VISIBLE AND DARK GROUPS OF SPACE-TIME

Sultan Catto

The Graduate School and University Center
The City University of New York
and
The Rockefeller University

Supersymmetries and Quantum Symmetries- 2017

A remarkable correspondence exists between lattices generated by discrete Jordan algebras and symmetries of superstrings, strongly suggesting that all known superstring theories are related and descend from a more general theory related to the Conway-Sloane transhyperbolic group.

Ten-dimensional space-time with Poincaré group as its "visible" group can be identified with the Cartan torus of the hyperbolic group E_{10} which may be interpreted as the invisible "dark" group surrounding space-time with boundary conditions and quantized momenta. This picture already holds for the 16-dimensional compactified space of heterotic string theory that is identified with the Cartan tori of the groups $E_8 \times E_8$ or $O(32)$ which are supported by the lattices of Euclidean spaces that admit $IO(16)$ as their local "visible" groups. The root lattice of E_{10} can then be combined with the $E_8 \times E_8$ root lattice to be embedded in the Conway-Sloane Lorentzian lattice which becomes associated with an infinite group combining the space-time and internal symmetries of a more general superstring.

The Conway-Sloane lattice and the E_{10} lattice are further associated with discrete Jordan algebras over octonions.

Cartan tori has visible spaces and G/H Cartan generators as dark builders of G and determined by root lattices. E_{10} as the dark group of the visible $(9+1)$ space time with the Lorentz group $O(9+1)$.

In the main geometrical approach to unified field theory of fundamental forces extra dimensions are added to space-time and then shrunk to a size too small to be observed directly. The Kaluza-Klein model of unification for electromagnetism and gravitation attaches a circle as a compactified one-dimension to the 4-dimensional Minkowski spacetime. The circle is the group space of $U(1)$, the gauge group of electromagnetism. How could we generalize the model to unify the $SU(2)$ Yang-Mills field with gravitation? Three options are open:

The first solution is to attach the group space S^3 of $SU(2)$ to the Minkowski space-time. This requires the three compactified dimensions of the 3-sphere as extra dimensions.

Another solution requires the two compactified dimensions of the two-sphere which is the coset space $SU(2)/U(1)$ on which $SU(2)$ acts non-linearly.

In the most economical third solution we still use the Kaluza-Klein circle S^1 as the single extra dimension but now we interpret it as the maximal abelian subgroup of $U(1)$ of $SU(2)$. The group $SU(2)$ has become invisible with its Cartan subgroup $U(1)$ being the visible but unobserved extra dimension.

The three scenarios of the five-dimensional Kaluza-Klein example can be transposed to ten and eleven-dimensional supergravity theories and to the 10-dimensional heterotic string which involves a right-moving 26-dimensional closed string.

In 11-dimensional supergravity seven dimensions must be compactified to reduce the manifold to 4-dimensional space-time.

Since there is no simple group of dimension seven, the first scenario is not applicable unless we introduce the group space of semi simple groups like $SU(2) \times SU(2) \times U(1)$. Applying the second scenario we would associate the compactified dimensions with the seven sphere S^7 which is the coset $SO(8)/SO(7)$ on which $SO(8)$ acts non-linearly and $SO(7)$ linearly. In this approach $SO(8)$ is the largest internal symmetry group that can be incorporated into the 11-dimensional gravity. It has the rank 4 but it is not large enough to accommodate the standard model based on $SU(3) \times SU(2) \times U(1)$ which is not a subgroup of $O(8)$.

Turning now to the third scenario let us compactify the seven dimensions on T^7, the seven dimensional torus $[U(1)]^7$. Since T^7, unlike S^7 is flat, the associated local visible group is $IO(7)$, the seven dimensional Euclidean group. On the other hand T^7 is the Cartan torus of a group of rank 7. As the quantized momenta on T^7 generate a lattice with roots of equal length the corresponding root lattice can be either $SU(8)$ or E_7. Both groups are large enough to admit the group of the standard model as subgroup. They have been considered to study the particle spectrum of 11-dimensional supergravity. We can call $SU(8)$ or E_7 the dark group built around the seven dimensions compactified as a torus in analogy with the dark matter in the spherical halo surrounding a pancake shaped spiral galaxy that can represent the Cartan torus T^7 of these groups, the visible local group of T^7 being $IO(7)$.

Unlike the 11-dimensional supergravity, 10-dimensional gravity is compatible with the presence of matter and chiral theories. The model becomes anomaly free when the Yang-Mills matter is based on the gauge groups $E_8 \times E_8$ or $O(32)$. Both groups share the same Cartan torus T^{16} and their root lattices are even and self dual. Hence, they can arise from the torus compactification of a 16-dimensional euclidean space with visible group $IO(16)$.

This is what happens in superstring theory shown for the right moving sector. In that case these dark groups arise from the torus compactification of the 26-dimensional string to 10 dimensions. Application of the second scenario would lead to compactification on the sphere S^{15} on which $O(16)$ acts non linearly and $O(15)$ linearly. Again the first scenario is not applicable as there is no simple group of dimension 16. Examples of semi-simple groups are $SU(3) \times SU(3)$ and $U(4)$.

All lattices do not represent the root lattices of Lie groups. The 24-dimensional Leech lattice is not a group lattice. Other special lattices are associated with infinite parameter groups like the affine extensions of simple groups which can be interpreted as current algebras on a circle. Some Lorentzian lattices correspond to infinite groups which are hyperbolic extensions of simple groups. We shall show that E_{10}, the hyperbolic extension of E_8 has a root lattice that coincides with a (9+1) Lorentzian lattice which is a discretized form of the 10-dimensional Minkowski space-time. E_{10} arises in supergravity as well as in superstring theories.

