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Positive Energy Unitary Irreducible Representations of the Superalgebras $osp(1|2n,\mathbb{R})$ and Character Formulae

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Introduction

The term positive energy representations was introduced by G.Mack in the study of the representations of the 4D conformal algebra $so(4,2) \cong su(2,2)$. The reason was that the conformal weight d of the class he studied was shown to be eigenvalue of the conformal Hamiltonian. Furthermore the unitary representations of that class turned to be characterized by the following conditions:

$$d \geq 2 + j_1 + j_2 \ , \quad j_1 j_2 \neq 0 \qquad (1a)$$

$$d \geq 1 + j_1 + j_2 \ , \quad j_1 j_2 = 0 \qquad (1b)$$

where j_1, j_2 are non-negative half-integers fixing the inducing representations of the Lorentz subalgebra of dimension $(2j_1+1)(2j_2+1)$. Hence, positive energy!

Of course, it was clear that positive energy representations is just a physics synonym of the mathematics term lowest weight representations of real semisimple groups. Actually not all real semisimple groups have lowest/highest weight representations, but only those of the class of *Hermitian symmetric spaces*. The practical criterion for a real semisimple Lie algebra $\mathcal{G}$ to be in that class is that the *maximal compact subalgebra* $\mathcal{K}$ of $\mathcal{G}$ is of the form:

$$\mathcal{K} \ = \ so(2) \oplus \mathcal{K}' \ . \tag{2}$$

There were many examples before Mack especially for the algebras $so(1,2)$, $so(3,2)$. Later these representations were completely classified by Enright-Howe-Wallach.

Such results are needed for the applications, especially with the development of superstring theories. Thus, superconformal field theories in various dimensions are attracting more interest. Thus, in the supersymmetry case such classification was needed and done first for the $D=4$ superconformal algebras $su(2,2|N)$ [FF] (for $N=1$), [DP] (for arbitrary N). Later, the classification for $D=3$ (for even N), $D=5$, and $D=6$ (for $N=1,2$) was given in [Minwalla] (though some of his results are conjectural), and then the $D=6$ case (for arbitrary N) was finalized in [D]. For the cases $D>6$ the relevant superalgebras turn out to be $osp(1|\,2n)$, cf. [Townsend, Ferrara, et al].

The study of the UIRs of $osp(1|\,2n,\mathbb{R})$ was started in [DZ] and [DMZZ].

In the present talk is presented the finalization of the unitarity classification achieved in [DS] and character formulae for $osp(1|\,6)$.

Representations of the superalgebras $osp(1|2n)$ and $osp(1|2n,\mathbb{R})$

The setting

We consider the superalgebra $\mathcal{G} = osp(1|2n,\mathbb{R})$ for $n > 1$. The even subalgebra of $\mathcal{G}$ is the algebra $sp(2n,\mathbb{R})$ with maximal compact subalgebra $\mathcal{K} = u(n) \cong su(n) \oplus u(1)$.

We label the relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [\, d\, ;\, a_1\, , ..., a_{n-1}\,] \tag{3}$$

where d is the conformal weight, and $a_1, ..., a_{n-1}$ are non-negative integers which label the finite-dimensional UIRs of the subalgebra $su(n)$ (the simple part of $\mathcal{K}$).

We classify the positive energy (lowest weight) UIRs of $\mathcal{G}$ following the methods used for the $D = 4, 6$ conformal superalgebras, cf. [DP,D], resp. The main tool is an adaptation of the Shapovalov form on the Verma modules V^χ over the complexification $\mathcal{G}^{\mathbb{C}} = osp(1|2n)$ of $\mathcal{G}$.

Root systems

The root system of $\mathcal{G}^{\mathbb{C}} = osp(1|\,2n)$ is given in terms of $\delta_1 \ldots, \delta_n\,,\ (\delta_i, \delta_j) = \delta_{ij}\,,\ i,j = 1,...,n$. The even and odd root systems are [Kac]:

$$\begin{aligned} \Delta_{\bar{0}} &= \{\pm\delta_i \pm \delta_j \ ,\ 1 \leq i < j \leq n \ , \\ &\qquad \pm 2\delta_i \ ,\ 1 \leq i \leq n\} \ , \\ \Delta_{\bar{1}} &= \{\pm\delta_i \ ,\ 1 \leq i \leq n\} \end{aligned} \tag{4}$$

(the signs $\pm$ are not correlated). We shall use the following distinguished simple root system [Kac]:

$$\Pi = \{\,\delta_1 - \delta_2\,, \,, \ldots, \delta_{n-1} - \delta_n\,, \,\delta_n\,\} \ , \tag{5}$$

or introducing standard notation for the simple roots:

$$\begin{aligned} \Pi &= \{\,\alpha_1\,, ..., \,\alpha_n\,\} \ , \\ \alpha_j &= \delta_j - \delta_{j+1} \ , \quad j = 1, ..., n-1 \ , \\ \alpha_n &= \delta_n \ . \end{aligned} \tag{6}$$

The root $\alpha_n = \delta_n$ is odd, the other simple roots are even. The Kac-Dynkin diagram is:

$$\underset{1}{\bigcirc} - - \cdots - - \underset{n-1}{\bigcirc} \Longrightarrow \underset{n}{\bullet} \tag{7}$$

The black dot is used to signify that the simple odd root is not nilpotent, otherwise a gray dot would be used [Kac]. In fact, the superalgebras $osp(1|\,2n)$ have no nilpotent generators unlike all other types of basic classical Lie superalgebras.

