$D(2,1;\alpha)$ superconformal symmetry and its applications

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Motivations

The exceptional supergroup $D(2,1;\alpha)$ describes the most general $N=4$ supersymmetric extension of the conformal group in one dimension $SO(2,1)$. It is of interest for several reasons:

- $d=1$, $N=4$ supersymmetry possesses some peculiar features which are not present in higher dimensions. In particular, some $d=1$, $N=4$ supermultiplets cannot be obtained by dimensional reduction from $d>1$. Classification of $d=1$, $N=4$ supermultiplets and their interactions is important from the standpoint of the formal development of supersymmetry in divers dimensions. $N=4$ is believed to the maximum value for which the construction of interacting multi–particle systems is feasible.
- It was conjectured that the study of many–body superconformal models in d=1 might provide a new insight into a microscopic description of extreme black holes. This proposal stimulated extensive recent studies of $N=4$ superconformal Calogero–type models. An interesting link to the Witten-Dijkgraaf-Verlinde-Verlinde equation

$$\partial_i\partial_k\partial_p F(x)\ \partial_j\partial_l\partial_p F(x) = \partial_j\partial_k\partial_p F(x)\ \partial_i\partial_l\partial_p F(x)$$

was revealed and explored.

Motivations

- Some superconformal models in one dimension can be linked to superparticles propagating on near horizon black hole backgrounds. In particular, the latter can be analyzed in terms of $d = 1$, $N = 4$ superconformal partners. The two descriptions are known as the AdS and conformal bases.
- Dynamical system associated with angular sector of a conformal mechanics is in general (super)integrable. Models with $D(2, 1; \alpha)$ superconformal symmetry are of interest in that context as well.
- Possible quantum mechanical applications of $D(2, 1; \alpha)$: new exactly solvable models, novel correlations, etc.
- Higher dimensional generalizations invoke non–relativistic conformal superalgebras and are of interest in the context of the non–relativistic AdS/CFT–correspondence.

Plan

1. Couplings in $D(2,1;\alpha)$ superconformal mechanics from the $SU(2)$ perspective
2. $D(2,1;\alpha)$ superparticles on near horizon black hole backgrounds. A link of the group parameter α to the cosmological constant Λ
3. $N=4$ supersymmetric extensions of the l–conformal Galilei algebra based upon $D(2,1;\alpha)$

The even part of the Lie superalgebra associated with $D(2,1;\alpha)$ includes the conformal algebra in one dimension $so(2,1)$

$$[H,D]=H, \quad [H,K]=2D, \quad [D,K]=K$$

where H, D, K generate time translations, dilatations and special conformal transformations, respectively, and two $su(2)$–subalgebras $\mathcal{J}_a$ and (I_+,I_-,I_3)

$$[\mathcal{J}_a,\mathcal{J}_b]=\epsilon_{abc}\mathcal{J}_c, \quad [I_-,I_3]=-iI_-, \quad [I_+,I_3]=iI_+, \quad [I_-,I_+]=2iI_3$$

The odd part involves four real supersymmetry charges $(Q_\alpha,\bar{Q}^\alpha)$, $\alpha=1,2$, and four real superconformal partners $(S_\alpha,\bar{S}^\alpha)$

$$\{Q_\alpha,\bar{Q}^\beta\}=-2iH{\delta_\alpha}^\beta, \qquad \{S_\alpha,\bar{S}^\beta\}=-2iK{\delta_\alpha}^\beta$$

$$\{Q_\alpha,\bar{S}^\beta\}=-2\alpha{(\sigma_a)_\alpha}^\beta\mathcal{J}_a+2iD{\delta_\alpha}^\beta+2(1+\alpha)I_3{\delta_\alpha}^\beta$$

$$\{Q_\alpha,S_\beta\}=2i(1+\alpha)\epsilon_{\alpha\beta}I_-, \qquad \{\bar{Q}^\alpha,\bar{S}^\beta\}=-2i(1+\alpha)\epsilon^{\alpha\beta}I_+$$

where α is an arbitrary real number and ${(\sigma_a)_\alpha}^\beta$ are the Pauli matrices.

At $\alpha = -1$ the $su(2)$–subalgebra generated by (I_+, I_-, I_3) decouples and one ends up with $su(1,1|2)$.