It is known that integer octonions generate the root lattices of E_8. We shall show that 2×2 discrete Hermitian octonionic matrices that are elements of the Jordan algebra J_8^2 generate the root system of E_{10}. Finally, discrete hermitian octonionic 3×3 matrices, elements of the exceptional Jordan algebra J_8^3 generate a Lorentzian 26-dimensional lattice associated with an infinite transhyperbolic extension of E_8, which admits $E_{10} \times E_8 \times E_8$ as a sub-lattice.

The lattice of the semi-simple group $E_8 \times E_8 \times E_8$ is another sublattice. This Lorentzian lattice was first introduced by Conway and Sloane in connection with the Monster group which is the largest exceptional finite group. This suggests a deep correspondence between discrete Jordan algebras and some remarkable lattices that are associated with groups, as well as their hyperbolic and transhyperbolic extensions arising in connection with the symmetries of superstring theories.

Relation of root lattices of $O(n)$ ($n = 3, 4, 8$) and E_8 to integer elements of the four division algebras:

Start with the Lie algebra of $O(3) \sim SU(2)$. $U(1)$ is the Cartan subgroup H and the ladder operators are $E_{\pm}$ defined by

$$H = J_3, \qquad E_{\pm} = J_1 \pm i\ J_2$$

with the commutation relations

$$[H, E_{\pm}] = \pm E_{\pm}, \qquad [E_{+}, E_{-}] = 2H.$$

Since the root vectors are the eigenvalues of the Cartan generators H_i with respect to the Lie bracket multiplication, the $SU(2)$ roots are ± 1. There are also as many zero roots as the number of Cartan generators which is equal to the rank of the Lie algebra.

The higher representations are labeled by an integer n equal to $2j$, j being the associated spin. Even integers form the root lattice generated by taking direct products of the adjoint representation. All representations (with dimension $n+1$) form the weight lattice. It is customary to normalize the $SU(n)$ roots to length $\sqrt{2}$. However, we shall give them unit length in order to keep a direct correspondence with integers. The root system consist of the points ± 1 with unit length.

Now consider Hurwitz integers r of unit norm. For real numbers $r = \pm 1$. These correspond to roots of $SU(2)$ and real integers form its root lattice. Gaussian integers of unit norm satisfy

$$m^2 + n^2 = r\bar{r} = 1, \qquad (r = m + i\ n)$$

so that its solutions $r = \pm 1$ and $r = \pm i$ represent the four roots of $O(4) \sim SU(2) \times SU(2)$ with rank two. The root lattice is a two-dimensional rectangular lattice with points labeled by the pair of integers (n, m) which can be represented by the complex gaussian integer r.

$O(8)$ has rank four, so that the root lattice is 4-dimensional. This suggests the generalization of complex numbers to quaternions. Integer quaternions can be defined through the quadratic equation any quaternion q satisfies. If $\bar{q}$ is the conjugate quaternion obtained by reversing the sign of the imaginary units e_i which satisfy

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k, \qquad (i = 1, 2, 3)$$

so that

$$q = q_4 + e_i q_i, \qquad \bar{q} = q_4 - e_i q_i$$

q_4 is the scalar part of q, components q_i refer to the vectorial part.

The quadratic norm $N(q)$ and q_4 are given by

$$N(q) = q\bar{q} = q_4^2 + \vec{q} \cdot \vec{q}$$

$$2\ Sc\ g = q + \bar{q} = 2\ q_4.$$

Multiplying by q we obtain the quadratic equation

$$q^2 - 2(Sc\ q)q + N(q) = 0$$

satisfied by any quaternion q. Since the quaternion algebra is associative q can be represented by 2×2matrix Q through the representation

$$e_j = -i\sigma_j, \qquad (j = 1, 2, 3)$$

with σ_j denoting the Pauli matrices.

Then above equation takes the form

$$Q^2 - (Tr\ Q)Q + Det\ Q = 0$$

and we have

$$[Q(q)]^\dagger = Q(\bar{q})$$

so that

$$Q + Q^\dagger = Tr\ Q, \qquad QQ^\dagger = Det\ Q$$

Now the quaternion q or its representative matrix Q will be called an integer element if the coefficients in above equations are integers and that such quaternions form a ring, with their sums and products also being integers in the sense of satisfying these equations with integer coefficients.

A finite subset is given by integer quaternions of unit norm. There are 24 of them given by

$$\pm 1, \pm e_i \qquad\qquad (8 \; integer \; elements)$$

$$\frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \qquad\qquad (2^4 = 16 \; integer \; elements).$$

These are points on the unit sphere S^3

$$q_4^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

with coordinates that are integers or half-odd integers. They represent the root vectors of $O(8)$, in the same way as the 4 points of S^1 with gaussian integers as coordinates represent the root system of $O(4)$.

In the $O(8)$ case there are 4 Cartan generators H_i (i=1,2,3,4) and we can form the quaternion operator

$$H = H_4 + \vec{e} \cdot \vec{H}$$

with eigenvalues that are root vectors r_i^α represented by the integer quaternions

$$r^{(\alpha)} = r_4^\alpha + \vec{e} \cdot \vec{r}^\alpha$$

and associated with the 24 generators of E_α of $O(8)$. In the Cartan theory a number of roots or E_α generators equal to the rank are selected among positive roots and called principal roots.