The corresponding to Π positive root system is:

$$\begin{aligned}\Delta^+_{\bar 0} &= \{\delta_i \pm \delta_j\ ,\ 1 \leq i < j \leq n\ ,\ 2\delta_i\ ,\ 1 \leq i \leq n\} \\ \Delta^+_{\bar 1} &= \{\delta_i\ ,\ 1 \leq i \leq n\} \end{aligned} \tag{8}$$

From the point of view of representation theory more relevant is the restricted root system, such that:

$$\bar\Delta^+ = \bar\Delta^+_{\bar 0} \cup \Delta^+_{\bar 1}\ , \tag{9}$$

$$\begin{aligned}\bar\Delta^+_{\bar 0} &\equiv \{\alpha \in \Delta^+_{\bar 0} \mid \tfrac{1}{2}\alpha \notin \Delta^+_{\bar 1}\} = \\ &= \{\delta_i \pm \delta_j\ ,\ 1 \leq i < j \leq n\}\end{aligned} \tag{10}$$

The above restricted root system is also the root system of the complex simple Lie algebra B_n (dropping the distinction between even and odd roots) with Dynkin diagram:

$$\underset{1}{\bigcirc} - - \cdots - - \underset{n-1}{\bigcirc} \Longrightarrow \underset{n}{\bigcirc} \tag{11}$$

$$\begin{aligned}\Delta^+_{B_n} = \{&\delta_i \pm \delta_j\ ,\ 1 \leq i < j \leq n\ , \\ &\delta_i\ ,\ 1 \leq i \leq n\} \cong \bar\Delta^+\end{aligned} \tag{12}$$

Lowest weight through the signature

We need the lowest weight $\Lambda \in \mathcal{H}^*$ (where $\mathcal{H}$ is the Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$) the values of which should be related to the signature (3):

$$(\Lambda, \alpha_k^\vee) = -a_k \ , \quad 1 \leq k \leq n \ , \tag{13}$$

where $\alpha_k^\vee \equiv 2\alpha_k/(\alpha_k, \alpha_k)$, and the minus signs are due to the fact that we use lowest weight Verma modules.

In fact, it remains to relate a_n to the conformal weight d which is a matter of overall normalization so as to correspond to some known cases. Our choice is:

$$a_n = -2d - a_1 - \cdots - a_{n-1} \ . \tag{14}$$

For the mathematically inclined reader we also give the Dynkin labelling:

$$m_k = (\rho - \Lambda, \alpha_k^\vee) \tag{15}$$

where $\rho \in \mathcal{H}^*$ is given by the difference of the half-sums $\rho_{\bar{0}}, \rho_{\bar{1}}$ of the even, odd, resp., positive roots (cf. (8)):

$$\begin{aligned} \rho \ \doteq \ \rho_{\bar{0}} - \rho_{\bar{1}} &= (n - \tfrac{1}{2})\delta_1 + (n - \tfrac{3}{2})\delta_2 + \cdots + \\ &+ \tfrac{3}{2}\delta_{n-1} + \tfrac{1}{2}\delta_n \ , \qquad (16) \\ \rho_{\bar{0}} &= n\delta_1 + (n-1)\delta_2 + \cdots + 2\delta_{n-1} + \delta_n \ , \\ \rho_{\bar{1}} &= \tfrac{1}{2}(\delta_1 + \cdots + \delta_n) \ . \end{aligned}$$

Unlike $a_k \in \mathbb{Z}_+$ for $k < n$ the value of a_n is arbitrary. In the cases when a_n is also a non-negative integer, and then $m_k \in \mathbb{N}$ $(\forall k)$ the corresponding irreps are the finite-dimensional irreps of $\mathcal{G}$ (and of B_n).

Verma modules

To introduce Verma modules we use the standard triangular decomposition:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^{+} \oplus \mathcal{H} \oplus \mathcal{G}^{-} \tag{17}$$

where $\mathcal{G}^{+}$, $\mathcal{G}^{-}$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^{\Lambda} \cong U(\mathcal{G}^{+}) \otimes v_0$, where $U(\mathcal{G}^{+})$ is the universal enveloping algebra of $\mathcal{G}^{+}$, and v_0 is a lowest weight vector v_0 such that:

$$\begin{aligned} Z\ v_0 &= 0\ , \quad Z \in \mathcal{G}^{-} \\ H\ v_0 &= \Lambda(H)\ v_0\ , \quad H \in \mathcal{H}\ . \end{aligned} \tag{18}$$

Adapting the criterion of [Kac] to lowest weight modules, one finds that a Verma module V^{Λ} is reducible w.r.t. the positive root β iff the following holds [DZ]:

$$(\rho - \Lambda, \beta) = m\,(\beta,\beta)/2, \qquad \beta \in \Delta^{+}\ , \quad m \in \mathbb{N}\ . \tag{19}$$

