Super 0-brane action on the coset space of $D(2,1;\alpha)$ supergroup

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Plan

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Motivations

- Classification of multiplets of d=1 supersymmetry E. Ivanov, S. Krivonos, O. Lechtenfeld, 2004; etc.
- Microscopic description of the extreme Reissner–Nordström black hole G. Gibbons, P. Townsend, 1999
- $d = 1$ $N = 4$ superconformal group $SU(1,1|2)$ and (super)conformal particles propagating on near horizon black hole backgrounds P. Claus, M. Derix, R. Kallosh, J. Kumar, P.K. Townsend, A. Van Proeyen, 1998; etc.
- $SU(1,1|2)$ as a particular instance of $D(2,1;\alpha)$
- Superconformal mechanics model of A. Galajinsky, 2017 and particle propagating in the near horizon of Reissner–Nordström–AdS-dS black hole
- Near horizon superparticle as a super 0–brane?
  (In the case with vanishing cosmological constant J. Zhou, 1999; M. Kreuzer, J. Zhou 1999)

Bosonic part of $D(2,1;\alpha)$ superalgebra contains conformal $so(2,1)$ and two copies of $su(2)$. Structure relations have the form:

bosonic sector

$$[H,D]=H,\qquad [H,K]=2D,\qquad [D,K]=K,\qquad [\mathcal{J}_a,\mathcal{J}_b]=\epsilon_{abc}\mathcal{J}_c$$

$$[I_-,I_3]=-iI_-,\qquad [I_+,I_3]=iI_+,\qquad [I_-,I_+]=2iI_3,$$

boson-fermionic sector

$$[D,Q_\alpha]=-\frac{1}{2}Q_\alpha\ ,\qquad [D,S_\alpha]=\frac{1}{2}S_\alpha\ ,\qquad [K,Q_\alpha]=S_\alpha\qquad [H,S_\alpha]=-Q_\alpha$$

$$[I_+,Q_\alpha]=-\epsilon_{\alpha\beta}\bar{Q}^\beta,\qquad [I_+,S_\alpha]=-\epsilon_{\alpha\beta}\bar{S}^\beta,\qquad [I_3,Q_\alpha]=\frac{i}{2}Q_\alpha,\quad [I_3,S_\alpha]=\frac{i}{2}S_\alpha,$$

$$[\mathcal{J}_a,Q_\alpha]=\frac{i}{2}(\sigma_a)_\alpha{}^\beta Q_\beta\ ,\quad [\mathcal{J}_a,S_\alpha]=\frac{i}{2}(\sigma_a)_\alpha{}^\beta S_\beta\ ,$$

fermionic sector

$$\{Q_\alpha,S_\beta\}=2i(1+\alpha)\epsilon_{\alpha\beta}I_-,\quad \{\bar{Q}^\alpha,\bar{S}^\beta\}=-2i(1+\alpha)\epsilon^{\alpha\beta}I_+,$$

$$\{Q_\alpha,\bar{Q}^\beta\}=-2iH\delta_\alpha{}^\beta,\qquad \{Q_\alpha,\bar{S}^\beta\}=-2\alpha(\sigma_a)_\alpha{}^\beta\mathcal{J}_a+2iD\delta_\alpha{}^\beta+2(1+\alpha)I_3\delta_\alpha{}^\beta,$$

$$\{S_\alpha,\bar{S}^\beta\}=-2iK\delta_\alpha{}^\beta,\qquad \{\bar{Q}^\alpha,S_\beta\}=2\alpha(\sigma_a)_\beta{}^\alpha\mathcal{J}_a+2iD\delta_\beta{}^\alpha-2(1+\alpha)I_3\delta_\beta{}^\alpha$$

Coset space and invariant action

We choose the subgroup H to be generated by the set $\{D, \mathcal{J}_3, I_\pm, I_3\}$ and construct the coset space G/H with $G = D(2,1;\alpha)$