These correspond to points on the Dynkin diagram which are joined by one line if the angle between the roots is $2\pi/3$ ($cos\theta = -1$) and not joined if the angle is $\pi/2$ ($cos\theta = 0$). One choice is

$$r^{(\alpha)} = e_\alpha \quad (\alpha = 1,2,3), \qquad r^{(0)} = \frac{1}{2}(1 - e_1 - e_2 - e_3).$$

The angles are given by

$$cos\theta_{(\alpha,\beta)} = Sc\ (r^{(\alpha)}\ \bar{r}^{(\beta)})$$

Hence, we have a Dynkin diagram with $r^{(0)}$ point in the middle joined to 3 points representing $r^{(1)}$,$r^{(2)}$, and $r^{(3)}$ which are not joined to each other. This is exactly the Dynkin diagram of $O(8)$ which exhibits a three-fold symmetry associated with the three quaternion units.

Once we have selected principal roots all the other roots can be obtained by Weyl reflections of one root $r^{(\alpha)}$ with respect to the hyperplane perpendicular to another root $r^{(\beta)}$. The new root $r^{(\alpha,\beta)}$ thus obtained is given by

$$r^{\mu}_{\alpha\beta} = r^{\mu}_{\alpha} - 2\ \frac{r^{\nu}_{\alpha} r_{\beta\nu}}{r^{\lambda}_{\beta} r_{\beta\lambda}}\ r^{\mu}_{\beta}$$

In the $O(8)$ case this is equivalent to the quaternion formula

$$r^{(\alpha,\beta)} = -r^{(\beta)}\ \bar{r}^{(\alpha)}\ r^{(\beta)}$$

each root having unit norm.

These results generalize directly to E_8 on replacing quaternions by octonions.

The E_8 root space and the complete root system of E_8

Integer octonion ω satisfies

$$\omega^2 + a\omega + b = 0,$$

where a and b are ordinary integers .

$$a = -(\omega + \bar{\omega}) = -2Sc\ \omega, \qquad b = N(\omega) = \omega\bar{\omega}.$$

Hence integer octonions have also integer or half odd integer components $\omega_1, \ldots \omega_8$.

Consider unit norm integer octonions

$$\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2+\omega_5^2+\omega_6^2+\omega_7^2+\omega_8^2=1$$

This is the equation of S^7. Thus we are selecting points on the unit S^7 that are integer octonions. Let

$$\begin{array}{lll} l_7=\frac{1}{2}(1+e_7) & l_1=\frac{1}{2}(e_1+e_4) & l_2=\frac{1}{2}(e_2+e_5) \\ l_3=\frac{1}{2}(e_3+e_6) & l_8=\frac{1}{2}(1-e_7) & l_4=\frac{1}{2}(e_1-e_4) \\ l_5=\frac{1}{2}(e_2-e_5) & l_6=\frac{1}{2}(e_3-e_6) & \end{array}$$

Note that these are related to the split octonion units

$$u_0=\frac{1+ie_7}{2}, \qquad u_1=\frac{e_1+ie_4}{2},$$

$$u_2=\frac{e_2+ie_5}{2}, \qquad u_3=\frac{e_3+ie_6}{2},$$

and their conjugates.

Consider expressions $\rho_{rs} = \pm l_r \pm l_s$ $(r \neq s)$, $|\rho_{rs}| = 1$. There are 112 such roots (roots of $O(16)$), also

$$\frac{1}{2}(\pm l_1 \pm l_2 \pm l_3 \pm l_4 \pm l_5 \pm l_6 \pm l_7 \pm l_8)$$

with any odd number of minus signs. There are 128 such roots associated with $E_8/SO(16)$. We have total of 240 roots of E_8. Complex octonions (products of Gaussian unit integers with octonion unit integers) give roots of $E_8 \times E_8$.

Another grouping of the roots with respect to nine l's (adjoining $l_0 = \frac{1}{2}(1 + e_1 + e_2 + e_3)$) gives the decomposition of E_8 with respect to $E_6 \times SU(3)$.

Complex octonions (products of Gaussian unit integers with octonion unit integers) give roots of $E_8 \times E_8$. All roots can be obtained from 8 principal roots $r^{(\alpha)}$ by Weyl reflections mentioned earlier, where the roots are now represented by integer octonions of unit norm. Each principal root corresponds to a point in the E_8 Dynkin diagram branching out of a center point. There are three branches of of length (number of links) $1, 2$ and 4. The center point and its three neighbors represent the $O(8)$ subgroup of E_8, so that the octonionic roots can be chosen to be e_3 for the short arm, e_2 for the first link in the arm of length two and e_1 for the first link of the long arm.

The center point corresponds to the root

$$B_0 = \frac{1}{2}(1 - e_1 - e_2 - e_3)$$

the octonionic analog of $r^{(0)}$. For the end point of the arm of length two we can choose the octonion

$$B_2 = -\frac{1}{2}(1 + e_2 + e_6 - e_4)$$

The second point in the long arm is now

$$B_1 = -\frac{1}{2}(1 + e_1 + e_5 - e_6)$$

which, together with B_2 and the four principal roots of $O(8)$ form the principal roots of E_6, subgroup of E_8. Note that E_6 diagram has a two-fold symmetry with respect to the short arm and does not involve the octonion unit e_7.

Now the E_7 diagram is obtained by adding e_5 to the long arm, next to B_1 and we complete the E_8 diagram by associating the end point of the long arm with the octonion

$$C = -\frac{1}{2}(e_7 + e_4 + e_5 + e_6)$$

This completes the root system of E_8. The weight lattice of E_8 is the same as its root lattice since the adjoint representation is the smallest. It is generated by all integer octonions.

Another grouping of the roots with respect to nine l's (adjoining $l_0 = \frac{1}{2}(1 + e_1 + e_2 + e_3)$) gives the decomposition of E_8 with respect to $E_6 \times SU(3)$.

The root space of $E_8 \times E_8$ has unique properties (self dualities of the Cartan matrix) directly connected with its octonionic structure used in heterotic string theory.