$d = 1$, $N = 4$ supermultiplets contain four real fermions, the number of physical bosons varying from 0 to 4

$$(0,4,4), \qquad (1,4,3), \qquad (2,4,2), \qquad (3,4,1), \qquad (4,4,0)$$

Some of these multiplets can be nicely associated with particular realizations of the $su(2)$–subalgebra generated by $\mathcal{J}_a$.

Consider a phase space parametrized by

$$\text{Bosons:} \qquad \{x,p\} = 1, \qquad \{\theta^A, p_{\theta B}\} = \delta_B{}^A, \quad A,B = 1,\ldots,n$$
$$\text{Fermions:} \qquad \{\psi_\alpha, \bar{\psi}^\beta\} = -i\delta_\alpha{}^\beta, \quad (\psi_\alpha)^* = \bar{\psi}^\alpha, \quad \alpha = 1,2$$

and assume that $J_a = J_a(\theta, p_\theta)$, $a = 1,2,3$ obey the structure relations of $su(2)$

$$\{J_a, J_b\} = \epsilon_{abc} J_c$$

Then any realization of $su(2)$ in terms of J_a can be extended to a dynamical realization of $D(2,1;\alpha)$

$$H=\frac{p^2}{2}+\frac{2\alpha^2}{x^2}J_aJ_a+\frac{2\alpha}{x^2}(\bar\psi\sigma_a\psi)J_a-\frac{(1+2\alpha)}{4x^2}\psi^2\bar\psi^2, \quad D=tH-\frac{1}{2}xp$$

$$K=t^2H-txp+\frac{1}{2}x^2, \qquad \mathcal{J}_a=J_a+\frac{1}{2}(\bar\psi\sigma_a\psi)$$

$$Q_\alpha=p\psi_\alpha-\frac{2i\alpha}{x}(\sigma_a\psi)_\alpha J_a-\frac{i(1+2\alpha)}{2x}\bar\psi_\alpha\psi^2\ , \qquad S_\alpha=x\psi_\alpha-tQ_\alpha$$

$$\bar Q^\alpha=p\bar\psi^\alpha+\frac{2i\alpha}{x}(\bar\psi\sigma_a)^\alpha J_a-\frac{i(1+2\alpha)}{2x}\psi^\alpha\bar\psi^2, \qquad \bar S^\alpha=x\bar\psi^\alpha-t\bar Q^\alpha$$

$$I_-=\frac{i}{2}\psi^2, \qquad I_+=-\frac{i}{2}\bar\psi^2, \qquad I_3=\frac{1}{2}\bar\psi\psi$$

- (1,4,3)–supermultiplet (the angular sector is empty)

$$J_a = 0$$

- (3,4,1)–supermultiplet (the angular sector: (Θ, p_Θ), (Φ, p_Φ))

$$J_1 = -p_\Phi \cot\Theta \cos\Phi - p_\Theta \sin\Phi + e\cos\Phi \sin^{-1}\Theta$$
$$J_2 = -p_\Phi \cot\Theta \sin\Phi + p_\Theta \cos\Phi + e\sin\Phi \sin^{-1}\Theta$$
$$J_3 = p_\Phi, \quad J_a J_a = p_\Theta^2 + (p_\Phi - e\cos\Theta)^2 \sin^{-2}\Theta + e^2$$

where e is a magnetic charge

- (4,4,0)–supermultiplet (the angular sector: (Θ, p_Θ), (Φ, p_Φ), (Ψ, p_Ψ))

$$J_1 = -p_\Phi \cot\Theta \cos\Phi - p_\Theta \sin\Phi + p_\Psi \cos\Phi \sin^{-1}\Theta$$
$$J_2 = -p_\Phi \cot\Theta \sin\Phi + p_\Theta \cos\Phi + p_\Psi \sin\Phi \sin^{-1}\Theta$$
$$J_3 = p_\Phi, \quad J_a J_a = p_\Theta^2 + (p_\Phi - p_\Psi\cos\Theta)^2 \sin^{-2}\Theta + p_\Psi^2$$

- (1,4,3) coupled to (0,4,4) ($\{\chi_\alpha, \bar\chi^\beta\} = -i\delta_\alpha{}^\beta$)

$$J_a = \frac{1}{2}(\bar\chi \sigma_a \chi)$$

Part I: Couplings of $D(2,1;\alpha)$ multiplets from the $SU(2)$ perspective

Consider a set of canonical pairs which involves bosons and fermions obeying the brackets $(i = 1, \ldots, M+1,\ A = 1, \ldots, N,\ \alpha = 1, 2)$

$$\{x^i, p^j\} = \delta^{ij}, \qquad \{\psi^i_\alpha, \bar\psi^{j\beta}\} = -i\delta^{ij}\delta_\alpha{}^\beta, \qquad \{\chi^A_\alpha, \bar\chi^{B\beta}\} = -i\delta^{AB}\delta_\alpha{}^\beta$$