- Bosonic part is $AdS_2 \times S^2$

For a coset representative $\tilde{G}$ define Maurer-Cartan(MC) one forms

$$\tilde{G}^{-1}d\tilde{G} = HL_H + KL_K + i\left(L_Q Q + \bar{Q}L_{\bar{Q}} + L_S S + \bar{S}L_{\bar{S}}\right) + \mathcal{J}_m L_m$$
$$+DL_D + i(I_+ L_{I_+} + I_- L_{I_-}) + I_3 L_{I_3} + \mathcal{J}_3 L_J$$

Consider the general case group G, its subgroup H and the coset space G/H. The left action of G on the coset space G/H yields the coordinate transformation $Z \to Z'$. Under this transformation MC one-forms on the coset space transform homogeneously

$$L^p(Z') = L^p(Z) + L^q(Z)\varepsilon^C W_C^I(Z) f_{Iq}^p$$

where f_{Iq}^p are the structure constants of a superalgebra at hand, ε^C are the infinitesimal transformation parameters, W_C^I is the H-compensator.

Transformation rules for the bosonic MC one-forms on the coset space

$$L_H \to L_H(1 - \varepsilon^C W_C^D), \quad L_K \to L_K(1 + \varepsilon^C W_C^D),$$
$$L_m \to L_n(\delta_{mn} + \varepsilon^C W_C^{J_3} \epsilon_{3nm})$$

where $W_C^{J_3}$ and W_C^D are H-compensators.
Invariant quadratic combinations

$$L_H L_K, \qquad L_m L_m$$

- Bosonic part of $L_H L_K \sim$ metric on AdS_2 and $L_m L_m \sim$ metric on S^2
- Building blocks for kinetic term

How to construct WZ-term?
MC one-forms on the subgroup H transform as connections

$$L_D \to L_D + df_D, \qquad L_J \to L_J + df_J$$

Invariant action on the coset space of $D(2,1;\alpha)$ supergroup

$$S = -\tilde{m} \int \sqrt{4 L_H L_K - c L_m L_m} - \int (a L_D + b L_J)$$

Kappa–symmetry

Variations of MC one-forms

$$\delta L_H = d[\delta x_H] + [\delta x_D]L_H - L_D[\delta x_H] - 2i\left([\delta\psi]L_{\bar{Q}} - L_Q[\delta\bar{\psi}]\right),$$
$$\delta L_K = d[\delta x_K] - [\delta x_D]L_K + L_D[\delta x_K] - 2i\left([\delta\eta]L_{\bar{S}} - L_S[\delta\bar{\eta}]\right),$$
$$\delta L_D = d[\delta x_D] - 2[\delta x_H]L_K + 2[\delta x_K]L_H + 2i\left([\delta\psi]L_{\bar{S}} - L_Q[\delta\bar{\eta}] + [\delta\eta]L_{\bar{Q}} - L_S[\delta\bar{\psi}]\right),$$
$$\delta L_a = d[\delta x_a] - \epsilon_{abc}[\delta x_b]L_c + 2\alpha\left([\delta\eta]\sigma_a L_{\bar{Q}} - L_S\sigma_a[\delta\bar{\psi}] - [\delta\psi]\sigma_a L_{\bar{S}} + L_Q\sigma_a[\delta\bar{\eta}]\right),$$

where $[\delta Z^A] = \delta Z^M L^A{}_M$ for MC one-form $L^A = dZ^M L^A{}_M$.

κ-symmetry transformations are characterized by vanishing of $[\delta Z^A]$ which are related to the bosonic one–forms on the coset space

$$[\delta x_H] = [\delta x_K] = [\delta x_m] = 0$$

Up to the boundary terms action variation has the form

$$\delta_\kappa S = 2\int\left\{\frac{2i\tilde{m}L_H[\delta\eta] - \alpha c\tilde{m}L_n[\delta\psi]\sigma_n}{\sqrt{4L_HL_K - cL_mL_m}} - i[\delta\psi](a + i\alpha b\sigma_3)\right\}L_{\bar{S}}$$
$$+2\int\left\{\frac{2i\tilde{m}L_K[\delta\psi] + \alpha c\tilde{m}L_n[\delta\eta]\sigma_n}{\sqrt{4L_HL_K - cL_mL_m}} - i[\delta\eta](a - i\alpha b\sigma_3)\right\}L_{\bar{Q}} + h.c.$$

Treating $\delta_\kappa S = 0$ as a linear algebraic equation on $[\delta\eta]$ and $[\delta\psi]$, one finds that it has a solution only when

$$c = \alpha^{-2}, \qquad \tilde{m}^2 = a^2 + (\alpha b)^2$$

- When $\alpha = -1$ our analysis reproduces that in J. Zhou, 1999.