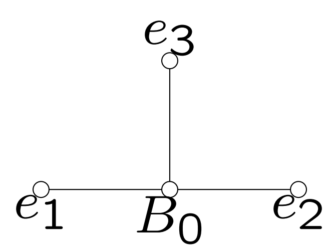

Fig. 1. The $O(8)$ Diagram

Here e_1, e_2 and e_3 are the quaternionic imaginary units or quaternionic subgroup of octonions.

Next we look at the extended $O(8)$ diagram. We will designate it as $\hat{O}(8)$.

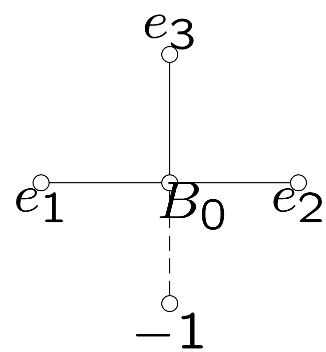

Fig. 2. The $\hat{O}(8)$ Diagram

The extended diagram has four-fold symmetry associated with a discrete subgroup of the norm group $O(4)$ of the quaternion units $1, e_1, e_2, e_3$. It is the diagram of the affine extension $\hat{O}(8)$ of $O(8)$, or $O(8)$ current algebra on S^1.

Next we look at the E_6 and the extended E_6 diagrams.

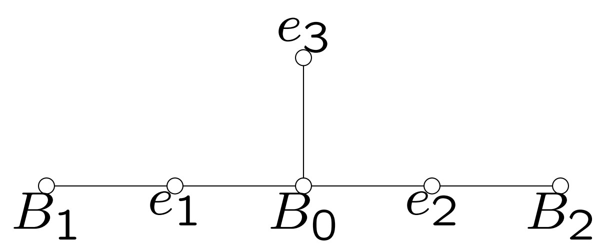

Fig. 3. The E_6 Diagram

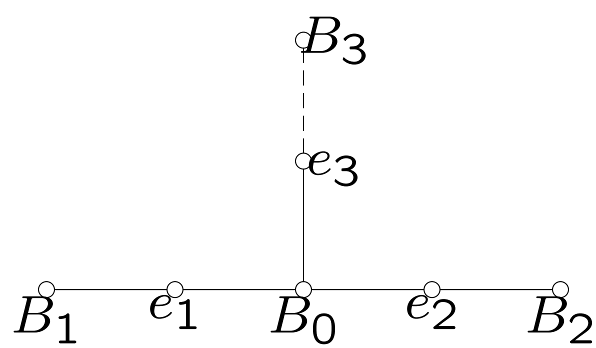

Fig. 4. The extended E_6 Diagram

This diagram has three-fold symmetry. Let

$$u_1 = \frac{1}{2}(e_1 + ie_4), \quad u_2 = \frac{1}{2}(e_2 + ie_5), \quad u_3 = \frac{1}{2}(e_3 + ie_6).$$

$\vec{u}$ is the (3) representation of $SU(3)$ which is the subgroup of the automorphism group G_2 of octonions that leaves e_7 invariant.

$$U\ u = u^{'}, \qquad U\ U^{\dagger} = 1,$$

$U \in SU(3)$ is a complex unitary 3×3 matrix. A special form is

$$U = e^{-\frac{2\pi i}{3}(\frac{\lambda_2 - \lambda_5 + \lambda_7}{\sqrt{3}})} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which leads to

$$B_1 \to B_3$$
$$B_2 \to B_1$$
$$B_3 \to B_2.$$

The E_7 diagram looks like

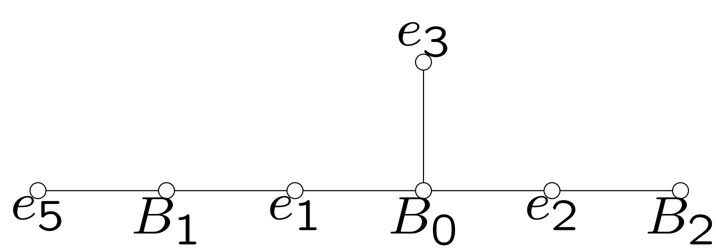

Fig. 5. The E_7 Dynkin Diagram

The E_8 Dynkin diagram is shown below. By Weyl reflections the eight principal roots generate all the 240 roots of E_8.

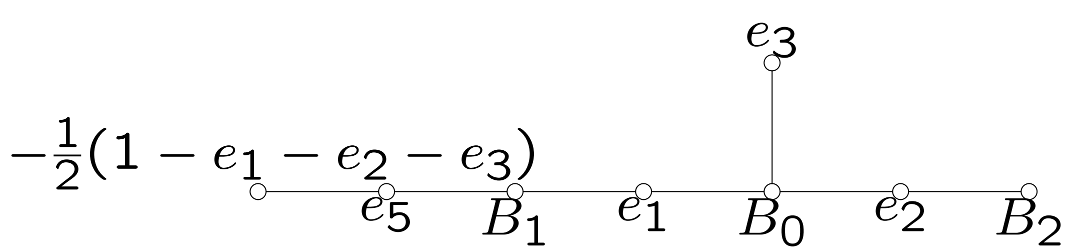

Fig. 6. The E_8 Dynkin Diagram

The extended E_8 Dynkin diagram is shown below. It will be referred as $\hat{E}_8$ or affine extension of E_8 (also called E_9).

