Complex structures and zero-curvature equations for σ-models

Dmitri Bykov

Max-Planck-Institut für Gravitationsphysik (Potsdam) &

Steklov Mathematical Institute (Moscow)

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Part I. General facts.

Two-dimensional σ-models serve as the theoretical underpinning of string theory. In this talk we will describe a new wide class of models, which are likely to be integrable (in the sense of the inverse scattering method, S-matrix factorization, etc.).

σ-models

The action of a σ-model describing maps X from a 2D worldsheet $\mathscr{C}$ to a target space $\mathscr{M}$ with metric h is given by

$$\mathcal{S} = \frac{1}{2} \int\limits_{\mathscr{C}} d^2 z \, h_{ij}(X) \, \partial_\mu X^i \, \partial_\mu X^j \tag{1}$$

We will assume $\mathscr{M}$ homogeneous:
$\mathscr{M} = G/H$, G compact and semi-simple. We will use the following standard decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{2}$$

where $\mathfrak{m} \perp \mathfrak{h}$ with respect to the Killing metric on $\mathfrak{g}$.

Symmetric target spaces

For a reductive homogeneous space one has the following relations:

$$\begin{aligned} [\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h} \quad &\Rightarrow \quad \mathfrak{h} \text{ is a subalgebra} \\ [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m} \quad &\Rightarrow \quad \mathfrak{m} \text{ is a representation of } \mathfrak{h} \end{aligned}$$

A homogeneous space G/H is called *symmetric* if

$$[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h} \tag{3}$$

Equivalently, there exists a $\mathbb{Z}_2$-grading on $\mathfrak{g}$,
i.e. a Lie algebra homomorphism σ of $\mathfrak{g}$, such that $\sigma(a) = a$ for $a \in \mathfrak{h}$ and $\sigma(b) = -b$ for $b \in \mathfrak{m}$.

Equations of motion. 1

The action of a σ-model with homogeneous target space G/H is globally invariant under the Lie group G. Therefore, there exists a conserved Noether current $K^\mu \in \mathfrak{g}$:

$$\partial_\mu K^\mu = 0 \tag{4}$$

Since the group G acts transitively on its quotient space G/H, the equations of motion are in fact *equivalent* to the conservation of the current.

Equations of motion. 2

It was observed by Pohlmeyer ('76) that in the case when the target space is *symmetric*, the current K is, moreover, flat (with proper normalization):

$$dK - K \wedge K = 0 \tag{5}$$

To get an idea, why this can be the case, recall that the Maurer-Cartan equation has the solution

$$K = -g^{-1}dg, \qquad g \in G \tag{6}$$

What is the relation between g and a point in the configuration space $[\tilde{g}] \in G/H$? The answer is given by Cartan's embedding $G/H \hookrightarrow G$:

$$g = \widehat{\sigma}(\tilde{g})\tilde{g}^{-1} \tag{7}$$

$\widehat{\sigma}$ is a Lie group homomorphism induced by the Lie algebra involution σ.

Equations of motion. 3

Another observation of Pohlmeyer was that the two conditions

$$\begin{aligned} d * K = 0 \qquad & (\text{Conservation}) \\ dK - K \wedge K = 0 \qquad & (\text{Flatness}) \end{aligned} \tag{8}$$

may be rewritten as an equation of flatness of a connection

$$A_u = \frac{1+u}{2} K_z dz + \frac{1+u^{-1}}{2} K_{\bar{z}} d\bar{z}, \tag{9}$$

where we have decomposed the current $K = K_z dz + K_{\bar{z}} d\bar{z}$. We have

$$dA_u - A_u \wedge A_u = 0 \tag{10}$$

This leads to an associated linear system (Lax pair)

$$(d - A_u)\Psi = 0 \tag{11}$$

Integrability

The existence of a linear system described above implies the existence of an infinite number of conserved charges and is often a sufficient condition for the classical integrability of the model.

The linear system was used by Zakharov & Mikhaylov ('79) to solve the equations of motion for the principal chiral model (target space G), with worldsheet $\mathbb{CP}^1$. A more rigorous approach was developed by Uhlenbeck ('89). Solutions of the e.o.m. for σ-models with symmetric target spaces may be obtained by restricting the solutions of the principal chiral model.

These constructions could not be directly generalized to the case of homogeneous, but not symmetric target spaces (no Cartan involution).

Part II. The new models.