Remark: The above criterion looks the same as the Bernstein-Gelfand-Gelfand criterion for the case of simple Lie algebras. The difference comes from the fact that in the case of simple Lie superalgebras – except our case of $osp(1/2n)$! – for the odd roots β holds $(\beta,\beta) = 0$ and then the reducibility criterion is just $(\rho - \Lambda, \beta) = 0$, thus, there is no positive integer m for the odd roots. $\diamondsuit$

If a condition from (19) is fulfilled then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair m, β : $\Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight vector v_0' of $V^{\Lambda'}$ to the **singular vector** $v_s^{m,\beta}$ in V^Λ which is completely determined by the conditions:

$$\begin{aligned} X\ v_s^{m,\beta} &= 0\ , \quad X \in \mathcal{G}^-\ , & (20) \\ H\ v_s^{m,\beta} &= \Lambda'(H)\ v_0\ , \quad H \in \mathcal{H}\ , \\ & \Lambda' = \Lambda + m\beta\ . & (21) \end{aligned}$$

Explicitly, $v_s^{m,\beta}$ is given by a polynomial in the positive root generators [D]:

$$v_s^{m,\beta} = P^{m,\beta}\, v_0\ , \quad P^{m,\beta} \in U(\mathcal{G}^+)\ . \tag{22}$$

Thus, the submodule I^β of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+)\, P^{m,\beta}\, v_0$.

Certainly, (19) may be fulfilled for several positive roots (even for all of them). Let Δ_Λ denote the set of all positive roots for which (19) is fulfilled, and let us denote: $\tilde{I}^\Lambda \equiv \cup_{\beta\in\Delta_\Lambda} I^\beta$. Clearly, $\tilde{I}^\Lambda$ is a proper submodule of V^Λ. Let us also denote $F^\Lambda \equiv V^\Lambda/\tilde{I}^\Lambda$.

The Verma module V^Λ contains a unique proper maximal submodule I^Λ ($\supseteq \widetilde{I^\Lambda}$) [Kac]. Among the lowest weight modules with lowest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e., $L_\Lambda = V^\Lambda/I^\Lambda$. (If V^Λ is irreducible then $L_\Lambda = V^\Lambda$.)

All above was valid in general. For our superalgebras in [DZ] it was established that from (19) follows that the Verma module $V^{\Lambda(\chi)}$ is reducible if one of the following relations holds:

$$\mathbb{N} \ni m^-_{ij} = j - i + a_i + \cdots + a_{j-1} \tag{23a}$$

$$\mathbb{N} \ni m^+_{ij} = 2n - i - j + 1 + a_j + \cdots + a_{n-1} - - a_1 - \cdots - a_{i-1} - 2d \tag{23b}$$

$$\mathbb{N} \ni m_i = 2n - 2i + 1 + a_i + \cdots + a_{n-1} - - a_1 + \cdots - a_{i-1} - 2d\ . \tag{23c}$$

For further use we introduce notation for the root vector $X^+_j \in \mathcal{G}^+$, $j = 1, \ldots, n$, corresponding to the simple root α_j.

Further, we notice that all reducibility conditions in (23a) are fulfilled by construction. The corresponding submodules $I^{\alpha_i} = U(\mathcal{G}^+)\, v^i_s$, where $\Lambda_i = \Lambda + m^-_i \alpha_i$ and $v^i_s = (X^+_i)^{1+a_i}\, v_0$. These submodules generate an invariant submodule which we denote by $I^\Lambda_c \subset \tilde{I}^\Lambda$. Since these submodules are nontrivial for all our signatures in the question of unitarity instead of V^Λ we shall consider also the factor-modules:

$$F^\Lambda_c \ = \ V^\Lambda / I^\Lambda_c \ \supset \ F^\Lambda \ . \qquad (24)$$

We shall denote the lowest weight vector of F^Λ_c by $|\Lambda_c\rangle$ and the singular vectors above become null conditions in F^Λ_c :

$$(X^+_i)^{1+a_i}\, |\Lambda_c\rangle \ = \ 0 \ , \quad i = 1, ..., n-1. \qquad (25)$$

If the Verma module V^Λ is not reducible w.r.t. the other roots, i.e., (23b,c) are not fulfilled, then $F^\Lambda_c = F^\Lambda$ is irreducible and is isomorphic to the irrep L_Λ with this weight.

In fact, for the factor-modules F^Λ_c reducibility is controlled by the value of d. For the unitarity we are interested in the maximal values of d.

The maximal d coming from the different possibilities in (23b) are obtained for $m^+_{ij} = 1$ and they are:

$$d_{ij} \equiv n + \tfrac{1}{2}(a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - i - j) \ , \qquad (26)$$

the corresponding root being $\delta_i + \delta_j$.

The maximal d coming from the different possibilities in (23c) are obtained for $m_i = 1$, and they are:

$$\begin{aligned} d_i \quad \equiv \quad & n - i + \tfrac{1}{2}(a_i + \cdots + a_{n-1} \ - \\ & - \ a_1 - \cdots - a_{i-1}) \ , \end{aligned} \qquad (27)$$

the corresponding roots being δ_i.