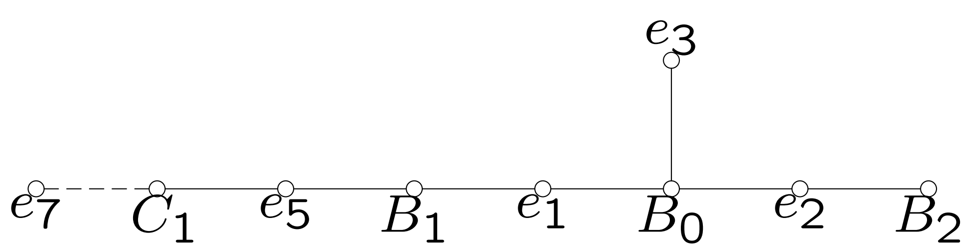

Fig. 7. The Extended E_8 Dynkin Diagram

Before we look deeper into root spaces let us look at the Jordan algebras of 2×2 matrices and continuous and discrete groups associated with them.

Jordan Algebra of 2×2 Hermitian Matrices (J_2):
Consider a 2×2 hermitian matrix of the form

$$A = A^{\dagger} = \begin{pmatrix} \alpha_+ & a \\ \bar{a} & \alpha_- \end{pmatrix}, \qquad \alpha_{\pm} \in \mathcal{R}$$

with $a \in \mathcal{R}, \mathcal{C}, \mathcal{H}$ or $\mathcal{O}$ (or $(J_2^1), (J_2^2), (J_2^4)$, or (J_2^8)).
Commutative product (Jordan multiplication) defined by

$$A.B = B.A = \frac{1}{2}\{AB\} = \frac{1}{2}(AB + BA).$$

Letting $A.B = C$, the associator is

$$[A, B, C] = (A.B).C - A.(B.C)$$

Then the Jordan identity

$$[A, B, A^2] = 0$$

ensures power associativity.

Forexample

$$(A^3)^2 = (A^2)^3 = A^6$$

unambigously, and

$$A^2.A^4 = A^6,$$

etc. We also note that

$$(A.A^2).A^3 = A.(A^2.A^3) = A^6$$

etc. Above equations for $A.B$ and $[A, B, A^2] = 0$ are also satisfied by 3×3 hermitian matrices over all the Hurwitz algebras (octonionic case being called exceptional) giving the algebras $J_3^1, J_3^2, J_3^4, J_3^8$. They are also satisfied by $n \times n$ hermitian matrices over the three associative algebras $\mathcal{R}, \mathcal{C}$ and $\mathcal{H}$. The multiplicative norm for the Jordan algebras is defined by

$$N(J) = Det\ J.$$

Now let us look at the integer elements for 2×2 Jordan algebras:

Consider the 2×2 hermitian matrix A, where we replace $\alpha_\pm$ by α_1 and α_2. The multiplicative norm of A is

$$||A|| = |Det\ A|, \qquad A\bar{A} = I\ Det\ A$$

where

$$\bar{A} = \begin{pmatrix} \alpha_2 & -a \\ -\bar{a} & \alpha_1 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The trace form is

$$\alpha = \alpha_1 + \alpha_2 = Tr\ A = Tr\ \bar{A}.$$

We have

$$A + \bar{A} = I\ Tr\ A.$$

Multiplying by A we have

$$A^2 - A\ Tr\ A + I\ Det\ A = 0.$$

A is an integer element when $Tr\ A$ and $Det\ A$ are integers. Thus $\alpha_1 + \alpha_2$ and $\alpha_1\alpha_2 - |a|^2$ are integers; a is a real integer, gaussian integer, quaternionic or octonionic integer.

When α_1, α_2 and the components of a are real, A represents a vector in the $(m+1,1)$ dimensional Minkowski space (like the momentum vector) which is a representation of $O(m+1,1)$ with $m = 1,2,4$ and 8.

An integer matric is a representation of the infinite discrete subgroup $\Gamma(m)$ of the Hurwitz algebra related to Lorentz group $O(m+1,1)$.

For $m = 1$, Γ is the modular group $SL(2, Z)$.

For $m = 2$, $\Gamma(2)$ is the complex modular (or Kleinian group).

In the case of $m = 4$ it is the discrete subgroup of $SL(2, \mathcal{H})$ or $O(5, 1)$.

Finally for $m = 8$ it is a discrete subgroup of $O(9, 1)$.

If we fix the norm (determinant) to be unity the integer hermitian 2×2 matrix N satisfies

$$N^2 - N \; Tr \; N + I = 0.$$

$$N = \begin{pmatrix} \nu_1 & n \\ \bar{n} & \nu_2 \end{pmatrix},$$

with $\nu_1 + \nu_2$ being an integer, and $\nu_1 \nu_2 - n\bar{n} = \pm 1$.

Hence ν_1 and ν_2 are integers and n is an integer element of Hurwitz algebras.

We have different cases:

First case:

$\nu_1\nu_2 = 0$, with $\nu_1 + \nu_2 \neq 0$. Hence $\nu_1 = 0$ or $\nu_2 = 0$. We can choose $||N|| = n\bar{n} \geq 0$, $n\bar{n} = 1$. Hence n is an integer Hurwitz element of norm 1. It represents a root vector of $O(3), O(4), O(8)$ or E_8. ν_1 (or ν_2) is an integer. Hence we have infinite solutions of above equation that can be associated with the winding number on a circle S^1, so that

$$\begin{pmatrix} \nu_1 & n \\ \bar{n} & 0 \end{pmatrix}, \qquad (n\bar{n} = 1)$$

$\nu_1 \in Z$ represents the loop algebras $\widetilde{O}(3), \widetilde{O}(4), \widetilde{O}(8)$, or $\widetilde{E}_8 = E_9$.

