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Noncommutative gravity and the relevance of the theta-constant deformation

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Overview

Review of different approaches to NC gravity

$SO(2,3)_\star$ NC gravity

- General
- Action
- Low energy expansion; equations of motion

One solution: NC Minkowski space-time

Discussion

NC geometry

Quantum space-time ??? NC space-time.
Quantum gravity ??? Gravity on NC spaces.

The first idea of noncommutative space-time: Heisenberg in 1930.
The first model: Snyder 1947.
Today different approaches: NC spectral geometry, deformation quantization, $\star$-product approach, frame formaslim,...

The main idea: coordinates do not commute. Instead

$$[\hat{x}^\mu, \hat{x}^\nu] = \hat{\theta}^{\mu\nu} \Rightarrow \Delta\hat{x}^\mu \Delta\hat{x}^\nu \geq \text{something}.$$

$\hat{\theta}^{\mu\nu}$ can be: constant matrix (Moyal-Weyl deformation), linear in $\hat{x}$ (κ-Minkowski,...), quadratic in coordinates (quantum groups), Lorentz tensor (DFR approach), dynamical ("Torino" group).

Different approaches to NC gravity

General relativity is based on the diffeomorphism symmetry. This concept (space-time symmetry) is difficult to generalize to NC spaces. Different approaches:

NC spectral geometry [Chamseddine, Connes, Marcolli '07; Chamseddine, Connes, Mukhanov '14].

Emergent gravity [Steinacker '10, '16]:

- dynamical quantum geometry arises from NC gauge theory (YM martix models).
-cosmological solution with a big bounce and an inflation like phase [Klammer, Steinacker '09].

Frame formalism, operator description [Burić, Madore '14; Fritz, Majid '16]:
-sphericaly symmetric solutions of NC Einstein equations

Twist approach [Wess et al. '05, '06; Ohl, Schenckel '09; Castellani, Aschieri '09; Aschieri, Schenkel '14]:

-action/equations of motion invariant under twisted diffeomprphisms; black hole and cosmology solutions (descrete time and radius, no initial singularity) [Ohl, Schenckel '09; Schupp, Solodukhin '09].
-Levi-Civita connection for a general twist deformation.

NC gravity as a gauge theory of Lorentz/Poincaré group [Chamseddine '01,'04, Cardela, Zanon '03, Aschieri, Castellani '09,'12; Dobrski '16]:

-based on NC gauge theories and the Seiberg-Witten (SW) map

-the NC space is usually the Moyal-Weyl (θ-constant) space; generalization to NC spaces coming from an Abelian twist are done by the "Torino" group.
-the NC gauge group is usually $SO(1,3)_\star$, but can be larger. Models with $SO(2,3)_\star$, $SO(1,4)_\star$, $SO(2,4)_\star$ are/can be considered.

$SO(2,3)_\star$ NC gravity: General

$SO(2,3)_\star$ NC gravity is based on:

-NC space-time $\rightarrow$ Moyal-Weyl deformation with small, constant NC parameter $\theta^{\alpha\beta} = -\theta^{\beta\alpha}$.
-gravity $\rightarrow$ $SO(2,3)$ gauge theory with symmetry broken to $SO(1,3)$, [Stelle, West '80; Wilczek '98].
-Seiberg-Witten (SW) map $\rightarrow$ relates NC fields to the corresponding commutative fields.

Our goal:

-construct NC gravity action, calculate equations of motion and find NC gravity solutions.
-diffeomorphism symmetry broken, give physical meaning to $\theta^{\alpha\beta}$.

