# Conformal Lagrangians from (formal) near boundary analysis of AdS gauge fields

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# Overview

- Standard approach: Lagrangian for conformal fields arise as logarithmically-divergent term in the on-shell action.

$$S[\phi_0] = \int d\rho d^d x \sqrt{g} S^{bulk}[\phi[\phi_0]]\,, \qquad \frac{1}{\rho^{\Delta_-/2}}\phi\Big|_{z=0} = \phi_0$$

  Drawback: need to know the Lagrangian for a field in AdS.
- Alternative approach: conformal equations can be seen as obstructions of extending an off-shell field on the conformal boundary of the AdS to a bulk on-shell field configuration. Can be formulated entirely at the level of EOM.
- That obstructions are found in explicit form (a simple generating procedure is proposed, to be precise) for conformal linear fields being the leading boundary values of AdS massive and unitary massless fields and Lagrangians (a generating procedure) for them is presented.

# Plan

- Review of the ambient space approach to boundary values
- Mixed symmetry gauge fields on AdS and their boundary values
- Conformal Lagrangians

# Ambient space

$AdS_{d+1}$ space can be realized as a quadric in a flat pseudo-Euclidean space $\mathbb{R}^{d,2}$ with Cartesian coordinates $X^A, A = 0, \ldots, d+1$ and the metric $\eta_{AB} = \text{diag}(- + \ldots + -)$.

$$X^2 = -1,$$

where notation $X \cdot X = \eta_{AB} X^A X^B = X^2$ was used.

Pros: $o(d,2)$ acts linearly.

The conformal boundary $\mathcal{X}$ of $AdS_{d+1}$ can be identified with the quotient of the hypercone $X^2 = 0$ by equivalence relation $X \sim \lambda X,\ \lambda \in \mathbb{R} \backslash 0$.
Can be seen as a surface. E.g. (Minkowski metric)

$$X^2 = 0 \quad X^+ = 1$$

# Ambient scalar

Scalar in $\mathrm{AdS}_{d+1} = \{X \in \mathbb{R}^{d,2} \,|\, X^2 = -1\}$

$$(\nabla^2 - m^2)\varphi(x) = 0$$

can be equivalently described in terms of $\mathbb{R}^{d,2}$

$$\partial_X \cdot \partial_X \Phi(X) = 0,$$

$$\left(X \cdot \frac{\partial}{\partial X} + \Delta\right) \Phi(X) = 0,$$

$$m^2 = \Delta(\Delta - d).$$

Different $\Delta$ correspond to different asymptotic behavior.

# Parent formulation

$$\left(\frac{\partial}{\partial X^A} - \frac{\partial}{\partial Y^A}\right)\Phi = 0,$$

$$\frac{\partial}{\partial Y}\cdot\frac{\partial}{\partial Y}\Phi = 0, \quad \left((X+Y)\cdot\frac{\partial}{\partial Y} + \Delta\right)\Phi = 0,$$

where $\Phi$ is now depends on $Y$.

Interpret the first equation as a covariant constancy condition determined by a particular $iso(d,2)$ connection.

$$\nabla = \mathbf{d} - E^A\frac{\partial}{\partial Y^A} - w^B{}_A Y^A \frac{\partial}{\partial Y^B},$$

namely the one where $E^A = dX^A$, $w^{AB} = 0$.

# Parent formulation

$$\nabla\Phi = 0,$$

$$\frac{\partial}{\partial Y}\cdot\frac{\partial}{\partial Y}\Phi = 0, \quad \left((V(X)+Y)\cdot\frac{\partial}{\partial Y}+\Delta\right)\Phi = 0,$$

where $V^A(X)$ are components of the section of the vector bundle s.t. in the suitable local frame coincide with Cartesian coordinates $X^A$. In particular $V^2 = X^2$.

Compatibility conditions are

$$\mathbf{d}w^A{}_B + w^A{}_C w^C{}_B = 0, \quad \mathbf{d}V^A + w^A{}_B V^B = E^A.$$

# Parent formulation

Idea: to use the ambient space construction in the fiber rather in spacetime.

By pulling back the bundle to the submanifold $X^2 = -1$ we get the system defined *explicitly* on $X^2 = -1$.

Identifying the conformal space $\mathcal{X}$ as a submanifold of the hypercone $X^2 = 0$ we arrive to the formulation in terms of fields defined on $\mathcal{X}$.

That formulation can be considered as a generating procedure for the equations satisfied by boundary values.