There is ordering between these maximal reduction points [DZ]:

$$\begin{aligned} d_1 \quad &> \quad d_2 \ > \ \cdots \ > \ d_n \ , \qquad (28) \\ d_{i,i+1} \quad &> \quad d_{i,i+2} \ > \ \cdots \ > \ d_{in} \ , \\ d_{1,j} \quad &> \quad d_{2,j} \ > \ \cdots \ > \ d_{j-1,j} \ , \\ d_i \quad &> \quad d_{jk} \ > \ d_\ell \ , \qquad i \le j < k \le \ell \ . \end{aligned}$$

Note that there is a biggest maximal reduction point, called First Reduction Point (FRP):

$$d_1 \ = \ n - 1 + \tfrac{1}{2}(a_1 + \cdots + a_{n-1}) \ . \qquad (29)$$

Unitarity

Theorem: [Dobrev-Zhang-Salom] All positive energy unitary irreducible representations of the superalgebras $osp(1|\,2n,\mathbb{R})$ characterized by the signature χ in (3) are obtained for real d and are given as follows:

$$
\begin{aligned}
d &\geq n-1+\tfrac{1}{2}(a_1+\cdots+a_{n-1}) \;=\; d_1\ , \qquad (30)\\
&\qquad a_1 \neq 0\ ,\\
d &\geq n-\tfrac{3}{2}+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) \;=\; d_{12}\ ,\\
&\qquad a_1=0,\ a_2\neq 0\ ,\\
d &= n-2+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) \;=\; d_2 > d_{13}\ ,\\
&\qquad a_1=0,\ a_2\neq 0\ ,\\
d &\geq n-2+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) \;=\; d_2 \;=\; d_{13}\ ,\\
&\qquad a_1=a_2=0,\ a_3\neq 0\ ,\\
&...\\
d &\geq n-1-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1})\ ,\\
&\qquad a_1=...=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0\ ,\\
&\qquad\qquad \kappa=\tfrac{1}{2},1,...,\tfrac{1}{2}(n-1)\ ,\\
d &= n-\tfrac{3}{2}-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1})\ ,\\
&\qquad a_1=...=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0\ ,\\
&...\\
d &= n-1-2\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1})\ ,\\
&\qquad a_1=...=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0\ ,\\
&...\\
&...\\
d &\geq \tfrac{1}{2}(n-1)\ , \quad a_1 \;=\; ... \;=\; a_{n-1} \;=\; 0\\
d &= \tfrac{1}{2}(n-2)\ , \quad a_1 \;=\; ... \;=\; a_{n-1} \;=\; 0\\
&...\\
d &= \tfrac{1}{2}\ , \quad a_1 \;=\; ... \;=\; a_{n-1} \;=\; 0\\
d &= 0 \;=\; d_n\ , \quad a_1 \;=\; ... \;=\; a_{n-1} \;=\; 0
\end{aligned}
$$

All UIRs are infinite-dimensional except the last case which is the trivial isolated one-dimensional irrep. $\diamondsuit$
The *Proof* is given in the arXiv paper.

Character formulae

Let $\hat{\mathcal{G}}$ be a simple Lie algebra of rank ℓ with Cartan subalgebra $\hat{\mathcal{H}}$, root system $\hat{\mathcal{D}}$, simple root system $\hat{\pi}$. Let Γ_+, be the set of all integral dominant elements of $\hat{\mathcal{H}}^*$, i.e., $\lambda \in \hat{\mathcal{H}}^*$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}_+$ for all simple roots α_i. Let V be lowest weight module with lowest weight Λ and lowest weight vector v_0. It has the following decomposition:

$$V = \bigoplus_{\mu \in \Gamma_+} V_\mu \ , \tag{31}$$

$$V_\mu = \{u \in V \mid Hu = (\lambda + \mu)(H)u, \ \forall \ H \in \hat{\mathcal{H}}\}$$

(Note that $V_0 = \mathbb{C}v_0$.) We use formal exponents $e(\mu)$ ($\mu \in \hat{\mathcal{H}}^*$) with properties: $e(0) = 1$, $e(\mu)e(\nu) = e(\mu + \nu)$.

Then the (formal) character of V is defined by:

$$\begin{aligned} ch_0 \ V &= \sum_{\mu \in \Gamma_+} (\dim \ V_\mu) \ e(\Lambda + \mu) = \\ &= e(\Lambda) \sum_{\mu \in \Gamma_+} (\dim \ V_\mu) \ e(\mu) \end{aligned} \tag{32}$$

(We shall use subscript '0' for the even case.)