Target spaces

We will consider a different class of models, with target spaces $\mathcal{M}$ of the following type:

- $\mathcal{M} = G/H$ is a homogeneous space; for simplicity we take G compact and semi-simple

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

- $\mathcal{M}$ has an integrable G-invariant complex structure $\mathscr{I}$

$$\mathfrak{m} = \mathfrak{m}_+ + \mathfrak{m}_-, \qquad [\mathfrak{h}, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm, \qquad [\mathfrak{m}_\pm, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm$$

- The Killing metric h is Hermitian (i.e. of type $(1,1)$) w.r.t. $\mathscr{I}$

$$h(\mathfrak{m}_\pm, \mathfrak{m}_\pm) = 0$$

Target spaces. 2

Complex homogeneous spaces were classified by Wang ('54) a long time ago. They are toric bundles over flag manifolds.

Consider for simplicity the case of $G = SU(N)$. Then the relevant manifolds are of the form

$$\mathcal{M} = \frac{SU(N)}{S(U(n_1) \times \ldots \times U(n_m))}, \qquad \sum_{i=1}^{m} n_i \leq N\,,$$

If $\sum_{i=1}^{m} n_i = N$, this is the manifold of partial flags in $\mathbb{C}^N$. Otherwise it is a $U(1)^{2s}$-bundle over a flag manifold, where $2s = N - \sum_{i=1}^{m} n_i$.

The action

Given a homogeneous space of the type just described, one can introduce the action of the model: [DB, '16]

$$\mathcal{S} = \int\limits_{\mathscr{C}} d^2 z \, \|\partial X\|^2 + \int\limits_{\mathscr{C}} X^* \omega =$$

$$= \int\limits_{\mathscr{C}} d^2 z \, \left(h_{ij} \partial_\mu X^i \partial_\mu X^j + \epsilon_{\mu\nu} \omega_{ij} \partial_\mu X^i \partial_\nu X^j \right),$$

where $\omega = h \circ \mathscr{I}$ is the Kähler form. Note, however, that, in general, the metric h is <u>not Kähler</u>, hence the form ω is not closed: $d\omega \neq 0$. Therefore the second term in the action contributes to the e.o.m.!

The action

Let K be the Noether current constructed using the above action. As we already discussed, the e.o.m. are equivalent to its conservation:

$$d * K = 0$$

The key observation is that, for the models considered, it is also flat:

$$dK - K \wedge K = 0$$

These two equations mean, in essence, that the described models are sub-models of the principal chiral model (PCM). In particular, the solutions of these models are a subset of solutions of the PCM. The Lax pair representation can be constructed in parallel with the Pohlmeyer procedure.

Relation to the case of symmetric spaces

Complex symmetric spaces fall in our category, with characteristic property $[\mathfrak{m}_+, \mathfrak{m}_+] = 0$. In fact, this implies $[\mathfrak{m}_+, \mathfrak{m}_-] \subset \mathfrak{h}$. Symmetric spaces of the group $SU(N)$ are the Grassmannians

$$\mathbb{G}_{n|N} := \frac{SU(N)}{S(U(n) \times U(N-n))}$$

In this case the canonical one-parametric family of flat connections is

$$\widetilde{A}_\lambda = \frac{1-\lambda}{2}\,\widetilde{K}_z dz + \frac{1-\lambda^{-1}}{2}\,\widetilde{K}_{\bar{z}} d\bar{z},$$

where $\widetilde{K}$ is the canonical Noether current, i.e. the one constructed using the standard action

$$\mathcal{S} = \frac{1}{2}\int\limits_{\mathscr{C}} d^2 z\, h_{ij}(X)\,\partial_\mu X^i\,\partial_\mu X^j \tag{12}$$

Relation to the case of symmetric spaces. 2

The models, which we described above, feature an additional term in their action: $\int\limits_{\mathscr{C}} X^*\omega$, the integral of the Kähler form. Therefore the Noether current K defined using this action will be different from $\widetilde{K}$, the difference being a 'topological' current:

$$K = \widetilde{K} + *dM$$

Nevertheless both K and $\widetilde{K}$ are flat. The one-parametric family of connections that we constructed earlier has the form

$$A_u = \frac{1+u}{2}\,K_z dz + \frac{1+u^{-1}}{2}\,K_{\bar{z}} d\bar{z},$$

A natural question arises: How are $\widetilde{A}_\lambda$ and A_u related?