$$
\begin{aligned}
f \cdot g \quad &\longrightarrow \quad f \star g = f \cdot g + \tfrac{i}{2}\theta^{\alpha\beta}(\partial_\alpha f)(\partial_\beta g) \\
&\qquad -\tfrac{1}{8}\theta^{\alpha\beta}\theta^{\kappa\lambda}(\partial_\alpha\partial_\kappa f)(\partial_\beta\partial_\lambda g) + \ldots \\
\alpha,\ \Phi,\ A_\mu,\ F_{\mu\nu} \quad &\longrightarrow \quad \hat{\alpha},\ \hat{\Phi},\ \hat{A}_\mu, \\
&\qquad \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu \stackrel{\star}{,} \hat{A}_\nu] \\
\delta_\alpha \Psi = i\alpha\Psi,\ \delta_\alpha\phi = i[\alpha, \phi] \quad &\longrightarrow \quad \delta^\star_\alpha \hat{\Psi} = i\hat{\alpha} \star \hat{\Psi},\ \delta^\star_\alpha \hat{\phi} = i[\hat{\alpha} \stackrel{\star}{,} \hat{\phi}] \\
\delta_\alpha A_\mu = \partial_\mu \epsilon + i[\alpha, A_\mu] \quad &\longrightarrow \quad \delta^\star_\alpha \hat{A}_\mu = \partial_\mu \hat{\alpha} + i[\hat{\alpha} \stackrel{\star}{,} \hat{A}_\mu] \\
\delta_\alpha F_{\mu\nu} = i[\alpha, F_{\mu\nu}] \quad &\longrightarrow \quad \delta^\star_\alpha \hat{F}_{\mu\nu} = i[\hat{\alpha} \stackrel{\star}{,} \hat{F}_{\mu\nu}]
\end{aligned}
$$

Idea of the Seiberg-Witten map: NC gauge transformations are induced by the commutative gauge transformations, $\delta_\alpha \to \delta^\star_\alpha$:

$$
\delta^\star_\alpha \hat{A}_\mu(A_\mu, \theta) = \hat{A}_\mu(A_\mu + \delta_\alpha A_\mu, \theta) - \hat{A}_\mu(A_\mu, \theta). \qquad (1)
$$

Recursive relations to all orders in $\theta^{\kappa\lambda}$ in [Ulker and Yapiskan '08, Aschieri and Castellani '12].

$$\hat{F}^{(n+1)}_{\mu\nu} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{\omega}_\kappa \overset{\star}{,} \partial_\lambda \hat{F}_{\mu\nu} + D_\lambda \hat{F}_{\mu\nu}\}\Big)^{(n)} + \frac{1}{2(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{F}_{\mu\kappa}, \overset{\star}{,} \hat{F}_{\nu\lambda}\}\Big)^{(n)}, \quad (2)$$

$$\hat{\phi}^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{\omega}_\kappa \overset{\star}{,} \partial_\lambda \hat{\phi} + D_\lambda \hat{\phi}\}\Big)^{(n)}. \quad (3)$$

The main result of the SW-map: No new degrees of freedom (no new fields), only new interaction terms appear.

$SO(2,3)_\star$ NC gravity: Action

$SO(2,3)_\star$ gauge theory: gauge field ω_μ and field strength tensor $F_{\mu\nu}$ of the $SO(2,3)$ gauge group:

$$\begin{aligned}
\omega_\mu &= \frac{1}{2}\omega_\mu^{AB}M_{AB} = \frac{1}{4}\omega^{ab}\sigma_{ab} + \frac{1}{2}\omega_\mu^{a5}\gamma_a\,, \qquad (4)\\
F_{\mu\nu} &= \frac{1}{2}F_{\mu\nu}^{AB}M_{AB} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_\mu,\omega_\nu]\\
&= \frac{1}{4}\Big(R_{\mu\nu}^{ab} - \frac{1}{l^2}(e_\mu^a e_\nu^b - e_\mu^b e_\nu^a)\Big)\sigma_{ab} + \frac{1}{2}F_{\mu\nu}^{a5}\gamma_a\,, \qquad (5)
\end{aligned}$$

with

$$\begin{aligned}
[M_{AB}, M_{CD}] &= i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC})\,, \qquad (6)\\
\eta_{AB} &= (+,-,-,-,+), \quad A,B = 0,\dots,3,5\\
M_{AB} &\to (M_{ab}, M_{a5}) = (\frac{i}{4}[\gamma_a,\gamma_b], \frac{1}{2}\gamma_a),\; a,b = 0,\dots 3\\
R_{\mu\nu}^{ab} &= \partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + \omega_{\mu c}^{a}\omega_\nu^{cb} - \omega_{\mu c}^{b}\omega_\nu^{ca},\\
F_{\mu\nu}^{a5} &= \frac{1}{l}\Big(\nabla_\mu e_\nu{}^a - \nabla_\nu e_\mu{}^a\Big), \quad \nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_{\mu b}^{a} e_\nu{}^b.
\end{aligned}$$

The decomposition in (4, 5) very much resembles the definitions of curvature and torsion in Einstein-Cartan gravity leading to General Relativity. Indeed, after introducing proper action(s) and the breaking of $SO(2,3)$ symmetry (gauge fixing) down to the $SO(1,3)$ symmetry one obtains this result [Stelle, West '80].