A realization of $D(2,1;\alpha)$ involves two prepotentials $V = V(x)$ and $F = F(x)$

$$H = \frac{1}{2}p^i p^i + \frac{1}{2}\partial^i V \partial^i V \tilde J_a \tilde J_a + \partial^i \partial^j V \tilde J_a (\bar\psi^i \sigma_a \psi^j) - \frac{1}{2}\partial^i W^{jkl}(\psi^i\psi^j)(\bar\psi^k\bar\psi^l)$$

$$Q_\alpha = p^i \psi^i_\alpha + i\partial^i V (\sigma_a \psi^i)_\alpha \tilde J_a + iW^{ijk}\bar\psi^i_\alpha(\psi^j\psi^k)$$

$$\tilde J_a = J_a + \frac{1}{2}(\bar\chi^A \sigma_a \chi^A)$$

where $W^{ijk} = \partial^i\partial^j\partial^k F(x)$.

Constraints on the prepotentials (S. Krivonos, O. Lechtenfeld, 2011)

$$\partial^i\partial^j V + \partial^i V \partial^j V - 2W^{ijk}\partial^k V = 0, \qquad W^{ijk}W^{klm} = W^{mjk}W^{kli}$$

$$x^i \partial^i V = -2\alpha, \qquad x^i W^{ijk} = -\frac{1}{2}(1+2\alpha)\delta^{jk}$$

Part I: Couplings of $D(2,1;\alpha)$ multiplets from the $SU(2)$ perspective

Given a particular solution to the master equations, the resulting model describes an interaction of M supermultiplets of the type $(1,4,3)$ with N $(0,4,4)$–supermultiplets and a single supermultiplet of either the type $(3,4,1)$, or $(4,4,0)$. Alternatively, one can regard this system as describing a coupling of $M+1$ copies of either $(3,4,1)$–, or $(4,4,0)$–supermultiplet, in which angular degrees of freedom are identified, to N supermultiplets of the type $(0,4,4)$.

Summary of Part I: Couplings in $D(2,1;\alpha)$–superconformal mechanics can be constructed by purely algebraic means in which $SU(2)$–subgroup plays a distinguished role.

Part II: A link to near horizon black hole geometries

Near horizon geometries of extreme black holes exhibit $so(2,1)$-symmetry. Massive (super)particles propagating on such backgrounds are automatically conformal invariant

$$H = r\left(\sqrt{(rp_r)^2 + L(\mu, p_\mu, p_\phi)} - q(p_\phi)\right), \quad D = tH + rp_r$$

$$K = \frac{1}{r}\left(\sqrt{(rp_r)^2 + L(\mu, p_\mu, p_\phi)} + q(p_\phi)\right) + t^2 H + 2trp_r$$

$$\{H, D\} = H, \qquad \{H, K\} = 2D, \qquad \{D, K\} = K$$

where (μ, p_μ) and (ϕ, p_ϕ) are latitudinal and azimuthal angular canonical pairs. $L(\mu, p_\mu, p_\phi)$ and $q(p_\phi)$ depend on details of a black hole configuration.

If $L(\mu, p_\mu, p_\phi)$ coincides with the Casimir element of $su(2)$

$$L(\mu, p_\mu, p_\phi) = \alpha^2 J_a J_a$$

where α is a constant, the bosonic conformal mechanics can be extended to accommodate the full $D(2,1;\alpha)$–superconformal symmetry.

