha_s) = \frac{3}{2}\sum_{\alpha=1}^{N_f} q_\alpha^2 \left(\dim(R) + \frac{\alpha_s}{\pi}\operatorname{tr} C(R) + \frac{\alpha_s^2}{\pi^2}\left[\frac{3}{2}C_2 \operatorname{tr} C(R)\left(\ln a_\varphi + 1\right) - \right.\right.$$
$$\left.\left. - N_f T(R) \mathrm{tr} C(R)\left(\ln a + 1\right) - \frac{1}{2}\operatorname{tr} C(R)^2\right]\right) + O(\alpha_s^3).$$

- We have calculated the Adler D-function at three loops and the anomalous dimension at two loops using their definition in terms of the bare charges for N=1 SQCD regularised by higher covariant derivatives in the complete form for matter lying in an arbitrary representation of the gauge group.
- We have verified the validity of the NSVZ-like relation for the RG functions defined in terms of the bare couplings.
- We have also found the renormalisation scheme in which the NSVZ-like relation is valid for the RG-functions defined in terms of the renormalised coupling constants.