To fix the gauge: the field ϕ transforming in the adjoint representation:

$$\phi = \phi^A \Gamma_A,\ \delta_\epsilon \phi = i[\epsilon, \phi]\,,$$

with $\Gamma^A = (i\gamma_a\gamma_5, \gamma_5)$ and γ_a and γ_5 are the usual Dirac gamma matrices in four dimensions.

Inspired by [Stelle, West '80; Wilczek '98] we define the NC gravity action as

$$S_{NC} = c_1 S_{1NC} + c_2 S_{2NC} + c_3 S_{3NC}\,, \tag{7}$$

with

$$\begin{aligned} S_{1NC} &= \frac{il}{64\pi G_N} \mathrm{Tr} \int \mathrm{d}^4x \epsilon^{\mu\nu\rho\sigma} \hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi}\,, \\ S_{2NC} &= \frac{1}{64\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4x \epsilon^{\mu\nu\rho\sigma} \hat{\phi} \star \hat{F}_{\mu\nu} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} + c.c.\,, \\ S_{3NC} &= -\frac{i}{128\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} D_\mu \hat{\phi} \star D_\nu \hat{\phi} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi}\,. \end{aligned}$$

The action is written in the 4-dimensional Minkowski space-time, as an ordianry NC gauge theory. It is invariant under the NC $SO(2,3)_\star$ gauge symmetry and the SW map guarantees that after the expansion it will be invariant under the commutative $SO(2,3)$ gauge symmetry.

Using the SW map solutions for the fields $\hat{F}_{\mu\nu}$ and $\hat{\phi}$ and the Moyal-Weyl $\star$-product, we expand the action (7) in the orders of NC parameter $\theta^{\alpha\beta}$. The expansion is done around the commutative vacuum $\phi^5 = l$, $\phi^a = 0$, that is the symmetry breaking is done after the expansion of NC fields and $\star$-products.

In the commutative limit, $\theta^{\alpha\beta} \to 0$ we obtain

$$S = S_1 + S_2 + S_3, \tag{8}$$

$$S_{1NC}^{(0)} = \frac{il}{64\pi G_N} \mathrm{Tr} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \phi \,, \tag{9}$$

Original Stelle-West action

$$S_{2NC}^{(0)} = \frac{1}{64\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} D_\rho \phi D_\sigma \phi \phi, \tag{10}$$

$$S_{3NC}^{(0)} = -\frac{i}{128\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} D_\mu \phi D_\nu \phi D_\rho \phi D_\sigma \phi \phi, \tag{11}$$

with $D_\mu \phi = \partial_\mu \phi - i[\omega_\mu, \phi]$.

After the gauge fixing, $\phi^a = 0, \phi^5 = l$, these actions reduce to

$$
\begin{aligned}
S_{1NC}^{(0)} &\to -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big(\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{\ \ ab}R_{\rho\sigma}^{\ \ cd} + e(R - \frac{6}{l^2})\Big),\\
S_{2NC}^{(0)} &\to -\frac{1}{16\pi G_N}\int \mathrm{d}^4x e\Big(R - \frac{12}{l^2}\Big),\\
S_{3NC}^{(0)} &\to -\frac{1}{16\pi G_N}\int \mathrm{d}^4x e\Big(-\frac{12}{l^2}\Big),
\end{aligned}
$$

with $e_\mu^{\ a} = \frac{1}{l}\omega_\mu^{a5}$, $e = \det e_\mu^a$, $R = R_{\mu\nu}^{\ \ ab}e_a^{\ \mu}e_b^{\ \nu}$. The constants c_1, c_2 and c_3 are arbitrary and can be determined from some consistency conditions.

Comments:

-main adventage of $SO(2,3)$ approach: basic fields are not metric and/or vielbeins but gauge fields of (A)dS group; consequences also in the NC setting.

-after the symmetry breaking: spin connection ω_μ and vielbeins e_μ. They are independent, 1st order formalsim.

-varying (8) with respect to ω_μ and vielbeins e_μ gives equations of motion for these fields. The spin connection in not dynamical (the equation of motion is algebraic, the zero-torsion condition) and can be expressed in terms of vielbeins, 2nd order formalism.

-(8) written in the 2nd order formalism has three terms: Gauss-Bonnet topological term, Einstein-Hilbert term and the cosmological constant term.