Part II: $D(2,1;\alpha)$ superparticles on near horizon black hole backgrounds

Consider a dynamical system governed by the Hamiltonian (α, b, M are constants)

$$H = \frac{x}{M^2}\left(\sqrt{b^2+(xp)^2+\alpha^2 J_a J_a}+b\right)$$
$$+\frac{x}{M^2}\left(\alpha(\bar\psi\sigma_a\psi)J_a-\frac{1}{8}(1+2\alpha)\psi^2\bar\psi^2\right)\left(\sqrt{b^2+(xp)^2+\alpha^2 J_a J_a}-b\right)^{-1}$$

which is characterized by the $D(2,1;\alpha)$–symmetry

$$D = tH+xp, \qquad K = t^2 H+2txp+\frac{M^2}{x}\left(\sqrt{b^2+(xp)^2+\alpha^2 J_a J_a}-b\right)$$

$$Q_\alpha = -\frac{2\left((xp)\psi_\alpha+i\alpha(\sigma_a\psi)_\alpha J_a+\frac{i}{4}(1+2\alpha)\bar\psi_\alpha\psi^2\right)}{\left(\frac{2M^2}{x}\left(\sqrt{b^2+(xp)^2+\alpha^2 J_a J_a}-b\right)\right)^{\frac{1}{2}}}$$

$$S_\alpha = \psi_\alpha\left(\frac{2M^2}{x}\left(\sqrt{b^2+(xp)^2+\alpha^2 J_a J_a}-b\right)\right)^{\frac{1}{2}}-tQ_\alpha$$

$$\mathcal{J}_a = J_a+\frac{1}{2}(\bar\psi\sigma_a\psi), \quad I_- = \frac{i}{2}\psi^2, \quad I_+ = -\frac{i}{2}\bar\psi^2, \quad I_3 = \frac{1}{2}\bar\psi\psi$$

$(3,4,1)$–multiplet of $D(2,1;\alpha)$ yields the background fields (near horizon RN)

$$ds^2 = \left(\frac{r}{M}\right)^2 dt^2 - \left(\frac{M}{r}\right)^2 dr^2 - \left(\frac{M}{\alpha}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$A = \frac{Q}{M^2} r dt + P\cos\theta d\phi$$

which solve the vacuum Einstein–Maxwell equation with a cosmological constant provided

$$M = \sqrt{\frac{2(Q^2+\alpha^4 P^2)}{1+\alpha^2}}, \quad \Lambda = \frac{\alpha^2-1}{2M^2}$$

Superparticle propagating on such background

$$S = -\int dt \left(m\sqrt{(r/M)^2 - (M/r)^2\dot{r}^2 - (M/\alpha)^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)} + eQr/M^2 + eP\cos\theta\dot{\phi} \right) + \text{fermions}$$

exhibits $D(2,1;\alpha)$–symmetry provided the BPS–like condition is satisfied

$$(eQ)^2 = (mM)^2 - (\alpha e P)^2$$

Part II: $D(2,1;\alpha)$ superparticle on $d=5$, $N=2$ supergravity background with Λ term

$(4,4,0)$–multiplet of $D(2,1;\alpha)$ yields the background fields

$$ds^2 = \left(\frac{r}{M}\right)^2 dt^2 - \left(\frac{M}{r}\right)^2 dr^2 -$$

$$-\left(\frac{M}{\alpha}\right)^2 \left(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2\right)$$

$$A = \frac{Qr}{M}dt, \qquad B = \frac{Qr}{\sqrt{2}M}dt, \qquad \varphi = 0$$

which solve the equations of motion describing the bosonic limit of the $d=5$, $N=2$ supergravity interacting with one vector multiplet in spacetime with cosmological constant provided

$$\Lambda = \frac{\alpha^2 - 1}{2M^2}, \qquad Q = \pm\sqrt{\frac{2+\alpha^2}{3}}, \qquad m^2 = \frac{(2+\alpha^2)e^2}{3}$$

Summary of Part II: Dynamical realizations of $D(2,1;\alpha)$ exist in which α is linked to a cosmological constant.

l-conformal Galilei algebra (J. Negro, M. del Olmo, A. Rodriguez-Marco, 1997)

$$[H,D]=H, \qquad [H,C_i^{(n)}]=nC_i^{(n-1)},$$
$$[H,K]=2D, \qquad [D,K]=K,$$
$$[D,C_i^{(n)}]=(n-l)C_i^{(n)}, \qquad [K,C_i^{(n)}]=\boxed{(n-2l)}\,C_i^{(n+1)},$$
$$[M_{ij},C_k^{(n)}]=-\delta_{ik}C_j^{(n)}+\delta_{jk}C_i^{(n)}, \qquad [M_{ij},M_{kl}]=-\delta_{ik}M_{jl}-\delta_{jl}M_{ik}+\dots$$

