

Short historical notes

Dislocations: V. Volterra (1905)
K. Kondo (1952)
J. F. Nye (1953)
B. Bilby, R. Bullough, E. Smith (1955)
E. Kröner, I. Dzyaloshinskii, G. Volovik, M. Kléman, I. Kunin,
J. Madore, A. Kadić, D. Edelen, H. Kleinert,
E. Bezerra de Mello, F. Moraes, C. Malyshev, M. Lazar, ...

Disclinations: F. Frank (1958)
I. Dzyaloshinskii, G. Volovik (1978)
J. Hertz (1978)

Torsion: E. Cartan (1922)

Torsion in gravity: A. Einstein, E. Schrödinger, H. Weyl,
T. Kibble, D. Sciama, R. Finkelstein,
F. Hehl, P. von der Heyde, G. Kerlich, J. Nester,
Y. Ne'eman, J. Nitsch, J. McCrea, J. Mielke, Yu. Obukhov
M. Blagoječić, I. Nikolić, M. Vasilić, K. Hayashi, T. Shirafuji,
E. Sezgin, P. van Nieuwenhuizen, I. Shapiro, ...

Chern-Simons term in the Geometric Theory of Defects

M. O. Katanaev, Steklov Mathematical Institute, Moscow

Katanaev, Volovich Ann. Phys. 216(1992)1; ibid. 271(1999)203

Katanaev Theor.Math.Phys.135(2003)733; ibid. 138(2004)163

Physics – Uspekhi 48(2005)675.

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Notation

\mathbb{R}^3 - continuous elastic media = Euclidean three-dimensional space

$x^i, y^i \quad i = 1, 2, 3$ - Cartesian coordinates

δ_{ij} - Euclidean metric

$u^i(x)$ - displacement vector field

$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ - strain tensor

σ^{ij} - stress tensor

Elasticity theory of small deformations

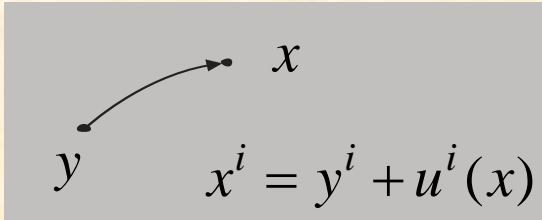
$\partial_i \sigma^{ij} + f^j = 0$ - Newton's law

$\sigma^{ij} = \lambda \delta^{ij} \varepsilon_k^k + 2\mu \varepsilon^{ij}$ - Hooke's law

$f^i(x)$ - density of nonelastic forces ($f^i = 0$)

λ, μ - Lamé coefficients

Differential geometry of elastic deformations



$$y^i \rightarrow x^i(y) \text{ - diffeomorphism: } \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$y^i \mapsto x^i$$

$$\delta_{ij} \quad g_{ij}$$

$$g_{ij}(x) = \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} \delta_{kl} \approx \delta_{ij} - \partial_i u_j - \partial_j u_i = \delta_{ij} - 2\varepsilon_{ij} \text{ - induced metric } (*)$$

$$\tilde{\Gamma}_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \neq 0 \text{ - Christoffel's symbols}$$

$$\tilde{R}_{ijk}{}^l = \partial_i \tilde{\Gamma}_{jk}{}^l - \tilde{\Gamma}_{ik}{}^m \tilde{\Gamma}_{jm}{}^l - (i \leftrightarrow j) = 0 \text{ - curvature tensor}$$