The root system of $\widetilde{O}(m+1,1)$ is obtained from the root system of $O(m)$ represented by the unit discrete transverse vector

$$\begin{pmatrix} 0 & n \\ \bar{n} & 0 \end{pmatrix}$$

by adjoining the longitudinal vector

$$\begin{pmatrix} \nu_1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $\nu_1 = \nu_0 + \nu_{m+1}$ and $\nu_2 = \nu_0 - \nu_{m+1} = 0$.

second case:

The 2×2 discrete matrix of norm

$$\begin{pmatrix} \nu_1 & n \\ \bar{n} & \nu 2 \end{pmatrix}$$

with $n\bar{n} - \nu_1\nu_2 = 1$ (unit integer), and $\nu_1 + \nu_2 =$ (integer), and there are an infinite number of solutions with $n\bar{n} \geq 1$. These correspond to the root systems of the hyperbolic extensions of $O(m)$ with $m = 1, 2, 4$ or 8.

For $m = 8$ we have the infinite parameter group called E_{10}.

Groups associated with 2×2 continuous and discrete Jordan algebras

Let

$$X = X^{\dagger} \in J_2^{(i)}, \qquad X' = L\ X\ L^{\dagger}$$

for $i = 1, 2, 4$. $Det\ L = 1$ and

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the elements belong to $\mathcal{R}, \mathcal{C}$ or $\mathcal{H}$.

Then $L \in SL(2, \mathcal{R})$ which is isomorphic to $SO(2, 1)$, or

$L \in SL(2, \mathcal{C})$ which is isomorphic to $SO(3, 1)$, or

$L \in SL(2, \mathcal{H})$ which is isomorphic to $SO(5, 1)$, which are the Lorentz groups $SO(i + 1, 1)$.

In the octonionic case the linear action of the group on X must also include associators. The ones that are not included in left and right multiplication are in the G_2 automorphism group of the octonion algebra. Dimension of G_2 is 14. A 2×2 unimodular octonionic matrix has $4 \times 8 - 1 = 31$ parameters. The unimodular linear action on $X \in J_2^8$ has $31 + 14 = 45$ parameters, which is $SO(9,1) \sim$ "$SL(2, \mathcal{O})$" (meaning linear action).

Now let us look at the *discrete case*: If X is a discrete element of J_2^i, then a discrete subgroup of L (Lorentz group) will transform it into another discrete element $X^{'}$. Then, for J_2^i, $i = 1, 2, 4$ L must have the form

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad Det\ L = 1,$$

where a, b, c, d are integers $Z(i = 1)$, Gaussian integers $Z_{\mathcal{C}} = m + in$ $(i = 2)$ or quaternionic integers $Z_{\mathcal{H}}(i = 4)$. These are the real modular group $\Gamma \subset SL(2, \mathcal{R})$, the complex modular group $\Gamma_{\mathcal{C}} \subset SL(2, \mathcal{C})$, or the quaternionic modular group $\Gamma_{\mathcal{H}} \subset SL(2, \mathcal{H})$.

In the octonionic case there is also a discrete subgroup Γ_ω of $SO(9, 1)$ that one can regard as subgroup of "$SL(2, \mathcal{O})$". Hence we have the sequence of modular groups $SL(2, Z^{(i)})$, $i = 1, 2, 4, 8$.

Root space of E_9 (affine extension of E_8)

Let

$$X = \begin{pmatrix} x_+ & \xi \\ \bar{\xi} & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} y_+ & \eta \\ \bar{\eta} & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} z_+ & \zeta \\ \bar{\zeta} & 0 \end{pmatrix},$$

and

$$\rho = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Then

$$\frac{1}{2}\{X\rho, Y\rho\} = Z\rho$$

or

$$Z = \frac{1}{2}(X\rho Y + Y\rho X) = \{X\rho Y\} = (X.\rho).Y + X.(\rho.Y) - (X.Y).\rho$$

This is Jacobson's isotopic (with respect to the hermitian metric ρ) Jordan product.

It is also valid for 3×3 matrices.

$$-X\bar{X} = -Det\ X = \xi\bar{\xi}$$

is same as the norm of the transverse part. If x_+ any integer X forms a modular group. If ξ is an octonionic integer then X generates the root system of the affine infinite group (Kac-Moody) E_9 obtained by adjoining to E_8 the light like orthogonal (to roots of E_8) vector

$$\begin{pmatrix} x_+ & 0 \\ 0 & 0 \end{pmatrix}$$

with x_+ an integer.

The E_{10} Diagram and Root Space of E_{10}

The principal roots now generate an infinite number of roots associated with an infinite algebra (hyperbolic extension of E_8). Its infinite subalgebra is E_9 (called affine extension of E_8). Root diagram is shown below:

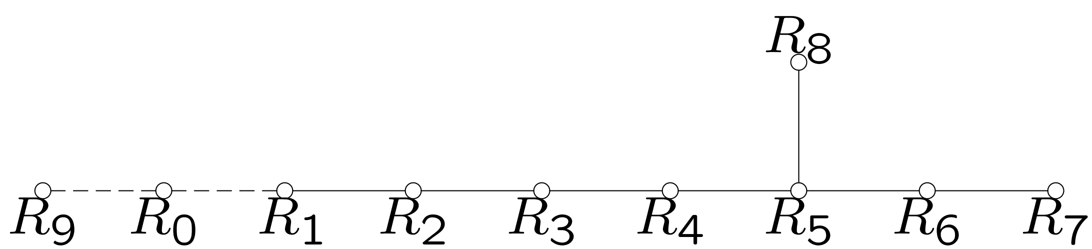

Fig. 8.

Here

$$R_a = \begin{pmatrix} 0 & \rho_a \\ \bar{\rho}_a & 0 \end{pmatrix} \qquad a = 1, \ldots, 8$$

$$R_0 = \begin{pmatrix} 1 & \rho_0 \\ \bar{\rho}_0 & 0 \end{pmatrix}, \qquad R_9 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The roots are R_A, $A = 0, 1, \ldots, 8, 9$;

ρ_0 and ρ_a are integer octonions.