Realization in spacetime

$$H=\partial_t, \quad D=t\partial_t+lx_i\partial_i, \quad K=t^2\partial_t+2ltx_i\partial_i, \quad M_{ij}=x_i\partial_j-x_j\partial_i$$
$$C_i^{(0)}=P_i=\partial_i, \qquad C_i^{(1)}=K_i=t\partial_i, \qquad \dots \qquad , \qquad C_i^{(n)}=t^n\partial_i$$

The algebra is finite–dimensional provided $n=0,1,\dots,2l$ which means that l is half–integer. $1/l$ is called the dynamical exponent. $C_i^{(n)}=t^n\partial_i$ are called the generators of accelerations. Current studies of the nonrelativistic version of the AdS/CFT correspondence stimulate a renewed interest in the l–conformal Galilei algebra.

l–conformal Galilei algebra in succinct notation

$$[L_m, L_n] = -\mathrm{i}\,(m-n)L_{n+m}, \qquad [L_n, \mathbf{U}_k] = -\mathrm{i}\,\left(\ell\, n - k\right)\mathbf{U}_{n+k}$$

where $n,m=-1,0,1$, with $H=L_{-1}$, $D=L_0$, $K=L_1$, $k=-\ell,\ldots,\ell$, and $\mathbf{U}_k = \mathrm{i}\,C_A^{(k+\ell)}$, where $A=1,\ldots,d$ is a vector index of $so(d)$.

In order to construct an $N=4$ supersymmetric extension of the l–conformal Galilei algebra, let us rewrite $D(2,1;\alpha)$ in the succinct notation

$$[L_m, L_n] = -\mathrm{i}(m-n)L_{n+m}, \quad \left[V^{ab}, V^{cd}\right] = -\mathrm{i}\left(\epsilon^{ac}V^{bd} + \epsilon^{bd}V^{ac}\right)$$

$$\left[W^{ij}, W^{kl}\right] = -\mathrm{i}\left(\epsilon^{ik}W^{jl} + \epsilon^{jl}W^{ik}\right), \quad \left[L_m, Q_r^{ai}\right] = -\mathrm{i}\left(\frac{m}{2} - r\right)Q_{r+m}^{ai}$$

$$\left[V^{ab}, Q_r^{ci}\right] = -\frac{\mathrm{i}}{2}\left(\epsilon^{ac}Q_r^{bi} + \epsilon^{bc}Q_r^{ai}\right), \quad \left[W^{ij}, Q_r^{ak}\right] = -\frac{\mathrm{i}}{2}\left(\epsilon^{ik}Q_r^{aj} + \epsilon^{jk}Q_r^{ai}\right)$$

$$\left\{Q_r^{ai}, Q_s^{bj}\right\} = -2\left(\epsilon^{ab}\epsilon^{ij}L_{r+s} + \alpha(r-s)\delta_{r+s,0}\epsilon^{ab}W^{ij} - \right.$$
$$\left. -(1+\alpha)(r-s)\delta_{r+s,0}\epsilon^{ij}V^{ab}\right).$$

where $a,b=1,2$, $i,j=1,2$, and $r=-\frac{1}{2},\frac{1}{2}$. In particular

$$Q^{1i}_{-\frac{1}{2}} = \bar{Q}^i, \quad Q^{2i}_{-\frac{1}{2}} = Q^i, \quad Q^{1i}_{\frac{1}{2}} = -\bar{S}^i, \quad Q^{2i}_{\frac{1}{2}} = -S^i$$

Part III: $N=4$ l–conformal Galilei superalgebra based upon $D(2,1;\alpha)$

For general reasons, the bracket of Q_r^{ai} and $\mathbf{U}_m$ should yield a fermionic superpartner $\mathbf{S}_r^{ai}$, while the structure relation involving Q_r^{ai} and $\mathbf{S}_r^{ai}$ must produce a bosonic superpartner $\mathbf{A}_m^{ij}=\mathbf{A}_m^{ji}$. Suppressing the lower index, one reveals the triplet $(\mathbf{U},\mathbf{S}^{ai},\mathbf{A}^{ij})$ which looks analogous to the irreducible $(1,4,3)$–supermultiplet of $D(2,1;\alpha)$. Assigning the conformal weights $(\ell,\ell+\frac{1}{2},\ell+1)$ to the triplet $(\mathbf{U}_n,\mathbf{S}_r^{ai},\mathbf{A}_m^{ij})$, one finds the superalgebra