The also has the reparametrization invariance which implies that the number of bosonic dynamical degrees of freedom is three. In its turn, the κ-symmetry reduces the number of (real) fermionic degrees of freedom from eight to four

Gauge fixing and global supersymmetry

Choose the coset parametrization

$$\tilde{G} = e^{tH} e^{zK} e^{i(\psi Q + \bar{Q}\bar{\psi})} e^{i(\eta S + \bar{S}\bar{\eta})} \tilde{G}_R,$$
$$\tilde{G}_R = e^{\phi \mathcal{J}_1} e^{(\theta - \pi/2)\mathcal{J}_2}.$$

Due to the κ–symmetry the number of fermionic degrees of freedom is reduced from eight to four. Hence we impose the gauge fixing condition

$$\eta = \bar{\eta} = 0$$

Gauge fixed MC one-forms

$$L_H = dt - i(\psi d\bar{\psi} - d\psi\bar{\psi}) - (1 + 2\alpha) L_K (\psi\bar{\psi})^2, \qquad L_K = z^2 dt + dz,$$
$$L_a = L_a^0 - 2\alpha L_K (\psi\sigma_b\bar{\psi}) R_{ba}, \qquad L_D = 2z dt$$

where

$$L_1^0 = \sin\theta d\phi, \qquad L_2^0 = d\theta, \qquad L_3^0 = -\cos\theta d\phi$$

and the matrix R_{ab} is defined by

$$\tilde{G}_R^{-1} J_a \tilde{G}_R = R_{ab} J_b$$

Gauge fixing and global supersymmetry

Gauge fixing is compatible with the bosonic symmetries only

$$\delta_H t = 1,$$
$$\delta_D t = -t, \quad \delta_D z = z, \quad \delta_D \psi = -\frac{1}{2}\psi, \quad \delta_D \psi = -\frac{1}{2}\bar{\psi},$$
$$\delta_K t = t^2, \quad \delta_K z = 1 - 2tz, \quad \delta_K \psi = t\psi, \quad \delta_K \psi = t\bar{\psi},$$
$$\delta_a \phi = \sin^{-1}\theta R_{a1}, \quad \delta_a \theta = R_{a2}, \quad \delta_a \psi = \frac{i}{2}\psi\sigma_a, \quad \delta_a \bar{\psi} = -\frac{i}{2}\sigma_a\bar{\psi},$$
$$\delta_{I_3}\psi = \frac{i}{2}\psi, \qquad \delta_{I_3}\bar{\psi} = -\frac{i}{2}\bar{\psi}, \qquad \delta_{I_+}\bar{\psi} = -\psi, \qquad \delta_{I_-}\psi = \bar{\psi}$$

where H, D, K, $\mathcal{J}_a$, $I_\pm$, I_3 stand for time translations, dilatations, special conformal transformations, rotations, and $su(2)$ transformations, respectively.

How to restore global supersymmetry transformations?
We need to find such transformations δ^*_G which leave the gauge invariant

$$\delta^*_G \eta = (\delta_{\tilde{\kappa}} + \delta_G)\eta = 0, \qquad \delta^*_G \bar{\eta} = (\delta_{\tilde{\kappa}} + \delta_G)\bar{\eta} = 0$$

where $\delta_{\tilde{\kappa}}$ is the compensating transformation.