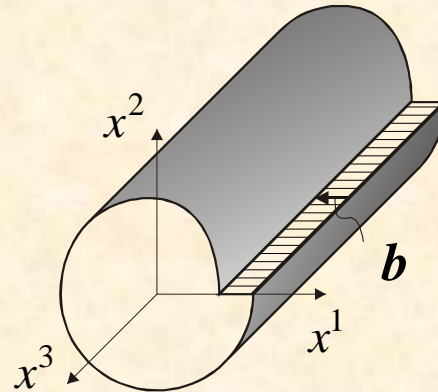
$$\ddot{x}^i = -\tilde{\Gamma}_{jk}{}^i \dot{x}^j \dot{x}^k \text{ - extremals (geodesics)}$$

$$R_{ijk}{}^l = 0 \text{ - Saint-Venant integrability conditions of } (*)$$

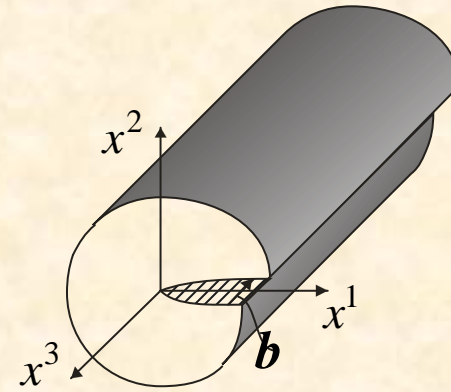
$$T_{ij}{}^k = \tilde{\Gamma}_{ij}{}^k - \tilde{\Gamma}_{ji}{}^k = 0 \text{ - torsion tensor}$$

Dislocations

Linear defects:



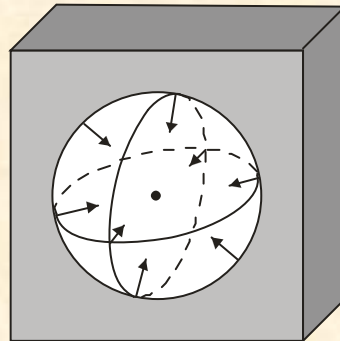
Edge dislocation



Screw dislocation

b - Burgers vector

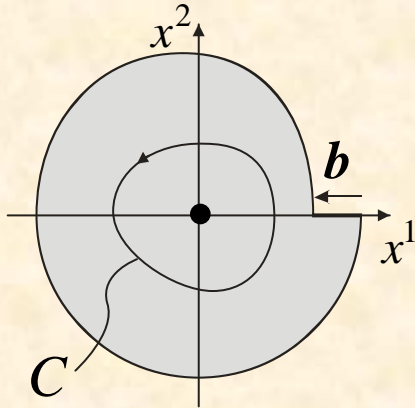
Point defects:



Vacancy

$u^i(x)$ $\left\{ \begin{array}{l} \text{is continuous} \quad = \text{elastic deformations} \\ \text{is not continuous} = \text{dislocations} \end{array} \right.$

Edge dislocation



$$\oint_C dx^\mu \partial_\mu u^i = -\oint_C dx^\mu \partial_\mu y^i = -b^i \quad (*)$$

$x^\mu, \mu = 1, 2, 3$ - arbitrary curvilinear coordinates

$y^i(x)$ - is not continuous !

$$e_\mu^i(x) = \begin{cases} \partial_\mu y^i & \text{- outside the cut} \\ \lim \partial_\mu y^i & \text{- on the cut} \end{cases}$$

- triad field
(continuous on the cut)

(*) $\Rightarrow b^i = \oint_C dx^\mu e_\mu^i = \iint_S dx^\mu \wedge dx^\nu (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i)$ - Burgers vector in elasticity

$$T_{\mu\nu}^i = \partial_\mu e_\nu^i - \omega_\mu^{ij} e_{\nu j} - (\mu \leftrightarrow \nu) \quad \text{- torsion}$$

$$R_{\mu\nu}^{ij} = \partial_\mu \omega_\nu^{ij} - \omega_\mu^{ik} \omega_{\nu k}^j - (\mu \leftrightarrow \nu) \quad \text{- curvature}$$

$$\omega_\mu^{ij} = -\omega_\mu^{ji}$$

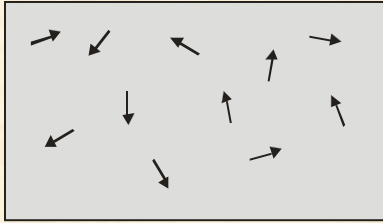
SO(3)-connection

$$b^i = \iint dx^\mu \wedge dx^\nu T_{\mu\nu}^i \quad \text{- definition of the Burgers vector in the geometric theory}$$