The scalar product is given by

$$(R_A, R_B) = \frac{1}{2} Tr(R_A . R_B)$$

so that

$$(R_a, R_b) = Sc(\rho_a \bar{\rho}_b), \qquad (R_0, R_a) = Sc(\rho_0 \bar{\rho}_a),$$

$$(R_9, R_a) = 0, \qquad (R_9, R_0) = -\frac{1}{2}, \qquad (R_a, R_b) = Sc(\rho_a \bar{\rho}_b).$$

$Sc(\rho_a, \bar{\rho}_b)$ is either $-\frac{1}{2}$ or 0. If $(R_A, R_B) = -\frac{1}{2}$ then A and B are joined; If $(R_A, R_B) = 0$ then A and B are not joined.

The ρ_0 and ρ_a are the principal roots of the extended E_8 diagram. They are given by:

$$\rho_0 = e_7, \qquad \rho_1 = -\tfrac{1}{2}(e_7 + e_4 + e_5 + e_6), \qquad \rho_2 = e_5,$$
$$\rho_3 = B_1 = -\tfrac{1}{2}(1 + e_1 + e_5 - e_6), \qquad \rho_4 = e_1,$$
$$\rho_5 = B_0 = \tfrac{1}{2}(1 - e_1 - e_2 - e_3), \qquad \rho_6 = e_2,$$
$$\rho_7 = B_2 = -\tfrac{1}{2}(1 + e_2 + e_6 - e_4), \qquad \rho_8 = e_3.$$

Integer elements for 3×3 Jordan algebras

Consider

$$J = \begin{pmatrix} \alpha_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \bar{a}_1 & \alpha_3 \end{pmatrix} = J^\dagger.$$

Define the Freudenthal product by

$$J \times J = J^{-1}\ Det\ J.$$

We have the formula that is valid even for $Det\ J = 0$:

$$J \times J = J^2 - J\ Tr\ J - \frac{1}{2} I (Tr\ J^2 - Tr\ J\ \ Tr\ J)$$

where I is the 3×3 identity matrix. Also

$$J.(J \times J) = I\ Det\ J.$$

We also have

$$Tr\ J \times J = -\frac{1}{2}(Tr\ J^2 - Tr\ J\ \ Tr\ J)$$

so that J satisfies the secular equation

$$J^3 - J^2\ Tr\ J + J\ Tr(J \times J) - I\ Det\ J = 0.$$

J will be an integer Jordan matrix if $Tr\ J$, $Tr\ J \times J$ and $Det\ J$ are integers. If $|Det\ J| = 1$ we have again infinite number of solutions representing infinite number of roots (representations of generalizations of the modular group $SL(3, Z)$).

Jacobson triple product for Jordan algebras: Define

$$D = \{A\ B\ C\} = (A.B).C + A.(B.C) - (A.C).B.$$

For ordinary matrices over reals or complex numbers one finds

$$\{A\ B\ C\} = \frac{1}{2}(A\ B\ C + C\ B\ A) = D.$$

D is obviously hermitian if A, B, C are hermitian. Special case when $C = A$

$$\{A\ B\ A\} = A\ B\ A = 2(A.B).A - (A.A).B.$$

Hence

$$Det\{A\ B\ A\} = (Det\ A)^2\ Det\ B.$$

The formula is true for all Jordan algebras including the exceptional Jordan algebra J_3^8.

Weyl reflections for discrete elements of 2×2 and 3×3 Jordan algebras:

Let A such that A^{-1} exists ($Det\ A \neq 0$). Consider elements A and X of J_2 or J_3. Let

$$X^{'} = -\lambda\{A\ \bar{X}\ \bar{A}^{-1}\} \qquad (\bar{X} = X^{-1}\ Det\ X).$$

$\lambda = 1$ for 2×2 matrix, and $\lambda = \frac{Det\ A}{Det\ X}$ for 3×3 matrix. We have

$$|Det\ X^{'}| = |Det\ A|\ |Det\ X|\ |Det\ A|^{-1} = Det\ X$$

so that if $|Det\ X| = 1$, $|Det\ X^{'}| = 1$.

If A and X are integral elements of J_2 and J_3 with unit determinant, they correspond to lattice elements. Then $X^{'}$ is also a lattice element. The special case when $A = X$. Then $X^{'} = -X$ is also a lattice element. Furthermore,

$$X^{'} = X - 2(A^{-1}.X).A$$

which is seen to be a generalization of the Weyl reflection formula

$$B\ \bar{A} + A\ \bar{B} = I\ Tr(A\ \bar{B})$$

and

$$A^{'} = -\frac{B\ \bar{A}\ B}{B\ \bar{B}} = A - \frac{B}{B\ \bar{B}}\ Tr\ (A\ \bar{B}) = -\{B\ \bar{A}\ \bar{B}^{-1}\}.$$

Embedding of $E_{10} \times E_8 \times E_8$ in the 3×3 integer lattice

We write the 3×3 integer lattice associated with the Conway-Sloane lattice and reduce it with respect to the 2×2 integer E_{10} lattice:

$$\begin{pmatrix} k_+ & \gamma & \alpha \\ \bar{\gamma} & k_- & \beta \\ \bar{\alpha} & \bar{\beta} & \ell \end{pmatrix}.$$

Here α, β and γ are integer octonions, and k_+, k_- and ℓ are real integers. We note that the E_{10} lattice sits in the upper left 2×2 part, i.e., in

$$\begin{pmatrix} k_+ & \gamma \\ \bar{\gamma} & k_- \end{pmatrix}$$

which also gives us the quantized space-time momenta in $(9+1)$ Minkowski space-time dimensions;

