# Holographic $SL(2,\mathbb{C})$-Toda RG flow

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based on work with
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(a) JINR BLTP

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# Outline

1. Introduction
   - Some facts about holographic RG-flow
2. Holographic $SL(2,\mathbb{C})$-Toda RG flow
   - Set up
   - How to integrate
   - Holographic $SL(2,\mathbb{C})$-Toda RG flow
   - Non-vacuum case
3. Outlook

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## The Domain Wall/QFT correspondence

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- $AdS$ + linear dilaton $\Leftrightarrow$ Domain Wall solution Lu et al'96

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$AdS \Leftrightarrow DW$, CFT$\Leftrightarrow$ QFT; AdS isometry group $\Leftrightarrow$Poincare isometry group Boonstra et. al.'98;Skenderis'99

## The Einstein-dilaton gravity

$$S = M_p^{d-1}\int d^dx\int dr\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right] + S_{YH}.$$

- Only one scalar $\leftrightarrow$ focus on a single operator $\mathcal{O}$ in the field theory.
- $V(\phi)$ encodes the dimension of the operator and the way the coupling runs.
- $V(\phi) < 0$

# Bulk field/ Operator correspondence

- An operator $\mathcal{O}(x) \Leftrightarrow \Phi(x, r)$.
- $\Phi(x, 0) \Leftrightarrow$ a source for $\mathcal{O}$ in CFT

$$\Phi(x,r) \sim \alpha(x) r^{(d-\Delta)} + \ldots r \to 0 \quad \Leftrightarrow \quad S_{CFT} = S_0 + \int d^4 x \alpha(x) \mathcal{O}(x)$$

- The on-shell action (BC fixed by $\alpha(x)$) $\Leftrightarrow$ the QFT generating func.

$$S[\Phi_\alpha(x,r)] = \text{functional of } \alpha(x), \quad \mathcal{Z}_{QFT}[\alpha(x)] = \exp iS[\Phi_\alpha(x,r)]$$

- $\Phi(x, r)$ – the scale-dep. coupling, renormalized at a scale $\mu \equiv \mu(r)$
- $\alpha(x)$ represents the bare UV coupling

$$\mu = 1/r, \quad \Phi(\mu) = \alpha \mu^{\Delta - d}$$

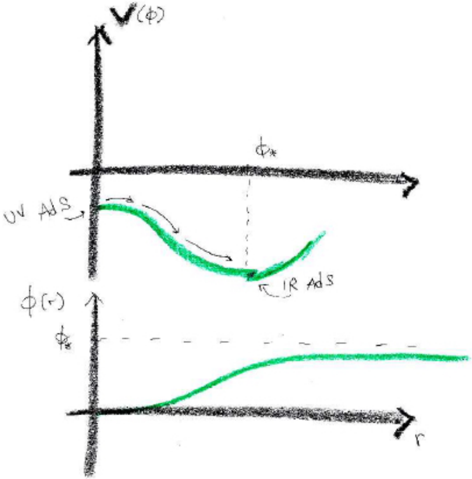



Figure: Pic. by Francesco Nitti

# Improved holographic QCD Gursoy,Kiritsis'07 Gubser'08

## Einstein-dilaton gravity

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left(R - \frac{4}{3}(\partial\phi)^2 + V(\phi)\right) - \frac{1}{\kappa^2}\int_\partial d^4x\sqrt{-\gamma}.$$

- $g_{\mu\nu}$ is dual to $T_{\mu\nu}$;
- $\phi$ is dual to $tr[F^2]$.