-arbitrary constants c_1, c_2 and c_3: EH term requires $c_1 + c_2 = 1$, while the absence of the cosmological constant is provided with $c_1 + 2c_2 + 2c_3 = 0$. Applying both constraints leaves one free parameter (can be used later in the NC generalization).

Calculations show that the first order correction $S_{NC}^{(1)} = 0$. Already known result [Chamseddine '01,'04, Cardela, Zanon '03, Aschieri, Castellani '09].

The first non-vanishing correction is of the second order in the NC parameter; it is long and difficult to calculate. However, the second order corrections can be anayzed sector by sector: high/low energy, high/low/zero cosmological constant, zero/non-zero torsion.

$SO(2,3)_\star$ NC gravity: low energy expansion

We are interested in the low energy expansion: we keep only the terms of the zeroth, the first and the second order in the derivatives of vierbeins (linear in $R_{\alpha\beta\gamma\delta}$, quadratic in $T^a_{\alpha\beta}$):

$$
\begin{aligned}
S_{NC} = &-\frac{1}{16\pi G_N}\int \mathrm{d}^4x\, e\Big(R - \frac{6}{l^2}(1+c_2+2c_3)\Big)\\
&+\frac{1}{128\pi G_N l^4}\int \mathrm{d}^4x\, e\theta^{\alpha\beta}\theta^{\gamma\delta}\Big((-2+12c_2+38c_3)R_{\alpha\beta\gamma\delta}\\
&+(4-18c_2-44c_3)R_{\alpha\gamma\beta\delta} - (6+22c_2+36c_3)g_{\beta\delta}R_{\alpha\mu\gamma}{}^{\mu} + \frac{6+28c_2+56c_3}{l^2}g_{\alpha\gamma}g_{\beta\delta}\\
&+(5-\frac{9}{2}c_2-7c_3)T^a_{\alpha\beta}T_{\gamma\delta a} + (-10+\frac{9}{2}c_2+14c_3)T^a_{\alpha\gamma}T_{\beta\delta a} + (3-3c_2-2c_3)T_{\alpha\beta\gamma}T_{\delta\mu}{}^{\mu}\\
&+(1+2c_2)T_{\alpha\beta\rho}T^{\rho}{}_{\gamma\delta} + 8T_{\alpha\gamma\delta}T_{\beta\mu}{}^{\mu} - (2c_2+4c_3)T_{\alpha\gamma\rho}T^{\rho}{}_{\delta\beta}\\
&+(2c_2+4c_3)g_{\beta\delta}T_{\gamma\sigma}{}^{\sigma}T_{\alpha\rho}{}^{\rho} - (2c_2+4c_3)T_{\alpha\rho\sigma}T_{\gamma}{}^{\sigma\rho}g_{\beta\delta} + (-2+4c_2+18c_3)T_{\alpha\beta\gamma}e^{\rho}_a\nabla_\delta e^a_\rho\\
&+(6-8c_2-8c_3)T_{\alpha\gamma\beta}e^{\rho}_a\nabla_\delta e^a_\rho + (2+4c_2+12c_3)T_{\alpha\gamma}{}^{\mu}e^a_\beta\nabla_\delta e^a_\mu - T_{\alpha\beta}{}^{\mu}e^a_\delta\nabla_\gamma e^a_\mu\\
&+(-6-8c_2-16c_3)T_{\delta\rho\beta}e^{\rho}_a\nabla_\alpha e^a_\gamma - (2c_2+4c_3)g_{\alpha\gamma}T_{\mu\beta}{}^{\mu}e^{\rho}_a\nabla_\delta e^a_\rho - (2c_2+4c_3)g_{\beta\delta}T_{\alpha\rho}{}^{\sigma}e^{\rho}_a\nabla_\gamma e^a_\sigma\\
&-(4+16c_2+32c_3)e^{\mu}_a e_{b\beta}\nabla_\gamma e^a_\alpha\nabla_\delta e^b_\mu + (4+12c_2+32c_3)e_{\delta a}e^{\mu}_b\nabla_\alpha e^a_\gamma\nabla_\beta e^b_\mu\\
&-(2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_a e^{\nu}_b\nabla_\gamma e^a_\mu\nabla_\alpha e^b_\nu + (2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_a e^{\rho}_c\nabla_\alpha e^a_\rho\nabla_\gamma e^c_\mu\Big).
\end{aligned}
\tag{12}
$$