# Parent formulation

Let us pick a local coordinate system $x^a$ on $\mathcal{X}$ and the local frame s.t. the only nonvanishing components of the flat connection $w$ are ${w^a}_+ = dx^a, {w^-}_a = -dx_a$ and $V^+ = 1, V^- = V^a = 0$.

$$\nabla = dx^a \left( \frac{\partial}{\partial x^a} - \left(Y^+ + 1\right) \frac{\partial}{\partial y^a} + y_a \frac{\partial}{\partial u} \right),$$

where $u \equiv Y^-$.

$$\nabla \Phi = 0,$$

$$\left( \frac{\partial}{\partial Y^+} \frac{\partial}{\partial u} + \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial y} \right) \Phi = 0, \quad \left( \frac{\partial}{\partial Y^+} + Y \cdot \frac{\partial}{\partial Y} + \Delta \right) \Phi = 0,$$

The first and the third equations have a unique solution for a given $\phi(x, u) = \Phi|_{y^a = Y^+ = 0}$. So in terms of $\phi$ the second implies

$$\frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x} \phi + \frac{\partial}{\partial u} \left( d - 2\Delta - 2u \frac{\partial}{\partial u} \right) \phi = 0$$

# Parent formulation: boundary values

$$\frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x}\phi + \frac{\partial}{\partial u}\left(d - 2\Delta - 2u\frac{\partial}{\partial u}\right)\phi = 0$$

This equation does not impose any constraints on $\phi_0(x) = \phi|_0$ for $\Delta \neq \frac{d}{2} - \ell$ with $\ell \in \mathbb{Z}^{>0}$.

However, if $\Delta = \frac{d}{2} - \ell,\ \ell \in \mathbb{Z}^{>0}$ then $\phi_0$ is subject to

$$\Box^\ell \phi = 0.$$

In other words the parent system is equivalent through the elimination of auxiliary fields to the system of two scalar fields $\phi_0$ subjected to $\Box^\ell \phi_0 = 0$ and unconstrained $\phi_\ell$ ($\ell$-th coefficient in the expansion of $\phi$ in powers of $u$).

# Ambient description of mixed symmetry fields

This picture can be generalized to the case of gauge fields.

Consider generating functions depending on $X^A$ and $P_i^A$, $i = 1, \ldots, n-1$

$$\Phi(X, P) \equiv \Phi(X, P_1, \ldots, P_{n-1})$$

$o(d, 2)$ algebra acts by

$$J_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} + \sum_{i=1}^{n-1} \left( P_{iA} \frac{\partial}{\partial P_i^B} - P_{iB} \frac{\partial}{\partial P_i^A} \right)$$

# Ambient description of mixed symmetry fields

Configurations of a unitary massless field of spin $\{s_1, s_2, \dots, s_{n-1}\}$ (it is assumed that $s_1 \geq s_2 \geq \dots s_{n-1}$ and $n-1 \leq \left[\frac{d}{2}\right]$) are determined by the following constraints:

Algebraic: $\partial_P^i \cdot \partial_P^j \Phi = 0, \quad P_i \cdot \partial_P^j \Phi = 0 \ \ i < j, \quad (P_i \cdot \partial_P^i - s_i)\Phi = 0,$

Tangent: $X \cdot \partial_P^i \Phi = 0,$

Radial: $(X \cdot \partial_X + \Delta)\Phi = 0 \quad \Delta = 1 + p - s,$

EOM and partial gauge: $\Box\Phi = 0, \quad \partial_P^i \cdot \partial_X \Phi = 0.$

Here $p$ denotes the height of the uppermost block in the Young tableau $(s_1, \dots, s_{n-1})$. I.e. $s_1 = \dots s_p > s_{p+1}$.

# Conformal equations

Leading boundary value is determined by boundary data $\phi_{00}(x,p) = \phi_0(x,p,w)|_{w_i=0}$, subjected to

$$(n_i - s_i)\phi_{00} = 0\,, \qquad (\partial_{p_i} \cdot \partial_{p_j})\phi_{00} = 0\,, \qquad (p_i \cdot \partial_{p_j})\phi_{00} = 0 \quad i < j\,.$$

Gauge invariant equations on $\phi_{00}$

$$(\tilde{\Box}^\ell \phi_0)|_{w_i=0} = 0\,, \qquad \phi_0|_{w_i=0} = \phi_{00}\,,$$

$$(\partial_{p_i} \cdot \partial)\phi_0 + \frac{\partial}{\partial w_i}(d + s_i - \Delta - i - \sum_{j \leq i} n_{w_j})\phi_0 + \sum_{i<j}(p_j \cdot \partial_{p_i})\frac{\partial}{\partial w_j}\phi_0 = 0$$

The last equation fixes the $w$-dependence.
So there is a bijection $\pi : \phi_0 \mapsto \phi_0|_{w_i=0}$ between solutions $\phi_0(x,p,w)$ and off-shell fields $\phi_{00}(x,p)$.