For a Verma module, i.e., $V = V^\Lambda$ one has $\dim \ V_\mu = P(\mu)$, where $P(\mu)$ is a generalized partition function,

$P(\mu) = \#$ of ways μ can be presented as a sum of positive roots β, each root taken with its multiplicity $\dim \mathcal{G}_\beta$ $(= 1$ here), $P(0) \equiv 1$. Thus, the character formula for Verma modules is:

$$\begin{aligned} ch_0\ V^\Lambda &= e(\Lambda) \sum_{\mu \in \Gamma_+} P(\mu) e(\mu) = \\ &= e(\Lambda) \prod_{\alpha \in \Delta^+} \frac{1}{1 - e(\alpha)} \end{aligned} \tag{33}$$

Example: For $sl(2, \mathbb{C})$ there is only one positive root α. For simplicity we denote $t = e(\alpha)$. The character formula for Verma modules is:

$$ch_0\ V^\Lambda = \frac{e(\Lambda)}{1 - t} = e(\Lambda) \sum_{k=0}^{\infty} t^k \tag{34}$$

Obviously, t^k corresponds to the basis element $(X^+)^k v_0$ of the Verma module V^Λ and the character formula encodes this basis, which is the general idea. $\diamondsuit$

Further we recall the standard reflections in $\hat{\mathcal{H}}^*$:

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha\,, \quad \lambda \in \hat{\mathcal{H}}^*\,, \quad \alpha \in \hat{\mathcal{D}}. \tag{35}$$

The Weyl group W is generated by the simple reflections $s_i \equiv s_{\alpha_i}$. Thus every element $w \in W$ can be written as the product of simple reflections. It is said that w is written in a reduced form if it is written with the minimal possible number of simple reflections; the number of reflections of a reduced form of w is called the length of w, denoted by $\ell(w)$.

The Weyl character formula for the finite-dimensional irreducible LWM L_Λ over $\hat{\mathcal{G}}$, i.e., when $\Lambda \in -\Gamma_+$, has the form:

$$ch_0\ L_\Lambda \ = \ \sum_{w\in W} (-1)^{\ell(w)}\ ch_0\ V^{w\cdot\Lambda}, \quad \Lambda \in -\Gamma_+ \qquad (36)$$

where the dot $\cdot$ action is defined by $w\cdot\lambda = w(\lambda-\rho)+\rho$. For future reference we note:

$$\begin{aligned} s_\alpha \ \cdot\ \Lambda \ &= \ \Lambda \ + \ n_\alpha \alpha\ , \quad \alpha \in \Delta^+ \qquad (37)\\ n_\alpha \ &= \ n_\alpha(\Lambda) \ \doteq \ (\rho - \Lambda, \alpha^\vee) \ = \ (\rho-\Lambda)(H_\alpha)\ . \end{aligned}$$

In the case of basic classical Lie superalgebras the first character formulae were given by Kac. They are more complicated than the bosonic case, except for the algebras we consider. Actually, for $osp(1/2n)$ the Verma module character formula is the same as (33):

$$ch\ V^\Lambda \ = \ e(\Lambda) \prod_{\alpha\in\bar{\Delta}^+} \frac{1}{1-e(\alpha)} \qquad (38)$$

using the restricted root system $\bar{\Delta}^+$. Naturally, the character formula for the finite-dimensional irreducible LWM L_Λ is again (36) using the Weyl group W_n of B_n.

Multiplets

A Verma module V^Λ may be reducible w.r.t. to many positive roots, and thus there may be many Verma modules isomorphic to its submodules. They themselves may be reducible, and so on.

One main ingredient of the approach of [D] is as follows. We group the (reducible) Verma modules with the same Casimirs in sets called **multiplets** [D]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible Verma modules and the lines between the vertices correspond to embeddings between them.

If a Verma module V^Λ is reducible w.r.t. to all simple roots (and thus w.r.t. all positive roots), i.e., $m_k \in \mathbb{N}$ for all k, then the irreducible submodules are isomorphic to the finite-dimensional irreps of $\mathcal{G}^\mathbb{C}$ [D]. (Actually, this is a condition only for m_n since $m_k \in \mathbb{N}$ for $k = 1, \ldots, n-1$.) In these cases we have the *main multiplets* which are isomorphic to the Weyl group of $\mathcal{G}^\mathbb{C}$ [D].

In the cases of non-dominant weight Λ the character formula for the irreducible LWM is [KL] :

$$ch\, L_\Lambda \ = \ \sum_{\substack{w\in W \\ w\leq w_\Lambda}} \ (-1)^{\ell(w_\Lambda w)}\, P_{w,w_\Lambda}(1)\, ch\, V^{w\cdot(w_\Lambda^{-1}\cdot\Lambda)}\,, \qquad \Lambda \in \tag{39}$$

where $P_{y,w}(u)$ are the Kazhdan–Lusztig polynomials $y, w \in W$ [KL] (for an easier exposition see [D]), w_Λ is a unique element of W with minimal length such that the signature of $\Lambda_0 \ = \ w_\Lambda^{-1}\cdot\Lambda$ is anti-dominant or semi-anti-dominant:

$$\chi_0 \ = \ (m'_1,\ldots,m'_n), \qquad m'_k \ = \ 1 - \Lambda_0(H_k) \ \in \mathbb{Z}_- \ . \tag{40}$$

When Λ_0 is semi-anti-dominant, i.e., at least one $m'_k = 0$, then in fact W is replaced by a reduced Weyl group W_R .

Most often the value of $P_{y,w}(1)$ is equal to 1, as in the character formula for the finite-dimensional irreps and incidentally in the cases presented below.

It is interesting to see how the reducible points relevant for unitarity fit in the multiplets. As shown in [DS] the Verma modules with weights related to positive energy would be somewhere in the main multiplet (or in a reduction of the main multiplet), and the first task for calculating the character is to find the w_Λ in the character formula (39). This we do in the next subsection in the case $n = 3$.