Consider the left action of the group element $e^{\epsilon Q}$ on the coset space with $\eta = \bar{\eta} = 0$

$$e^{\epsilon Q}\tilde{G}|_{\eta=\bar{\eta}=0} = e^{H(t+\delta_Q t)}e^{K(z+\delta_Q z)}e^{iQ(\psi+\delta_Q\psi)+i(\bar{\psi}+\delta_Q\bar{\psi})\bar{Q}}e^{-iz(\epsilon S)}\tilde{G}'_R h$$

with

$$\tilde{G}'_R = e^{J_1(\phi+\delta_Q\phi)}e^{J_2(\theta+\delta_Q\theta+\pi/2)},$$
$$\delta_Q t = z(\epsilon\bar{\psi})(\psi\bar{\psi}) + i(\epsilon\bar{\psi}), \qquad \delta_Q z = -z^2\delta_Q t,$$
$$\delta_Q\phi = 2\alpha z\sin^{-1}\theta(\epsilon\sigma_a\bar{\psi})S_{a1}, \qquad \delta_Q\theta = 2\alpha z(\epsilon\sigma_a\bar{\psi})S_{a2},$$
$$\delta_Q\psi = \epsilon\left(1+iz(\psi\bar{\psi})\right) + i(1+\alpha)z(\epsilon\psi)\psi + z\psi\delta_Q t, \qquad \delta_Q\bar{\psi} = i\alpha z\bar{\psi}(\epsilon\bar{\psi})$$

where ϵ is an infinitesimal parameter of transformation and h is stability subgroup element. From this expression one obtains the transformation rules for the fermionic coordinates

$$\delta_Q\eta|_{\eta=\bar{\eta}=0} = -z\epsilon, \qquad \delta_Q\bar{\eta}|_{\eta=\bar{\eta}=0} = 0$$

which imply compensating transformations

$$\delta_{\tilde{\kappa}}\eta = z\epsilon, \qquad \delta_{\tilde{\kappa}}\bar{\eta} = 0$$

Gauge fixing and global supersymmetry

Using the κ–symmetry conditions

$$[\delta x_H] = [\delta x_K] = [\delta x_m] = 0,$$
$$[\delta\eta] = \kappa, \qquad [\delta\psi] = [\delta\eta]\Omega$$

with

$$\Omega = \frac{\sqrt{4L_H L_K - \alpha^{-2} L_m L_m}}{2\tilde{m} L_K}(a - i\alpha b\sigma_3) + i\alpha^{-1}\sigma_m \frac{L_m}{2L_K}$$

we find compensating transformations for supersymmetry generator Q

$$\delta_{\tilde{\kappa}} t = -iz\left(\epsilon\Gamma\Omega\Gamma^\dagger\bar{\psi}\right), \qquad \delta_{\tilde{\kappa}} z = -z^2\delta_{\tilde{\kappa}} t, \qquad \delta_{\tilde{\kappa}}\phi = \delta_{\tilde{\kappa}}\theta = 0,$$
$$\delta_{\tilde{\kappa}}\psi = z\left(\epsilon\Gamma\Omega\Gamma^\dagger\right) + z\delta_{\tilde{\kappa}} t\psi, \qquad \delta_{\tilde{\kappa}}\bar{\psi} = z\delta_{\tilde{\kappa}} t\bar{\psi},$$

where $\Gamma\Omega\Gamma^\dagger$ is the following matrix:

$$\Gamma\Omega\Gamma^\dagger = \frac{\sqrt{4L_H L_K - \alpha^{-2} L_m L_m}}{2m L_K}\left(a + ib\sigma_a R_{a3}\right) - i\alpha^{-1}\sigma_a R_{am}\frac{L_m}{2L_K}$$

Bosonic part of the action

Making coordinate redefinition

$$t \to \frac{1}{2}\left(t + \frac{1}{r}\right), \qquad z \to r$$

one rewrites the action in the conventional AdS basis

$$S = -\tilde{m}\int dt\left(r^2 - \dot{r}^2/r^2 - \alpha^{-2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right)^{1/2} - \int dt\left(ar - b\cos\theta\dot{\phi}\right)$$

On the other hand, the action of a particle probe propagating near the horizon of the extreme Reissner–Nordström–AdS–dS black hole reads

$$S = -m\left(\frac{2}{V''(r_+)}\right)^{1/2}\int\left(r^2 - \frac{\dot{r}^2}{r^2} - V''(r_+)\frac{r_+^2}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right)^{1/2} -$$
$$-q\int\left(\frac{2Qr}{r_+^2 V''(r_+)} - P\cos\theta\dot{\phi}\right), \qquad V(r) = 1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2} + \frac{1}{3}\Lambda r^2$$

where Q and P are the electric and magnetic charges of the black hole, while m and q are the mass and electric charge of a test particle and r_+ refers to the horizon radius.