-equations of motion: variation with respect to the vierbeins and the spin connection

$$
\begin{aligned}
\delta e^a_\mu : \qquad & R_{\alpha\gamma}{}^{cd} e^\gamma_d e^\alpha_a e^\mu_c - \frac{1}{2} e^\mu_a R + \frac{3}{l^2}(1 + c_2 + 2c_3) e^\mu_a \\
& = \tau_a{}^\mu = -\frac{8\pi G_N}{e} \frac{\delta S^{(2)}_{NC}}{\delta e^a_\mu}, \qquad (13) \\
\delta \omega^{ab}_\mu : \qquad & T_{ac}{}^c e^\mu_b - T_{bc}{}^c e^\mu_a - T_{ab}{}^\mu \\
& = S_{ab}{}^\mu = -\frac{16\pi G_N}{e} \frac{\delta S^{(2)}_{NC}}{\delta \omega^{ab}_\mu}. \qquad (14)
\end{aligned}
$$

Effective energy-momentum tensor $\tau_a{}^\mu$ and effective spin tensor $S_{ab}{}^\mu$ depend on $\theta^{\alpha\beta}$: noncommutativity as a source of curvature and torsion!
Equations are written assuming that (after variation) $T^a_{\alpha\beta} = 0$ in the zeroth order.

NC corrections to Minkowski space-time

Minkowski space-time is a solution of vacuum Einstein equations without the cosmological constant. We are interested in corrections to this solution induced by our NC gravity model. First, we assume that $1 + c_2 + 2c_3 = 0$, that is that the cosmological constant in not present in the zeroth odred. Then we assume that the NC metric is of the form:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu}$ is a small correction that is second order in the deformation parameter $\theta^{\alpha\beta}$.
Insterting this ansatz into the action (12) and varying with respect to $h_{\mu\nu}$ leads to:

$$\frac{1}{2}(\partial_\sigma\partial^\nu h^{\sigma\mu} + \partial_\sigma\partial^\mu h^{\sigma\nu} - \partial^\mu\partial^\nu h - \Box h^{\mu\nu}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h)$$
$$= \frac{11}{4l^6}(2\eta_{\alpha\gamma}\theta^{\alpha\mu}\theta^{\gamma\nu} + \frac{1}{2}\eta_{\alpha\gamma}\eta_{\beta\delta}\eta^{\mu\nu}\theta^{\alpha\beta}\theta^{\gamma\delta})\,. \qquad (15)$$

A solution of the form:

$$
\begin{aligned}
g_{00} &= 1 - R_{0m0n} x^m x^n \,, \\
g_{0i} &= -\frac{2}{3} R_{0min} x^m x^n \,, \\
g_{ij} &= -\delta_{ij} - \frac{1}{3} R_{imjn} x^m x^n \,,
\end{aligned}
\tag{16}
$$

where $R_{\mu\nu\rho\sigma} \sim \theta^{\alpha\beta}\theta^{\gamma\delta}$ is the Reimann tensor for this solution. This shows that the coordinates x^μ we started with, are Fermi normal coordinates: inertial coordinates of a local observer moving along a geodesic; can be constructed in a small neighborhood along the geodesic (cylinder), [Manasse, Misner'63; Chicone, Mashoon'06; Klein, Randles '11].

The measurements performed by the local observer moving along the geodesic are described in the Fermi normal coordinates. He/she is the one that measures $\theta^{\alpha\beta}$ to be constant! In any other reference frame, observers will measure $\theta^{\alpha\beta}$ different from constant.

Discussion

- SW-expanded NC gravity models have no new fields. Unless one promotes $\theta^{\alpha\beta}$ to a proper tensor/siplectic form [Dobrski '16] or a dynamical field [Aschieri, Castellani '12].
- First nonzero correction terms are second order in NC parameter, very small.
- The breaking of diffeomorphism invariance (due to constant $\theta^{\alpha\beta}$) can be understood as gauge fixing: there is a prefered refernce system defined by the Fermi normal coordinates and the NC parameter $\theta^{\mu\nu}$ is constant in that particular reference system. Not obvious if one can write an action invariant under the full diffeomorphism symetry...
- Phenomenological implications: modification of the G_N, in relation to DM problem a la MOND theories, cosmological solution will obtain NC corrections and (very likely) non-zero torzion.
- Coupling of matter fields: scalars (DE, inflation), spinors (cosmological neutrinos, CνB radiation)