Back to elasticity: if $R_{\mu\nu}^{ij} = 0$ then $\omega_\mu^{ij} \rightarrow 0$

Disclinations

Ferromagnets



$n^i(x)$ - unit vector field

n_0^i - fixed unit vector

$$n^i = n_0^j S_j^i(\omega)$$

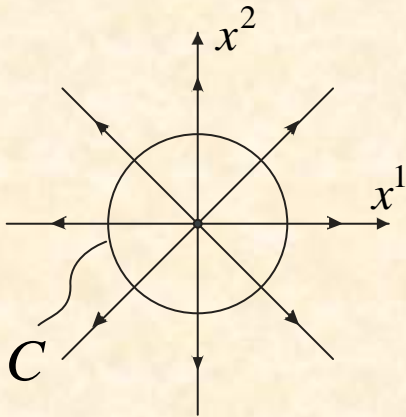
$S_i^j \in \mathbb{SO}(3)$ - orthogonal matrix

$\omega^{ij} = -\omega^{ji} \in \mathfrak{so}(3)$ - Lie algebra element (spin structure)

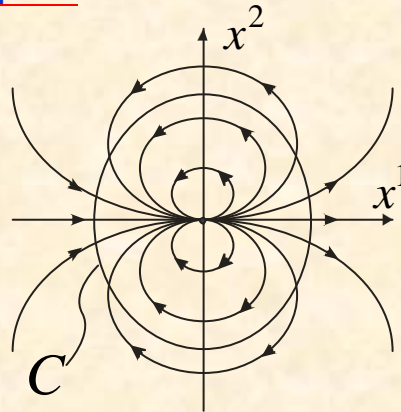
$$\omega_i = \frac{1}{2} \varepsilon_{ijk} \omega^{jk} \text{ - rotational angle}$$

ε_{ijk} - totally antisymmetric tensor ($\varepsilon_{123} = 1$)

Examples



$$\Theta = 2\pi$$



$$\Theta = 4\pi$$

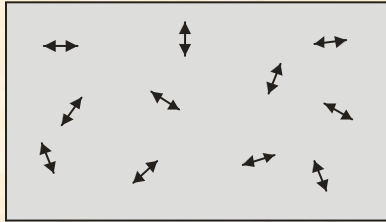
$$\Omega^{ij} = \oint_C dx^\mu \partial_\mu \omega^{ij}$$

$\Theta_i = \varepsilon_{ijk} \Omega^{jk}$ - Frank vector
(total angle of rotation)

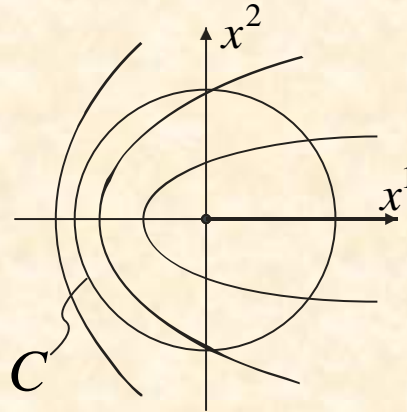
$$\Theta = \sqrt{\Theta^i \Theta_i}$$

More examples

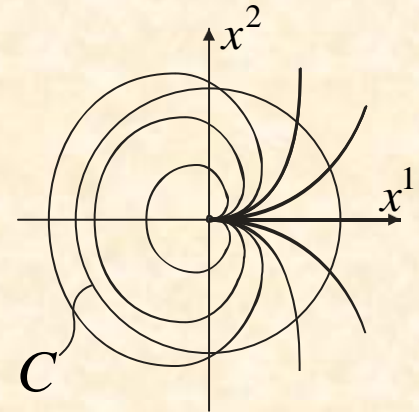
Nematic liquid crystals



$$n^i \sim -n^i$$



$$\Theta = \pi$$



$$\Theta = 3\pi$$

Model for a spin structure:

$\omega^i(x) \in \mathfrak{so}(3)$ - basic variable

$$S_i^j = \delta_i^j \cos \omega + \frac{\omega^k \varepsilon_{ki}^j}{\omega} \sin \omega + \frac{\omega_i \omega^j}{\omega^2} (1 - \cos \omega) \in \mathbb{S}\mathbb{O}(3), \quad \omega = \sqrt{\omega^i \omega_i}$$

$l_{\mu i}^j = (\partial_{\mu} S^{-1})_i^k S_k^j$ - trivial $\mathbb{S}\mathbb{O}(3)$ -connection (pure gauge)

$$\partial^{\mu} l_{\mu}^{ij} = 0$$

- principal chiral $\mathbb{S}\mathbb{O}(3)$ -model

Frank vector

$\omega^{ij}(x)$ - is not continuous !

$$\omega_{\mu}^{ij}(x) = \begin{cases} \partial_{\mu} \omega^{ij} & \text{- outside the cut} \\ \lim \partial_{\mu} \omega^{ij} & \text{- on the cut} \end{cases}$$

- SO(3)-connection
(continuous on the cut)

$$\Omega^{ij} = \oint dx^{\mu} \omega_{\mu}^{ij} = \iint dx^{\mu} \wedge dx^{\nu} (\partial_{\mu} \omega_{\nu}^{ij} - \partial_{\nu} \omega_{\mu}^{ij}) \quad \text{- the Frank vector}$$

$$R_{\mu\nu}^{ij} = \partial_{\mu} \omega_{\nu}^{ij} - \omega_{\mu}^{ik} \omega_{\nu k}^j - (\mu \leftrightarrow \nu) \quad \text{- curvature}$$

$$\Omega^{ij} = \iint dx^{\mu} \wedge dx^{\nu} R_{\mu\nu}^{ij}$$

- definition of the Frank vector
in the geometric theory

Back to the spin structure: if $n \in \mathbb{R}^2$ then $\text{SO}(3) \rightarrow \text{SO}(2)$

Summary of the geometric approach (physical interpretation)

Media with dislocations and disclinations =

= \mathbb{R}^3 with a given Riemann-Cartan geometry

Independent variables $\left\{ \begin{array}{l} e_{\mu}^i \text{ - triad field} \\ \omega_{\mu}^{ij} \text{ - SO(3)-connection} \end{array} \right.$

$T_{\mu\nu}^i = \partial_{\mu} e_{\nu}^i - \omega_{\mu}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$ - torsion (surface density of the Burgers vector)

$R_{\mu\nu}^{ij} = \partial_{\mu} \omega_{\nu}^{ij} - \omega_{\mu}^{ik} \omega_{\nu k}^j - (\mu \leftrightarrow \nu)$ - curvature (surface density of the Frank vector)

Elastic deformations: $R_{\mu\nu}^{ij} = 0, \quad T_{\mu\nu}^i = 0$

Dislocations: $R_{\mu\nu}^{ij} = 0, \quad T_{\mu\nu}^i \neq 0$

Disclinations: $R_{\mu\nu}^{ij} \neq 0, \quad T_{\mu\nu}^i = 0$

Dislocations and disclinations: $R_{\mu\nu}^{ij} \neq 0, \quad T_{\mu\nu}^i \neq 0$

Pontryagin characteristic class

4-dimensional manifold \mathbb{M} with coordinates x^α , $\alpha = 1, 2, 3, 4$

$\mathbb{P}(\mathbb{M}, \pi, \mathbb{G})$ - principal fiber bundle

$\omega = \omega^A L_A = dx^\alpha \omega_\alpha^A L_A$ - local connection form

L_A , $A = 1, \dots, \dim \mathbb{G}$ - generators of Lie algebra

$R^A := d\omega^A - \frac{1}{2} \omega^B \wedge \omega^C f_{BC}^A$ - curvature 2-form

$\mathfrak{g} \times \mathfrak{g} \ni X, Y \rightarrow X^A Y^B \eta_{AB} \in \mathbb{R}$ - invariant (degenerate) quadratic form in Lie algebra

$$\text{Theorem. } R^A \wedge R^B \eta_{AB} = d \left(d\omega^A \wedge \omega^B \eta_{AB} - \frac{1}{3} \omega^A \wedge \omega^B \wedge \omega^C f_{ABC} \right)$$