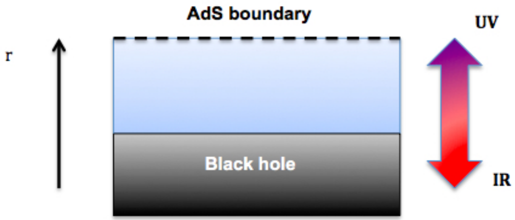
The potential can be tuned to reproduce $\beta$-function.

| | | | | |
|---|---|---|---|---|
| For asymptotically AdS UV | $\lambda \to 0$ | $V(\lambda)$ | $=$ | $V_0 + v_1\lambda + v_2\lambda^2 + \dots$ |
| For confinement in the IR | $\lambda \to \infty$ | $V(\lambda)$ | $\sim$ | $\lambda^Q(\log\lambda)^P$ |

Confinement $\Leftrightarrow$ finite-$T$ transition between thermal gas and Black Hole

## Thermal gas solution

$$ds^2 = e^{2A(r)}\eta_{ij}dx^i dx^j + dr^2, \quad \phi = \phi(r).$$




## The Chamblin-Reall black hole

$$ds^2 = e^{2A(r)}\left(-f(r)dt^2 + \delta_{ij}dx^i dx^j\right) + \frac{dr^2}{f(r)}, \quad f(r) = 1 - C_2\lambda^{-\frac{4(1-X^2)}{3X}}.$$

# Outline

# Set up

The action reads

$$S = \frac{1}{2\kappa^2}\int d^4x \int du\sqrt{-g}\left(R - \frac{4}{3}(\partial\phi)^2 + V(\phi)\right) - \frac{1}{\kappa^2}\int_{\partial} d^4x\sqrt{-\gamma},$$

the potential $V < 0$ – $AdS$ fixed points; $V(\phi) = C_1 e^{2k_1\phi} + C_2 e^{2k_2\phi}$, with $C_i$, $k_i$ with $i = 1, 2$ are some constants.

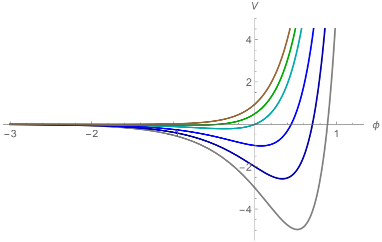



Figure: The behaviour of the potential $V(\phi)$ for $C_1 < 0$, $C_2 > 0$.

## The ansatz for the metric

$$ds^2 = -e^{2A(u)}dt^2 + e^{2B(u)}\sum_{i=1}^{3} dy_i^2 + e^{2C(u)}du^2,$$

$A$, $B$, $C$ – log's of scale factors, smooth functions.

The gauge $C = A + 3B$.

## The dynamical system $x^1 = A$, $x^2 = B$, $x^3 = \phi$, $x = C$

$$L = \frac{1}{2}G_{MN}\dot{x}^M\dot{x}^N - V, \quad V = -\frac{1}{2}\sum_{s=1}^{2} C_s e^{2(x^1+3x^2+k_s x^3)}, \dot{} \equiv \frac{d}{du}.$$

$$(G_{MN}) = \begin{pmatrix} 0 & -3 & 0 \\ -3 & -6 & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}, (G^{MN}) = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{3}{4} \end{pmatrix}, \quad i,j = 1,2,3.$$

$(G_{MN})$ – minisuperspace metric on the target space $\mathcal{M}$

$$L = \frac{1}{2}\langle \dot{x}, \dot{x}\rangle - \tilde{C}_1 e^{\langle V, x\rangle} - \tilde{C}_2 e^{\langle W, x\rangle}, \quad \tilde{C}_1 = -\frac{1}{2}C_1, \tilde{C}_2 = -\frac{1}{2}C_2.$$

$V$ and $W$ are some vectors on $\mathcal{M}$ (the original basis is $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$)

$$(V^i) = (-\frac{2}{3}, -\frac{2}{3}, \frac{3}{2}k_1), \quad (W^i) = (-\frac{2}{3}, -\frac{2}{3}, \frac{3}{2}k_2)$$

$\langle V, V\rangle = 3\left(k_1^2 - \frac{16}{9}\right)$, $\langle W, W\rangle = 3\left(k_2^2 - \frac{16}{9}\right)$, $\langle V, W\rangle = 3\left(k_1 k_2 - \frac{16}{9}\right)$.

LET $\langle V, W\rangle = 0 \Leftrightarrow k_1 k_2 = \frac{16}{9}$.