The case n=3.

For $n = 3$ the ordering (28) simplifies to:

$$d_1 \;>\; d_{12} \;>\; d_2 \;>\; d_{23} \;>\; d_3$$

$$\hookrightarrow \;>\; d_{13} \;>\; \nearrow$$

The DZS Theorem now reads:

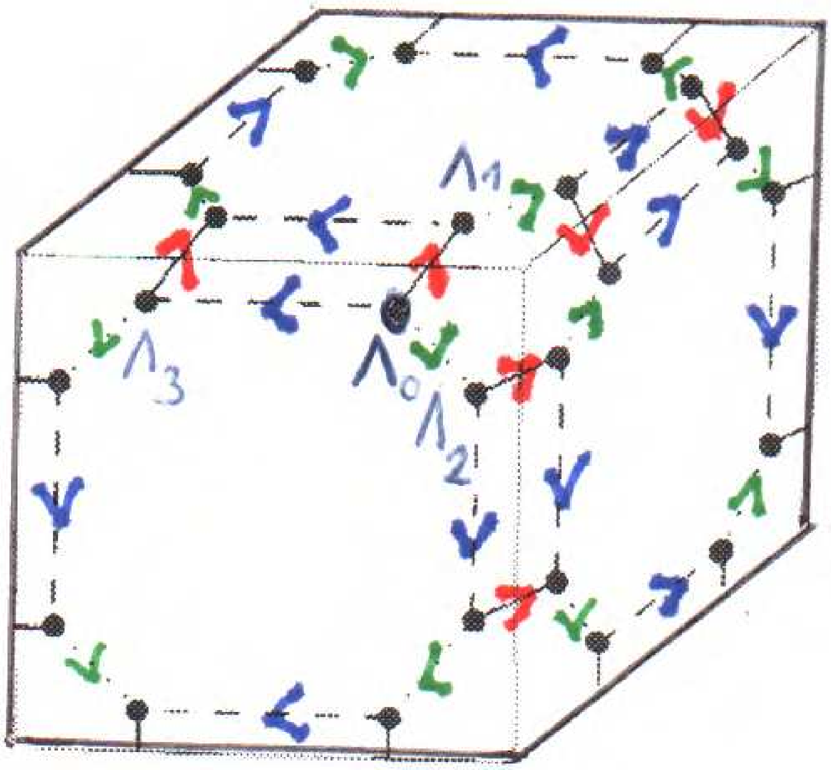
$$\begin{aligned}
d &\geq 2 + \tfrac{1}{2}(a_1 + a_2) \;=\; d_1 \;, \quad a_1 \neq 0 \;, \\
d &\geq \tfrac{3}{2} + \tfrac{1}{2}a_2 \;=\; d_{12} \;, \quad a_1 = 0,\; a_2 \neq 0 \;, \\
d &= 1 + \tfrac{1}{2}a_2 \;=\; d_2 > d_{13} \;, \quad a_1 = 0,\; a_2 \neq 0 \;, \\
d &\geq 1 \;=\; d_2 = d_{13} \;, \quad a_1 = a_2 = 0 \;, \\
d &= \tfrac{1}{2} \;=\; d_{23} \;, \quad a_1 = a_2 = 0 \;, \\
d &= 0 \;=\; d_3 \;, \quad a_1 = a_2 = 0 \;.
\end{aligned} \tag{41}$$

The Weyl group W_n of B_n has $2^n n!$ elements, i.e., 48 for B_3. Let $S = (s_1, s_2, s_3)$, $s_i \equiv s_{\alpha_i}$, be the simple reflections. They fulfill the following relations:

$$\begin{aligned} & s_1^2 = s_2^2 = s_3^2 = e, \ (s_1 s_2)^3 = e, \ (s_2 s_3)^4 = e, \\ & s_1 s_3 = s_3 s_1 , \end{aligned} \tag{42}$$

e being the identity of W_3. The 48 elements may be listed as:

$$\begin{aligned} & e\,,\ s_1\,,\ s_2\,,\ s_3 \\ & s_1s_2\,,\ s_1s_3\,,\ s_2s_1\,,\ s_2s_3\,,\ s_3s_2\,, \\ & s_1s_2s_1\,,\ s_1s_2s_3\,,\ s_1s_3s_2\,,\ s_2s_1s_3\,,\ s_2s_3s_2\,, \\ & \ s_3s_2s_1\,,\ s_3s_2s_3\,, \\ & s_1s_2s_1s_3\,,\ s_1s_2s_3s_2\,,\ s_1s_3s_2s_1\,,\ s_1s_3s_2s_3\,, \\ & s_2s_3s_2s_1\,,\ s_2s_1s_3s_2\,,\ s_3s_2s_3s_1\,,\ s_3s_2s_3s_2\,, \\ & s_1s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3\,,\ s_1s_2s_1s_3s_2\,, \\ & s_1s_3s_2s_3s_2\,, s_2s_1s_3s_2s_1\,,\ s_2s_1s_3s_2s_3\,, \\ & s_3s_2s_3s_1s_2\,,\ s_3s_2s_3s_2s_1\,, \\ & s_1s_3s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3s_2\,,\ s_1s_2s_1s_3s_2s_1\,, \\ & s_2s_1s_3s_2s_1s_3\,, s_2s_1s_3s_2s_3s_2\,,\ s_3s_2s_3s_1s_2s_1\,, \\ & s_3s_2s_3s_1s_2s_3\,, s_2s_1s_3s_2s_3s_2s_1\,, \\ & s_2s_1s_3s_2s_3s_1s_2\,, s_3s_2s_1s_2s_3s_2s_1\,, \\ & s_3s_2s_3s_1s_2s_1s_3\,,\ s_3s_2s_3s_1s_2s_3s_2\,, \\ & s_2s_3s_2s_1s_2s_3s_2s_1\,,\ s_3s_2s_1s_3s_2s_3s_2s_1\,, \\ & s_3s_2s_1s_3s_2s_3s_1s_2\,, s_2s_3s_2s_1s_3s_2s_3s_2s_1\ . \end{aligned} \tag{43}$$