$\omega^A, \bar{\omega}^A$ - two connection forms

$$\text{Theorem. } R^A \wedge R^B \eta_{AB} - \bar{R}^A \wedge \bar{R}^B \eta_{AB} = dQ$$

$$I := \int_{\mathbb{M}} R^A \wedge R^B \eta_{AB}$$

- Pontryagin topological invariant

There is no metric on \mathbb{M}

Chern-Simons action

3-dimensional manifold \mathbb{M} with coordinates x^μ , $\mu = 1, 2, 3$

$$S_{\text{CS}} := \int_{\mathbb{M}} \left(d\omega^A \wedge \omega^B \eta_{AB} - \frac{2}{3} \omega^A \wedge \omega^B \wedge \omega^C f_{ABC} \right) \quad \text{- the Chern-Simons action}$$

Hilbert-Einstein action

Cartan variables: e_μ^i, ω_μ^{ij} - triad and $\mathbb{SO}(3)$ connection $i, j = 1, 2, 3$

$$\omega_\mu^{ij} = \omega_{\mu k} \varepsilon^{kij} \Leftrightarrow \omega_{\mu k} = \frac{1}{2} \omega_\mu^{ij} \varepsilon_{ijk} \quad \text{- parametrization of connection}$$

$$e^i := dx^\mu e_\mu^i, \quad \omega^i := dx^\mu \omega_\mu^i \quad \text{- 1-forms}$$

Riemann-Cartan geometry: $R^i = d\omega^i + \frac{1}{2} \varepsilon^i{}_{jk} \omega^j \wedge \omega^k$ - curvature

$$T^i = de^i + \varepsilon^i{}_{jk} e^j \wedge \omega^k \quad \text{- torsion}$$

$$S_{\text{HE}\Lambda} = \int \left(2R^i \wedge e^j \delta_{ij} - \frac{1}{6} \Lambda e^i \wedge e^j \wedge e^k \varepsilon_{ijk} \right) \quad \text{- the Hilbert-Einstein action with cosmological constant } \Lambda$$

$$\delta\omega_i^j = d\lambda_i^j + \lambda_i^k \omega_k^j - \omega_i^k \lambda_k^j \quad \text{- local rotations with parameter } \lambda_i^j(x)$$

$$\delta e^i = -e^j \lambda_j^i$$

Gauge fields for the $\text{ISO}(3)$ group

$\eta_{ab} := \text{diag}(+++ \nu)$, $\nu \geq 0$ - the invariant metric on group manifold

$$a, b = 1, 2, 3, 4$$

$$\mathbb{G} = \begin{cases} \text{SO}(4), & \nu = 1 \\ \text{ISO}(3), & \nu = 0 \end{cases} \quad \text{- the gauge group}$$

$$\{L_{ab}\} \mapsto \{L_{ij}, P_i := L_{i4}\} \quad \begin{array}{l} L_{ij} \quad \text{- rotations} \\ P_i \quad \text{- translations} \end{array}$$

$$[L_{ij}, L_{kl}] = -\delta_{ik} L_{jl} + \delta_{il} L_{jk} + \delta_{jk} L_{il} - \delta_{jl} L_{ik}$$

$$[L_{ij}, P_k] = -\delta_{ik} P_j - \delta_{jk} L_i \quad \text{- Lie algebra}$$

$$[P_i, P_j] = -\nu L_{ij}$$

$$\lambda^{ab} = \begin{pmatrix} \lambda^{ij} & \lambda^i \\ -\lambda^j & 0 \end{pmatrix}, \quad \lambda^{ij} = -\lambda^{ji}, \quad \lambda^i := \lambda^{i4} \quad \text{- parameters of local rotations and translations}$$

$$\omega_\mu^{ab} L_{ab} = \omega_\mu^{ij} L_{ij} + e_\mu^i P_i \quad \text{- } \mathbb{G} \text{ connection}$$

$$\{F_{\mu\nu}^{ab}\} = \left\{ F_{\mu\nu}^{ij} := R_{\mu\nu}^{ij} + \nu(e_\mu^i e_\nu^j - e_\nu^i e_\mu^j), \quad F_{\mu\nu}^{i4} := T_{\mu\nu}^i \right\} \quad \text{- } \mathbb{G} \text{ curvature}$$