# Conformal equations

That conformal equations above have the form $\mathcal{A}\phi_{00} = 0$ for the operator $\mathcal{A}$ that makes the following diagram commutative

$$\begin{array}{ccc} \Phi_0 & \xrightarrow{\tilde{\Box}^{\ell}} & \Phi_0 \\ {\scriptstyle \pi^{-1}}\uparrow & & \downarrow{\scriptstyle \pi} \\ \Phi_{00} & \overset{\mathcal{A}}{\dashrightarrow} & \Phi_{00} \end{array}$$

where $\Phi_{00}$ denotes the space of Lorentz irreducible tensor fields and $\Phi_0$ the space of polynomials in $w_i$ variables with coefficients being smooth functions.
E.g. for $d = 4$, spin 1:

$$\mathcal{A} : \phi_{00} \mapsto (\Box - (p \cdot \partial)(\partial_p \cdot \partial))\phi_{00}$$

In components:

$$\mathcal{A} : p^a \varphi_a \mapsto p^a(\Box \varphi_a - \partial_a \partial^b \varphi_b)$$

# Conformal Lagrangians

Let us consider the inner product

$$\langle \phi, \chi \rangle = \int dx^d \langle \phi, \chi \rangle',$$

where $\langle \cdot, \cdot \rangle'$ is the standard inner product on polynomials, defined by the metric $\eta_{ab}$.

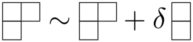
*$\mathcal{A}$ acts on the space of Lorentz-irreducible tensor fields and is formally symmetric.*

$$f(x)^\dagger = f(x), \qquad \partial_a{}^\dagger = -\partial_a, \qquad p_i^{a\dagger} = \eta^{ab} \frac{\partial}{\partial p_i^b}.$$

(Gauge invariant) equations $\mathcal{A}\phi_{00} = 0$ follow from the (gauge invariant) Lagrangian

$$L = \langle \phi_{00}, \mathcal{A}\phi_{00} \rangle = \langle \phi_{00}, (\tilde{\Box}^\ell \phi_0)|_{w_i=0} \rangle = \langle \phi_0, \tilde{\Box}^\ell \phi_0 \rangle|_{w_i=0}.$$

# Example: "hook"-type field



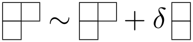
The equations of motion

$$\Box^2\phi_{ab,c} - \Box\partial^e\left(\partial_a\phi_{eb,c} + \partial_b\phi_{ea,c}\right) + \frac{1}{2}\Box\partial^e\left(\partial_a\phi_{bc,e} + \partial_b\phi_{ac,e}\right)$$
$$- 2\Box\partial^e\partial_c\phi_{ab,e} + \frac{1}{2}\left(\eta_{ab}\Box + 2\partial_a\partial_b\right)\partial^e\partial^f\phi_{ef,c}$$
$$- \frac{1}{4}\partial^e\partial^f\left[\left(\eta_{ac}\Box + 2\partial_a\partial_c\right)\phi_{ef,b} + \left(\eta_{bc}\Box + 2\partial_b\partial_c\right)\phi_{ef,a}\right] = 0$$

The gauge transformation

$$\delta\phi_{ab,c} = \partial_a\lambda_{bc} + \partial_b\lambda_{ac} - \frac{1}{3}\partial^e\left(2\eta_{ab}\lambda_{ec} - \eta_{ac}\lambda_{eb} - \eta_{bc}\lambda_{ea}\right)$$

# Example: "hook"-type field



$$L = \int d^4x \Big\langle \phi_{00}, \Big( \Box^2 - \Box(p_1\cdot\partial)(\partial_{p_1}\cdot\partial) - \frac{5}{2}\Box(p_2\cdot\partial)(\partial_{p_2}\cdot\partial)$$

$$+ \frac{5}{3}(p_1\cdot\partial)(\partial_{p_1}\cdot\partial)(p_2\cdot\partial)(\partial_{p_2}\cdot\partial) + \frac{1}{3}(p_1\cdot\partial)^2(\partial_{p_1}\cdot\partial)^2\Big)\phi_{00}\Big\rangle$$

In components ($\phi_{00} = p_1^a p_1^b p_2^c \phi_{abc}$, $\phi_{abc} = \phi_{bac}$, $\phi_{abc} + \phi_{acb} + \phi_{bca} = 0$):

$$\frac{1}{2}L = \phi^{abc}\Box^2\phi_{abc} + 2\partial_e\phi^{ebc}\Box\partial^f\phi_{fbc}$$

$$+ \frac{5}{2}\partial_e\phi^{abe}\Box\partial^f\phi_{abf} + \frac{3}{2}\partial_a\partial_b\phi^{abc}\partial^e\partial^f\phi_{efc}$$

# Fin

Thanks for attention!