or they may be pictorially represented on a cube as in the figure, where we have given only the simple root reflections, namely, continuous **red** arrows represent action of reflection s_1, dashed **blue** arrows represent action of reflection s_2, dotted **green** arrows represent action of reflection s_3. Each **face** of the cube contains an **octet** of elements related by blue and green arrows representing the Weyl group of B_2 generated by s_2 and s_3. The figure contains also 8 **sextets** (around the eight corners of the cube) from red and green arrows representing the Weyl group of A_2 generated by s_1 and s_2. Finally there are 12 **quartets** (straddling the edges of the cube) formed by red and blue arrows representing the Weyl group of $A_1 \times A_1$ generated by the commuting reflections s_1 and s_3.

The character formula for the Verma modules in our case is given explicitly by:

$$ch\ V^\Lambda = \frac{e(\Lambda)}{(1-t_1)(1-t_2)(1-t_1t_2)}\ \times$$
$$\times \frac{1}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times$$
$$\times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \qquad (44)$$

where $t_j \equiv e(\alpha_j)$.

Now we give character formulae for the UIRs at boundary or isolated unitarity cases. Below we shall denote the signature of the dominant weight Λ_0 which determines the main multiplet by (m'_1, m'_2, m'_3), $m'_k \in \mathbb{N}$, using primes to distinguish from the signatures of the weights we are interested. We shall use also reductions of the main multiplet when the weights are semi-dominant, i.e., when some $m'_k = 0$.

• In the case of $d = d_1 = 2 + \frac{1}{2}(a_1 + a_2)$ there are twelve members of the multiplet which is a submiltiplet of a main multiplet. They are grouped into two standard $sl(3)$ submultiplets of six members. The first submultiplet starts from $V^{\Lambda_0^{d_1}}$, where $\Lambda_0^{d_1} = w \cdot \Lambda_0 = \Lambda_{32312}$, $w = w_{\Lambda_0^{d_1}} = s_2 s_1 s_3 s_2 s_3$, with signature:

$$\Lambda_0^{d_1} \quad : \quad (m_1, m_2, m_3' = 1 - 2m_{12}) \ , \qquad m_1, m_2 \in \mathbb{N} \ , \quad m_{12} \equiv m_1 + m_2 \ . \tag{45}$$

The other submultiplet starts from $V^{\Lambda_0'}$ with $\Lambda_0' = \Lambda_0^{d_1} + \delta_1 = \Lambda_0^{d_1} + \alpha_1 + \alpha_2 + \alpha_3$, with signature: $\Lambda_0' : (m_1 - 1, m_2, m_3' = 1 - 2m_{12})$, $m_1 > 1$. The character formula is (39) with $w_\Lambda = w_{\Lambda_0^{d_1}}$:

$$\begin{aligned} \mathrm{ch}\,\Lambda_0^{d_1} = & \frac{e(\Lambda_0^{d_1})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \\ & \times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\ & \times \{ \ \mathrm{ch}\,\Lambda_{m_1,m_2}(t_1,t_2) \ - \\ & - \ t_1t_2t_3\,\mathrm{ch}\,\Lambda_{m_1-1,m_2}(t_1,t_2) \ \} \ , \ m_1 > 1 \end{aligned} \tag{46}$$

where $\mathrm{ch}\,\Lambda_{m_1,m_2}(t_1,t_2)$ is the normalized character of the finite-dimensional $sl(3)$ irrep with Dynkin labels

(m_1, m_2) (and dimension $m_1 m_2 (m_1 + m_2)/2$):

$$\text{ch } \Lambda_{m_1,m_2}(t_1, t_2) = \tag{47}$$

$$= \frac{1 - t_1^{m_1} - t_2^{m_2} + t_1^{m_1} t_2^{m_{12}} + t_1^{m_{12}} t_2^{m_2} - t_1^{m_{12}} t_2^{m_{12}}}{(1 - t_1)(1 - t_2)(1 - t_1 t_2)}$$

Naturally, the latter formula is a polynomial in t_1, t_2 with positive coefficients, e.g., $\text{ch } \Lambda_{1,1}(t_1, t_2) = 1$, $\text{ch } \Lambda_{2,1}(t_1, t_2) = 1 + t_1 + t_1 t_2$.