Gauge fields for the $\text{ISO}(3)$ group

Elements of algebra $X = X^i L_i + X^m P_m$

Two invariant metrics: $\eta = \begin{pmatrix} 4\delta_{ij} & 0 \\ 0 & 4\nu\delta_{mn} \end{pmatrix}$ - the Killing-Cartan form

$\gamma = \begin{pmatrix} 0 & \delta_{in} \\ \delta_{jm} & 0 \end{pmatrix}$ - second invariant metric (breaks parity)

$$S_{\text{CS}}^\eta = \int \left(4R^i \wedge \omega_i + 4\nu T^i \wedge e_i - \frac{2}{3} \omega^i \wedge \omega^j \wedge \omega^k \varepsilon_{ijk} \right)$$

$$S_{\text{CS}}^\gamma = \int \left(\underbrace{R^i \wedge e_i + T^i \wedge \omega_i}_{\text{the Hilbert-Einstein action}} - \frac{1}{2} \omega^i \wedge \omega^j \wedge e^k \varepsilon_{ijk} + \frac{1}{3} \nu \underbrace{e^i \wedge e^j \wedge e^k \varepsilon_{ijk}}_{\text{cosmological constant}} \right) \quad \text{Witten (1989)}$$

$$\delta e^i = d\lambda^i + \varepsilon^i_{jk} \omega^j \lambda^k$$

- local translations $\lambda^i(x)$ (additional gauge transformations)

$$\delta \omega^i = \nu \varepsilon^i_{jk} e^j \lambda^k$$

The free energy

$$S = S_{\text{HE}}[e] + S_{\text{CS}}[\omega] \quad - \text{the total free energy}$$

No dislocations: $g_{\mu\nu} = \delta_{\mu\nu} \quad (e_{\mu}^i = \delta_{\mu}^i)$ - Euclidean metric $S_{\text{HE}} \rightarrow 0$

Three dimensions:

$$\omega_{\mu}^{ij} = \omega_{\mu k} \varepsilon^{kij}, \quad \omega_{\mu k} := \frac{1}{2} \omega_{\mu}^{ij} \varepsilon_{ijk}, \quad -\text{SO}(3) \text{ connection} \\ \text{(1-form, the only variable)}$$

$\mu, \nu, \dots = 1, 2, 3; \quad i, j, \dots = 1, 2, 3$ - indices

$$R_{\mu\nu k}(\omega) := R_{\mu\nu}^{ij} \varepsilon_{ijk} = 2 \left(\partial_{\mu} \omega_{\nu k} - \partial_{\nu} \omega_{\mu k} + \omega_{\mu}^i \omega_{\nu}^j \varepsilon_{ijk} \right) \quad - \text{curvature}$$

$$S_{\text{CS}} = \int_{\mathbb{R}^3} \left(\frac{1}{2} \omega^i \wedge d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^i \wedge \omega^j \wedge \omega^k - \omega_i \wedge J^i \right) \quad - \text{free energy for disclinations}$$

J^i - the source term (2-form)

$$R_{\mu\nu}^k = J_{\mu\nu}^k \quad - \text{equations of equilibrium}$$

One linear disclination

$q^\mu(t) \in \mathbb{R}^3$, $t \in \mathbb{R}$ - the core of disclination

$S_{\text{int}} = \int dq^\mu \omega_{\mu i} J^i = \int dt \dot{q}^\mu \omega_{\mu i} J^i =$ - the interaction term

$$= \int dt d^3x \dot{q}^\mu \omega_{\mu i} J^i \delta^3(x - q) = \int d^3x \frac{\dot{q}^\mu}{\dot{q}^3} \omega_{\mu i} J^i \delta^2(\mathbf{x} - \mathbf{q})$$

$$\frac{\delta S_{\text{int}}}{\delta \omega_{\mu i}} = \frac{\dot{q}^\mu}{\dot{q}^3} J^i \delta^2(\mathbf{x} - \mathbf{q})$$