$$[L_n,\mathbf{U}_m]=-\mathrm{i}\,(\ell\,n-m)\,\mathbf{U}_{n+m},\ \left[L_n,\mathbf{S}_r^{ai}\right]=-\mathrm{i}\,\left(\left(\ell+\frac{1}{2}\right)n-r\right)\mathbf{S}_{n+r}^{ai}$$

$$\left[L_n,\mathbf{A}_m^{ij}\right]=-\mathrm{i}\,\left(\left(\ell+1\right)n-m\right)\mathbf{A}_{n+m}^{ij},\ \left[Q_r^{ai},\mathbf{U}_m\right]=\mathrm{i}\mathbf{S}_{r+m}^{ai}$$

$$\left[Q_r^{ai},\mathbf{A}_m^{jk}\right]=\mathrm{i}\,\left(2\left(\ell+1\right)r-m\right)\left(\epsilon^{ij}\mathbf{S}_{r+m}^{ak}+\epsilon^{ik}\mathbf{S}_{r+m}^{aj}\right)$$

$$\left\{Q_r^{ai},\mathbf{S}_s^{bj}\right\}=\left(\left(2\ell+1\right)r-s\right)\epsilon^{ab}\epsilon^{ij}\mathbf{U}_{r+s}-\epsilon^{ab}\mathbf{A}_{r+s}^{ij}$$

where $\alpha=\ell+1$ is fixed by the Jacobi identities and the index range is

$$\mathbf{S}_r^{ai}:r=-\ell-\frac{1}{2},\ldots,\ell+\frac{1}{2};\quad \mathbf{A}_m^{ij}:m=-\ell-1,\ldots,\ell+1$$

Alternatively, one may choose the descending sequence of conformal weights $(\ell, \ell-\frac{1}{2}, \ell-1)$ which yields the algebra

$$\begin{aligned}
&[L_n, \mathbf{U}_m] = -\mathrm{i}\,(\ell\, n - m)\,\mathbf{U}_{n+m},\ \left[L_n, \mathbf{S}^{ai}_r\right] = -\mathrm{i}\,\left(\left(\ell - \frac{1}{2}\right) n - r\right)\mathbf{S}^{ai}_{n+r},\\
&\left[L_n, \mathbf{A}^{ij}_m\right] = -\mathrm{i}\,((\ell-1)\,n - m)\,\mathbf{A}^{ij}_{n+m},\ \left[Q^{ai}_r, \mathbf{U}_m\right] = \mathrm{i}\,(2\ell\, r - m)\,\mathbf{S}^{ai}_{r+m},\\
&\left[Q^{ai}_r, \mathbf{A}^{jk}_m\right] = \mathrm{i}\,\left(\epsilon^{ij}\mathbf{S}^{ak}_{r+m} + \epsilon^{ik}\mathbf{S}^{aj}_{r+m}\right),\\
&\left\{Q^{ai}_r, \mathbf{S}^{bj}_s\right\} = \epsilon^{ab}\epsilon^{ij}\mathbf{U}_{r+s} - ((2\ell-1)\,r - s)\,\epsilon^{ab}\mathbf{A}^{ij}_{r+s}.
\end{aligned}$$

where $\alpha = -\ell$ is fixed by the Jacobi identities and the index range is

$$\mathbf{S}^{ai}_r : r = -\ell + \frac{1}{2}, \ldots, \ell - \frac{1}{2}; \qquad \mathbf{A}^{ij}_m : m = -\ell + 1, \ldots, \ell - 1$$

In a similar fashion one can build $N=4$ l–conformal Galilei superalgebras based upon $(3,4,1)$–, $(2,4,2)$–, and $(4,4,0)$–supermultiplets of $D(2,1;\alpha)$.

Part III: $N = 4$ l–conformal Galilei superalgebra based upon $D(2, 1; \alpha)$

Consider an analog of a general real superfield: bosonic generators $\mathbf{U}_n$, $\mathbf{A}^{ab}_m$, $\mathbf{B}^{ij}_m$, $\mathbf{G}_n$ with the conformal weights ℓ, $\ell+1$, $\ell+1$, $\ell+2$, and their fermionic partners $\mathbf{S}^{ai}_r$, $\mathbf{R}^{ai}_r$, which have the conformal weights $\ell+\frac{1}{2}$ and $\ell+\frac{3}{2}$. No constraint on α arises for the superalgebra