The new basis

$$\left\langle e_i^{'}, e_j^{'}\right\rangle = \eta_{ij}, \quad (\eta_{ij}) = diag\,(-1, 1, 1)\,, \quad e_1^{'} = \frac{V}{||V||}, \quad e_2^{'} = \frac{W}{||W||}.$$

$$X^i = \eta_{ii}\left\langle e_i^{'}, x\right\rangle, \quad x^i = \sum_{j=1}^{3} S_j^i X^j, \quad e_j^{'} = \sum_{i=1}^{3} S_j^i e_i.$$

$S_j^i$ – components of general Lorentz transformations.

## The $A_1 \times A_1$-mechanical model

$$L = \frac{1}{2}\sum_{i,j=1}^{3} \eta_{ij}\dot{X}^i\dot{X}^j - \tilde{C}_1 e^{\eta_{11}|\langle V,V\rangle|^{1/2}X^1} - \tilde{C}_2 e^{\eta_{22}|\langle W,W\rangle|^{1/2}X^2},$$

$$E_0 = \frac{1}{2}\sum_{i,j=1}^{3} \eta_{ij}\dot{X}^i\dot{X}^j + \tilde{C}_1 e^{\eta_{11}|\langle V,V\rangle|^{1/2}X^1} + \tilde{C}_2 e^{\eta_{22}|\langle W,W\rangle|^{1/2}X^2}.$$

## Liouville equations or $SL(2,\mathbb{C})$-Toda chains

$$\begin{aligned}\ddot{X}^s &= -\sqrt{|\langle R_s,R_s\rangle|}\tilde{C}_s e^{\eta_{ss}|\langle R_s,R_s\rangle|^{1/2}X^s}, \quad s=1,2,\\ \ddot{X}^3 &= 0, \quad \text{with} \quad \langle R_1,R_1\rangle = \langle V,V\rangle, \quad \langle R_2,R_2\rangle = \langle W,W\rangle.\end{aligned}$$

Gavrilov, Ivashchuk, Melnikov'94

# The solution to mechanical model

The solution reads

$$\begin{aligned} X^s &= -\eta_{ss}|\left\langle R_s, R_s\right\rangle|^{-1/2}\ln\left(F_s^2(u-u_{0s})\right), \quad s=1,2 \\ X^3 &= p^3 u + q^3, \end{aligned}$$

with

$$F_s(u-u_{0s}) = \begin{cases} \sqrt{|\frac{\tilde{C}_s}{E_s}|}\cosh\left[\sqrt{\frac{|E_s\langle R_s,R_s\rangle|}{2}}(u-u_{0s})\right], & \eta_{ss}\tilde{C}_s>0, \eta_{ss}E_s>0, \\ \sqrt{|\frac{\tilde{C}_s|}{E_s}|}\sin\left[\sqrt{\frac{|E_s\langle R_s,R_s\rangle|}{2}}(u-u_{0s})\right], & \eta_{ss}\tilde{C}_s<0, \eta_{ss}E_s<0, \\ \sqrt{|\frac{\tilde{C}_s}{E_s}|}\sinh\left[\sqrt{\frac{|E_s\langle R_s,R_s\rangle|}{2}}(u-u_{0s})\right], & \eta_{ss}\tilde{C}_s<0, \eta_{ss}E_s>0, \\ \sqrt{\frac{|\langle R_s,R_s\rangle\tilde{C}_s|}{2}}(u-u_{0s}), & \eta_{ss}\tilde{C}_s<0, E_s=0, \end{cases}$$

$u_{0s}$, $E_s$, $E_s$, $p^3$, $q^3$ are constants of integration. The component of transformations are

# Holographic $SL(2,\mathbb{C})$-Toda RG flow

$$C_1 < 0, \quad C_2 > 0, \quad 0 < k < 4/3.$$

$S_1^i = \frac{V^i}{|\langle V,V\rangle|^{1/2}}$, $S_2^i = \frac{W^i}{\langle W,W\rangle^{1/2}}$

$$\alpha^i = S_3^i p^3, \quad \beta^i = S_3^i q^3$$

1. $\alpha^1 = \alpha^2 = \alpha^3 = 0$ –Vacuum solutions
2. $\alpha^1 = -3\alpha^2$, $\alpha^3 = 0$. – Non-zero temperature solutions