γ corresponds to E_8 lattice lifting $O(8)$ helicity group; α and β each contain the E_8 internal symmetry groups. $(\alpha \quad \beta)$ part corresponds to $E_8 \times E_8$ lattice (or $O(32)$); ℓ is the lattice of $SL(2, \mathcal{R})$. We note that when $|\alpha| = 1$, and $|\beta| = 1$ give $E_8 \times E_8$. If α is restricted to 112 roots of $O(16)$, then $(\alpha \quad \beta)$ give $O(16) \times O(16)$. If $\alpha\ \bar{\alpha} + \beta\ \bar{\beta} = 1$ and α and β restricted, we have $O(32)$ roots. Generally speaking

$$\begin{pmatrix} 0 & \gamma & \alpha \\ \bar{\gamma} & 0 & \beta \\ \bar{\alpha} & \bar{\beta} & 0 \end{pmatrix}$$

can represent other Niemeier lattices like E_8^3, $E_8 \times O(32)$, etc., or the Leech lattice.

Use of Jordan matrices for the heterotic string

To obtain the heterotic string one starts from two 26 dimensional vectors that arise in closed string theory. Let X_μ^L be left moving and X_μ^R be the right moving that are both solutions of the closed string equations of motion. X_μ^R is compactified as above, while X_μ^L is turned into a superstring by turning 16 of its components X_a^L into the 16 components of a Weyl-Majorana spinor in $D = 10$. This can be done because $X_a^L(\tau, \sigma)$ is a two-dimensional field theory by a process of fermionization.

In the end one has the 10 dimensional bosonic vectors X_α^L, X_α^R ($\alpha = 1, \ldots, 10$), the spinor in $D = 10$: ψ_A ($A = 1, \ldots, 16$) and the group G_r describing an internal local gauge symmetry. $G_r = E_8 \times E_8$ gives the best phenomenology.

Further compactification of $10 - 4 = 6$ dimensions (on Calabi-Yau surfaces or orbifolds) breaks one of the E_8 groups further to E_6 which is known to be one of the possible grand unified theories that include $SO(10)$ and $SU(5)$, thus making contact with particle phenomenology.

There remain fundamental unsolved problems in the scenario, namely the principle for the right compactification (true vacuum), and the cosmological constant problem ($\Lambda = 0$ after supersymmetry breaking).
In the present approach X_μ^L and X_μ^R are identified with the components X^L and X^R of the 27 and $\bar{27}$ dimensional representation of $E_{6,-26}$. The 27th dimension is interpreted as the bosonic ghost dimension introduced by Siegel for a covariant quantization of the 26 dimensional string. $E_{6,-26}$ is a real form of E_6.

Then $X^{L,R}$ have the form

$$X = \begin{pmatrix} V & \psi \\ \psi^\dagger & v \end{pmatrix}$$

where V is a 2×2 matrix representing the 10 dimensional vectors V_L and V_R, ψ is a 2×1 matrix representing the 16 dimensional spinor for X_L and the roots of $E_8 \times E_8$ or $O(32)$ for X_R, and v the ghost variable.

Now a new possibility arises: If V is also compactified on a torus of large radii V can be interpreted as associated with a group. The quantized momenta P_α ($\alpha = 1, \ldots, 10$) would be then identified with the Cartan torus of the hyperbolic infinite group E_{10}.

One then obtains the root lattice of $E_{10} \times E_8 \times E_8$ or $E_{10} \times O(32)$. These are sub lattices of a $26+1$ (Conway-Sloane) lattice or a $26+2$ lattice associated with a new infinite group $\tilde{G}$, shown to exist by Conway as the minimal extension of E_{10}.

This points out to the possibility of a new string theory as a representation of $\tilde{G}$ which would have as special solutions the 3 known heterotic strings ($E_8 \times E_8$, $O(32)$ and $O(16) \times O(16)$) obtained through various breaking of $\widehat{G}$ into its subgroup that contain $E_{10} = \tilde{E}_8$.

The 4 dimensional theory could be obtained by breaking $\tilde{E}_8$ into $\tilde{O}_4$ which contains the Lorentz group $SL(2,\mathcal{C})$ as a subgroup.

Thus E_{10} acquires a new importance if its Cartan subgroup is identified with the $(9+1)$ Minkowski space-time with periodic conditions.

Space time was previously associated with $O(3,1)$ or the coset $O(4,2)/Inh.O(3,1) \times Dilatations$. Now there is the possibility of regarding it as the Cartan subgroup of the hyperbolic group $\tilde{O}(4)$ with rank 4.

This new space is nested in the $(9+1)$ space-time that is identified with the Cartan subgroup of $\tilde{E}_8 = E_{10}$ (hyperbolic extension of E_8) which also appears in the internal sector twice.

Conclusions and Prospects

A new point of view emerges. E_{10} is interpreted as the dark halo groups of the visible $(9+1)$ dimensional space-time allowing it to be integrated with the internal symmetry groups $E_8 \times E_8$, $O(32)$ and $O(16) \times O(16)$ of the heterotic string.

Representations of the Lorentzian root lattice of E_{10} by integer 2×2 octonionic Jordan matrices that also represent quantized momenta in periodic space-time is developed.

Root system of the Conway-Sloane lattice as integer 3×3 exceptional Jordan matrix which has $E_{10}\times E_8\times E_8$, etc. as sublattices.

Expression of the Weyl reflections for E_{10} and the Conway-Sloane lattice by means of Jordan products.

Prospects for writing right moving string and left moving superstring Lagrangians by means of Jordan matrices $J_{L,R}(z)$ in 27 and $\bar{27}$ representations of $E_{6,-26}$ and as cosets G/H conformal fields ($G = E_7$, $H = E_6 \times U(1)$).

THANK YOU!