Bosonic part of the action

Black hole extremality condition implies

$$V(r_+) = V'(r_+) = 0 \quad \rightarrow \quad Q^2 + P^2 = r_+^2(1 + \Lambda r_+^2), \qquad M = r_+\left(1 + \frac{1}{3}\Lambda r_+^2\right)$$

One thus sees that the bosonic part of the super 0-brane action describes the propagating particle in the near horizon region provided the identification

$$\tilde{m} = m\left(\frac{2}{V''(r_+)}\right)^{1/2}, \qquad a = \frac{2Qq}{r_+^2 V''(r_+)}, \qquad b = qP, \qquad \alpha^{-2} = \frac{r_+^2}{2}V''(r_+)$$

holds, while the κ-symmetry restriction takes the form

$$m^2 = \frac{2q^2Q^2}{r_+^4 V''(r_+)} + \frac{q^2P^2}{r_+^2}$$

Note, that the Reissner–Nordström–AdS–dS black hole can be viewed as a supergravity solution only for the vanishing magnetic charge $P = 0$ and the negative cosmological constant (L. Romans, 1992; M. Caldarelli, D. Klemm, 1999). Thus the test particle propagates on a non-BPS background.

The gauge fixed supersymmetric action

$$S=-\tilde{m}\int\Big[4(z^2-\dot{z})-\alpha^{-2}(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)-4i(\psi\dot{\bar{\psi}}-\dot{\psi}\bar{\psi})-4\alpha^{-1}L^0_m R_{bm}(\psi\sigma_b\bar{\psi})$$

$$-4(2\alpha-1)(\psi\bar{\psi})^2\Big]^{1/2}dt-\int\Big(2az-b\cos\theta\dot{\phi}-2\alpha b(\psi\sigma_a\bar{\psi})R_{a3}\Big)\,dt$$

Introducing the momenta (p_z,p_θ,p_ϕ) canonically conjugate to the bosonic variables (z,θ,ϕ), one finds the Hamiltonian

$$H=z^2p_z+\frac{a^2}{p_z}+2az+\frac{\alpha^2}{p_z}J_aJ_a+2\alpha(\psi\sigma_a\bar{\psi})J_a-p_z(1+2\alpha)(\psi\bar{\psi})^2$$

where

$$J_1=p_\phi,\qquad J_2=p_\theta\cos\phi-p_\phi\cot\theta\sin\phi+b\frac{\sin\phi}{\sin\theta},$$

$$J_3=p_\theta\sin\phi+p_\phi\cot\theta\cos\phi-b\frac{\cos\phi}{\sin\theta}$$

Momenta $(p_\psi,p_{\bar{\psi}})$ canonically conjugate to the fermionic variables $(\psi,\bar{\psi})$ give rise to the second class constraints

$$p_\psi-i\bar{\psi}p_z=0,\qquad p_{\bar{\psi}}-i\psi p_z=0$$

We demonstrate that this model is related with the model constructed in A. Galajinsky, 2017 by a canonical transformation. First, let us change the fermions

$$\psi \to \frac{\psi}{\sqrt{2p_z}}, \qquad \bar{\psi} \to \frac{\bar{\psi}}{\sqrt{2p_z}}, \qquad p_\psi \to \sqrt{2p_z}p_\psi, \qquad p_{\bar{\psi}} \to \sqrt{2p_z}p_{\bar{\psi}}$$

This transformation is a canonical on the constraint surface. Second, we canonically redefine the bosonic coordinate z and its conjugate momentum p_z

$$z \to -\frac{p}{x} - \frac{2a}{x^2}, \qquad p_z \to \frac{x^2}{2}$$

These transformations bring the Hamiltonian and the constraint surface to those in A. Galajinsky, 2017. As a by–product at $\alpha = -1$ one obtains a canonical transformation which links the $SU(1,1|2)$–invariant models in J. Zhou, 1999; M. Kreuzer, J. Zhou 2000 and S. Bellucci, A. Galajinsky, E. Ivanov, S. Krivonos, 2003; A. Galajinsky, 2008.