# Holographic RG-flow. Vacuum case

## The metric and the dilaton

$$ds^2 = -F_1^{\frac{8}{9k^2-16}} F_2^{\frac{9k^2}{2(16-9k^2)}} dt^2 + F_1^{\frac{8}{9k^2-16}} F_2^{\frac{9k^2}{2(16-9k^2)}} \sum_{i=1}^{3} dy_i^2 + F_1^{\frac{32}{9k^2-16}} F_2^{\frac{18k^2}{16-9k^2}} du^2.$$

$$\phi = -\frac{9k}{9k^2-16} \ln F_1 + \frac{9k}{9k^2-16} \ln F_2,$$

## The energy condition

$$E_1 + E_2 = 0$$

The functions $F_1$ and $F_2$ are given by

$$F_1 = \pm\sqrt{\left|\frac{\tilde{C}_1}{E_1}\right|} \sinh(\mu_1 u),\ \mu_1 = \sqrt{\left|\frac{3E_1}{2}(k^2 - \frac{16}{9})\right|}\ for\ E_1 < 0,$$

$$F_2 = \pm\sqrt{\left|\frac{\tilde{C}_2}{E_2}\right|} \sinh(\mu_2 u),\ \mu_2 = \sqrt{\left|\frac{3E_2}{2}\left(\left(\frac{16}{9}\right)^2 \frac{1}{k^2} - \frac{16}{9}\right)\right|}\ for\ E_2 > 0.$$

$$E_1 = -E_2.$$

# Check the fixed-point

$$\phi' = 0, \quad \Rightarrow \mu_1 \frac{\cosh(\mu_1 u_0)}{\sinh(\mu_1 u_0)} = \mu_2 \frac{\cosh(\mu_2 u_0)}{\sinh(\mu_2 u_0)} \Rightarrow \frac{\tanh(\mu_1 u_0)}{\tanh(\mu_2 u_0)} = \frac{\mu_1}{\mu_2}.$$

This equation has a solution $u_0 = 0$. So, with $\frac{\mu_1}{\mu_2} = \frac{3k}{4}$

$$\phi|_{u_0=0} = \frac{9k}{16-9k^2} \log \frac{3k}{4} + \frac{1}{2} \frac{9k}{9k^2-16} \log \left| \frac{C_2}{C_1} \right| . \quad (1)$$

Let $C_1 C_2 < 0$, then $V'_\phi = C_1 2k_1 e^{2k_1\phi} + 2k_2 C_2 e^{2k_2\phi}$, and at the critical point one obtains

$$C_1 2k_1 e^{2k_1\phi} + 2k_2 C_2 e^{2k_2\phi} = 0 \Rightarrow \frac{C_1 k_1}{k_2 C_2} = -e^{2(k_2-k_1)\phi}.$$

For $k_1 = k$, $k_2 = \frac{16}{9k}$ we get $\frac{9C_1k^2}{16C_2} = -e^{2(\frac{16}{9k}-k)\phi_c}$.

$$\phi_c = \frac{9k}{(16-9k^2)} \log \frac{3k}{4} + \frac{9k}{2(16-9k^2)} \log \left| \frac{C_1}{C_2} \right| . \quad (2)$$

$$\phi_c = \phi(u_0 = 0)$$

# AdS-fixed point

One can represent the metric in the following form

$$ds^2 = M_1\left(-dt^2 + dy_1^2 + dy_2^2 + dy_3^2\right) + M_3 du^2,$$

where

$$M_1 = F_1^{\frac{8}{9k^2-16}} F_2^{\frac{9k^2}{2(16-9k^2)}}, \quad M_3 = F_1^{\frac{32}{9k^2-16}} F_2^{\frac{18k^2}{16-9k^2}}$$