$$\left[Q^{ai}_r, \mathbf{U}_m\right] = \mathrm{i}\, \mathbf{S}^{ai}_{r+m}, \qquad \left[Q^{ai}_r, \mathbf{G}_m\right] = \mathrm{i}\; (2(\ell+2)r - m)\, \mathbf{R}^{ai}_{r+m}$$

$$\left\{Q^{ai}_r, \mathbf{S}^{bj}_s\right\} = ((2\ell+1)r - s)\, \epsilon^{ab}\epsilon^{ij}\mathbf{U}_{r+s} - \frac{2+\ell+\alpha}{3+2\ell}\epsilon^{ab}\mathbf{B}^{ij}_{r+s} - \frac{1+\ell-\alpha}{3+2\ell}\epsilon^{ij}\mathbf{A}^{ab}_{r+s}$$

$$\left\{Q^{ai}_r, \mathbf{B}^{jk}_m\right\} = \mathrm{i}\,(2(\ell+1)r - m)\left(\epsilon^{ij}\mathbf{S}^{ak}_{r+m} + \epsilon^{ik}\mathbf{S}^{aj}_{r+m}\right) - \mathrm{i}\frac{1+\ell-\alpha}{3+2\ell}\left(\epsilon^{ij}\mathbf{R}^{ak}_{r+m} + \epsilon^{ik}\mathbf{R}^{aj}_{r+m}\right)$$

$$\left\{Q^{ai}_r, \mathbf{A}^{bc}_m\right\} = \mathrm{i}\,(2(\ell+1)r - m)\left(\epsilon^{ab}\mathbf{S}^{ci}_{r+m} + \epsilon^{ac}\mathbf{S}^{bi}_{r+m}\right) + \mathrm{i}\frac{2+\ell+\alpha}{3+2\ell}\left(\epsilon^{ab}\mathbf{R}^{ci}_{r+m} + \epsilon^{ac}\mathbf{R}^{bi}_{r+m}\right)$$

$$\left\{Q^{ai}_r, \mathbf{R}^{bj}_s\right\} = ((2\ell+3)r - s)\left(\epsilon^{ab}\mathbf{B}^{ij}_{r+s} - \epsilon^{ij}\mathbf{A}^{ab}_{r+s}\right) + \epsilon^{ij}\epsilon^{ab}\mathbf{G}_{r+s}$$

As usual, the conformal weights determine the index range

$$\mathbf{U}_n : n=-\ell,\ldots,\ell; \qquad \mathbf{S}^{ai}_r : r=-\ell-\frac{1}{2},\ldots,\ell+\frac{1}{2}$$

$$\mathbf{A}^{ab}_m : m=-\ell-1,\ldots,\ell+1; \qquad \mathbf{B}^{ij}_m : m=-\ell-1,\ldots,\ell+1$$

$$\mathbf{R}^{ai}_s : s=-\ell-\frac{3}{2},\ldots,\ell+\frac{3}{2}; \qquad \mathbf{G}_n : n=-\ell-2,\ldots,\ell+2$$

The simplest realization arises at $\alpha=-\frac{1}{2}$

$$t^n\frac{\partial}{\partial x_i}, \qquad n=0,\ldots,2l$$

$$\theta_\alpha t^n\frac{\partial}{\partial x_i}, \quad \bar{\theta}^\alpha t^n\frac{\partial}{\partial x_i} \qquad n=0,\ldots,2l-1$$

$$\theta_\alpha\bar{\theta}^\beta t^n\frac{\partial}{\partial x_i}, \quad \theta^2 t^n\frac{\partial}{\partial x_i}, \quad \bar{\theta}^2 t^n\frac{\partial}{\partial x_i} \qquad n=0,\ldots,2l-2$$

$$\theta_\alpha\bar{\theta}^2 t^n\frac{\partial}{\partial x_i}, \quad \bar{\theta}^\alpha\theta^2 t^n\frac{\partial}{\partial x_i} \qquad n=0,\ldots,2l-3$$

$$\theta^2\bar{\theta}^2 t^n\frac{\partial}{\partial x_i}, \qquad n=0,\ldots,2l-4$$

Open problems

Summary of Part III: $D(2,1;\alpha)$ facilitates the construction of $N=4$ supersymmetric extensions of the l–conformal Galilei algebra.

Open problems:

- Off-shell superfield Lagrangian formulations
- Dynamical systems enjoying the $N=4$ l–conformal Galilei symmetry
- Newton–Hooke counterparts of the $N=4$ l–conformal Galilei superalgebras and their dynamical realizations

Thanks for your attention!