- the source term $\mathbf{x} = (x^1, x^2)$

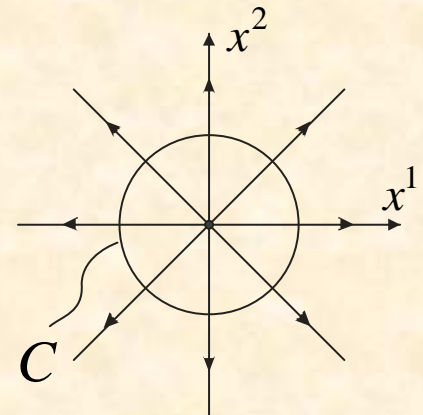
One disclination along x^3 axis

Notation: $x^1 = x$, $x^2 = y$, $z := x + iy$

ω_x^3, ω_y^3 - the only nontrivial components

$\omega_z^3 = \frac{1}{2} \omega_x^3 - \frac{i}{2} \omega_y^3$ - one complex component

$R_{z\bar{z}}^3 = 2 \left(\partial_z \omega_{\bar{z}}^3 - \partial_{\bar{z}} \omega_z^3 \right)$ - the curvature tensor



One straight linear disclination

Fixing the source term:

$$R_{z\bar{z}}^3 = 4\pi i A \delta(z), \quad A \in \mathbb{R} \quad - \text{new kind of defect}$$

The solution: $\omega_z^3 = -\frac{iA}{z}$ $\partial_z \frac{1}{\bar{z}} = \pi \delta(z)$ - important formula

$$\omega_x^3 = -\frac{2Ay}{x^2 + y^2}, \quad \omega_y^3 = \frac{2Ax}{x^2 + y^2} \quad - \text{real components}$$

Rotational angle field $\omega(\mathbf{x})$

$$\partial_x \omega = -\frac{2Ay}{x^2 + y^2}, \quad \partial_y \omega = \frac{2Ax}{x^2 + y^2}$$

The integrability conditions $\partial_{xy} \omega = \partial_{yx} \omega$ are fulfilled

$$\omega = -2A \arctan \frac{x}{y} + C \quad - \text{a general solution}$$

$$C = \pi A \quad \Rightarrow \quad \tan \varphi = \frac{x}{y}, \quad \varphi := \frac{\omega}{2A} \quad - \text{polar angle}$$

One straight linear disclination

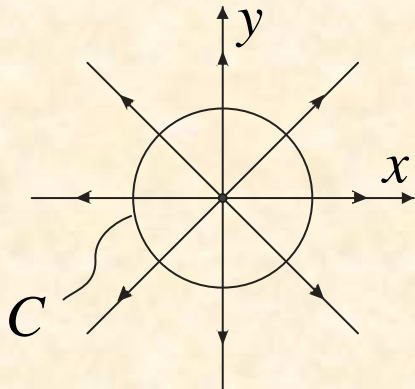
To make the rotational angle field $\omega(x, y)$ well defined, we must impose the quantization condition:

$$A = \frac{n}{2}, \quad n \in \mathbb{Z} \quad \Rightarrow \quad \omega = n\varphi$$

SO(3)-connection:

$$\omega_x^{12} = -\frac{ny}{x^2 + y^2} = -\frac{n}{r} \sin\varphi,$$

$$\omega_y^{12} = \frac{nx}{x^2 + y^2} = \frac{n}{r} \cos\varphi.$$



Conclusion

- 1) The first example of disclination is described within the geometric theory of defects.
- 2) The Chern-Simons term is well suited for disclinations in the geometric theory of defects.
- 3) Linear disclinations correspond to a new type of geometric singularity of the $SO(3)$ connection.