Let us consider $u \to 0$, then

$$M_1 \sim u^{-1/2}, \quad M_3 \sim u^{-2}.$$

$$v = \int \sqrt{\frac{M_3}{M_1}} du = \int u^{-3/4} du = -4u^{1/4}.$$

Then we have

$$M_1 \sim v^{-1/2}, \quad M_3 \sim v^{-8}, \quad du^2 = v^6 dv^2.$$

For $\alpha_1 = \alpha_2 = 0$ we obtain the $AdS$-spacetime

$$ds^2 \sim v^{-2}(-dt^2 + dy_1^2 + dy_2^2 + dy_3^2 + dv^2)$$

# Geodesics

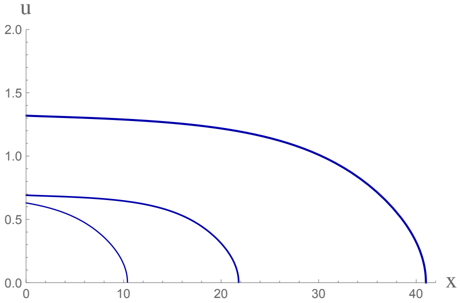



Figure: Space-like geodesics for $C_1 = -1, C_2 = 1$, $E_1 = 0.5 = -E_2$ and different $k = 0.3$(bottom), $k = 1$ (middle), $k = 4/3 - 0.1$ (top), $J = 0.1$.

# Non-vacuum case

$\alpha_1 = -3\alpha_2$, $\alpha_3 = 0$.

The metric reads

$$ds^2 = -F_1^{\frac{8}{9k^2-16}} F_2^{\frac{9k^2}{2(16-9k^2)}} e^{2\alpha^1 u} dt^2 + F_1^{\frac{8}{9k^2-16}} F_2^{\frac{9k^2}{2(16-9k^2)}} e^{2\alpha^2 u} \sum_{i=1}^{3} dy_i^2 + F_1^{\frac{32}{9k^2-16}} F_2^{\frac{18k^2}{16-9k^2}} du^2. \quad (3)$$

The dilaton reads

$$\phi = -\frac{9k}{9k^2-16} \ln F_1 + \frac{9k}{9k^2-16} \ln F_2,$$

where $F_1$ and $F_2$ are the same as in the vacuum case, but

$$|E_1| \neq |E_2|$$

.

$$\mu_1 = \sqrt{|\frac{3E_1}{2}|}\sqrt{\frac{16}{9} - k^2}, \quad \mu_2 = \sqrt{|\frac{3E_2}{2}|}\,\frac{4}{3k}\sqrt{\frac{16}{9} - k^2} = \frac{4}{3k}\sqrt{\frac{E_2}{E_1}}\mu_1.$$

# Attempt to a black brane

$$ds^2 = \Phi\left(-f dt^2 + d\vec{x}^2 + \frac{dr^2}{f}\right), \quad f(r_0) = 0.$$

We have $e^{2A} = \Phi f$, $e^{2B} = \Phi$, $e^{2C}\left(\frac{du}{dr}\right)^2 = \Phi f^{-1}$.

$$\frac{du}{dr} = e^{-2C}\left(\Phi f^{-1}\right)^{1/2} \Rightarrow r = \int M_1^{3/2} e^{\frac{5}{3}\alpha^1 u} du.$$

For $u \to 0$

$$r = \Gamma\left(\frac{1}{4}\right) - 4\left(-\frac{5\alpha_1}{3}u\right)^{1/4}.$$

Let us take $\alpha_1 = -3/5$ and $r - r_0 = 4u^{1/4}$.

$$f = e^{-\frac{8}{5}u} = e^{-\frac{8}{5}(\frac{1}{4}(r-r_0))^4} \approx 1 - \frac{8}{5}(\frac{1}{4}(r-r_0))^4.$$

The black brane metric takes the form

$$ds^2 = \Phi\left(-(1 - \frac{8}{5}(\frac{1}{4}(r-r_0))^4)dt^2 + d\vec{x}^2 + \frac{dr^2}{1 - \frac{8}{5}(\frac{1}{4}(r-r_0))^4}\right),$$

with $\Phi$ given by $\Phi = u^{-1/2}e^{2/5u} \sim u^{-1/2}\left(1 + \frac{2}{5}u\right)$.

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## TO DO!!

- Behavior of the superpotential $W$ for both cases
- Apply solution to $T \neq 0$ QCD (non-conformal plasmas). Policastro'15
- Trace anomaly of the stress energy tensor.

# Thank you for attention!