In the case $m_1 = 2, m_2 = 1$ the character formula (46) simplifies to:

$$\text{ch}\, \Lambda_0^{d_1} = \frac{e(\Lambda_0^{d_1})}{(1 - t_3)(1 - t_2 t_3)} \times \tag{48}$$

$$\times \frac{1}{(1 - t_2 t_3^2)(1 - t_1 t_2 t_3^2)(1 - t_1 t_2^2 t_3^2)} \times$$

$$\times \left\{ 1 + \frac{t_1(1 + t_2)}{1 - t_1 t_2 t_3} \right\}, \quad m_1 = 2, m_2 = 1$$

• In the case of $d = d_{12} = \frac{1}{2}(3 + a_2)$, $(a_1 = 0)$, there are again twelve members of the multiplet. Omitting the details [DS] the character f-la is:

$$\begin{aligned}
\mathrm{ch}\, \Lambda_0^{d_{12}} = & \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \\
\times & \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \\
\times & \{ \ \mathrm{ch}\, \Lambda_{1,m_2}(t_1,t_2) \ - \\
- & \ (t_1t_2^2t_3^2)^{m_2}\, \mathrm{ch}\, \Lambda_{1,m_2-1}(t_1,t_2) \ \} \ , \ m_2 > 1
\end{aligned} \tag{49}$$

In the case $m_2 = 2$ it simplifies to:

$$\begin{aligned}
\mathrm{ch}\, \Lambda_0^{d_{12}} = & \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \\
\times & \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)} \\
\times & \{ \ 1 + t_1t_2^2t_3^2 + \frac{t_2(1+t_1)}{1-t_1t_2^2t_3^2} \ \}
\end{aligned} \tag{50}$$

• In the case $d = d_2 = d_{13} = 1$ and $a_1 = a_2 = 0$, $m_1 = m_2 = 1$, the signature is:

$$\Lambda_0^{d_2=d_{13}} \ : \ (1,1,-1) \ . \tag{51}$$

Again there are twelve members of the multiplet. Omitting the details [DS] the character f-la is:

$$\begin{aligned} \mathrm{ch}\,\Lambda_0^{d_2=d_{13}} \ &= \\ &= \ \frac{e(\Lambda_0^{d_2=d_{13}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \ \times \\ &\times \ \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\ &\times \ \{ \ 1 \ - \ t_1t_2^2t_3^3 \ \} \ = \\ &= \ \frac{e(\Lambda_0^{d_2=d_{13}})}{(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)} \times \\ &\times \left(\ \frac{1}{1-t_1t_2^2t_3^2} \ + \ \frac{t_3}{1-t_3} \ \right) \end{aligned} \tag{52}$$

• In the case of $d = d_2 = 1 + \frac{1}{2}a_2 > d_{13} = 1$, i.e., $m_1 = 1$, $m_2 = 1 + a_2 > 1$. The multiplet has 24 members for $m_2 > 2$. Omitting the details [DS] the character f-la is:

$$\mathrm{ch}\, \Lambda_0^{\prime d_2} = \frac{e(\Lambda_0^{\prime d_2})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times$$
$$\times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$
$$\times \{ \mathrm{ch}\, \Lambda_{1,m_2}(t_1,t_2) -$$
$$- t_2t_3\, \mathrm{ch}\, \Lambda_{2,m_2-1}(t_1,t_2) +$$
$$+ t_1t_2^3t_3^3\, \mathrm{ch}\, \Lambda_{2,m_2-2}(t_1,t_2) -$$
$$- t_1^2t_2^4t_3^4\, \mathrm{ch}\, \Lambda_{1,m_2-2}(t_1,t_2) \} \qquad (53)$$

When $m_2 = 2$ $(a_2 = 1)$ the multiplet reduces to only 12 members, and the character formula simplifies to:

$$\mathrm{ch}\, \Lambda_0^{\prime d_2} = \frac{e(\Lambda_0^{\prime d_2})}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times$$
$$\times \{ \frac{1}{(1-t_3)(1-t_1t_2t_3)} + \frac{t_2}{(1-t_3)(1-t_2t_3)}$$
$$+ \frac{t_1t_2}{(1-t_2t_3)(1-t_1t_2t_3)} \} \qquad (54)$$

• In the case of $d = d_{23} = \frac{1}{2}$, $a_1 = a_2 = 0$, i.e., $m_1 = m_2 = 1$, again we have a multiplet with 24 members. Omitting the details [DS] the character formula is:

$$\mathrm{ch}\,\Lambda_0^{d_{23}} \;=\; \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \;\times$$

$$\times\; \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$

$$\times\;\{\; 1 \;-\; t_2t_3^2\,(1+t_1+t_1t_2) \;+$$

$$+\; t_1t_2^2t_3^4\,(1+t_2+t_1t_2) \;-\; t_1^2t_2^4t_3^6 \;\} \;=$$

$$=\; \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)}$$

$$=\; e(\Lambda_0^{d_{23}}) \sum_{k_1,k_2,k_3\in\mathbb{Z}_+} t_1^{k_1}\, t_2^{k_1+k_2}\, t_3^{k_1+k_2+k_3} \qquad (55)$$

Note that this is a **singleton** representation, since each weight can appear only once as is easily seen from the last line of (55).

Thank You!