Relation to the case of symmetric spaces. 3

The answer is: $\widetilde{A}_\lambda$ and A_u are related by a gauge transformation Ω:

$$\widetilde{A}_\lambda = \Omega A_u \Omega^{-1} - \Omega d\Omega^{-1}$$

Ω can be written out explicitly ($\tilde{g}$ is the 'dynamical' group element):

$$\Omega = \tilde{g}\Lambda\tilde{g}^{-1}, \quad \text{where} \quad \Lambda = \mathrm{diag}(\underbrace{\lambda^{-1/2}, \ldots, \lambda^{-1/2}}_{n}, \underbrace{\lambda^{1/2}, \ldots, \lambda^{1/2}}_{N-n})$$

Rather important is the nontrivial relation between the spectral parameters:

$$\lambda = u^{1/2}$$

This relation may be confirmed by analyzing the limiting behavior of the holonomies of the connection as $u \to 0$ (such analysis can be borrowed from Hitchin ('90)).

Part III. Relation to η-deformations.

η-deformed models.

The action of the η-deformed ($\eta \in \mathbb{C}$) principal chiral model has the following form ($J := -g^{-1}dg, \;\; g \in G$): [Klimcik, '02, '09]

$$\mathscr{S}_\eta = \frac{1}{2} \int d^2x \, \langle J_+, \frac{1+\eta^2}{1-\eta \mathscr{R}} \circ J_- \rangle, \tag{13}$$

where $\mathscr{R}$ is a linear operator on the Lie algebra $\mathfrak{g}$, satisfying two equations:
1) "Modified classical Yang-Baxter equation" (MCYBE)

$$[\mathscr{R} \circ a, \mathscr{R} \circ b] - \mathscr{R} \circ ([\mathscr{R} \circ a, b] + [a, \mathscr{R} \circ b]) - [a, b] = 0 \qquad \forall \, a, b \in \mathfrak{g}$$

2) Anti-symmetry condition

$$\langle \mathscr{R} \circ a, b \rangle = -\langle a, \mathscr{R} \circ b \rangle$$

η-deformed models. 2.

There are zero-curvature representations for the e.o.m. of these models. In recent years many attempts were made to apply the deformation to the $AdS_5 \times S^5$ (super)-σ-model [Delduc, Magro, Vicedo; Arutyunov, Borsato, Frolov; van Tongeren; Hoare, Tseytlin; ..., '13^+]

Our principal observation in this direction [DB, '16] is that there is a simple geometric class of solutions to the above two equations: simply take for $\mathscr{R}$ an integrable complex structure $\mathscr{J}$ on the Lie group G, compatible with the Killing metric (for compact simple even-dimensional groups it always exists).

η-deformed models. 3.

We set $\mathscr{R} = \mathscr{J}$. This means $\mathscr{R}^2 = -\mathbb{1}$. Then:

1) MCYBE $\Rightarrow$ Vanishing of the Nijenhuis tensor (integrability of $\mathscr{J}$)

2) Anti-symmetry condition $\Rightarrow$ Compatibility of $\mathscr{J}$ with the metric

As a result, one obtains the deformation of the principal chiral model by a term proportional to the Kähler form on the group G (which is not closed, so it is not a topological term!):

$$\mathcal{S}_\eta = \int\limits_{\mathscr{C}} d^2x \, \|\partial X\|^2 + \eta \int\limits_{\mathscr{C}} X^* \omega$$

η-deformed models. 4.

For a different choice of $\mathscr{R}$ (when $\mathscr{R}^3 = -\mathscr{R}$, and the null space of $\mathscr{R}$ is the Cartan subalgebra of $\mathfrak{g}$), some of the models discussed before (in Part II) may be seen as limits of the η-deformed models as $\eta \to \pm i$.

This limit is somewhat degenerate, as it changes the target space of the model (and even its dimension): $G \to G/H$.

As a result of such a limit, however, one can only obtain target-spaces of the type G/H with *abelian* 'gauge group' H. In our original approach, there is no such constraint on H.

Part IV. Models with graded target spaces.

$\mathbb{Z}_m$-graded spaces. 1.

A $\mathbb{Z}_m$-graded (m-symmetric) space G/H is characterized by the relations

$$\mathfrak{g} = \oplus_{k=0}^{m-1} \mathfrak{g}_k, \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j \bmod m} \tag{14}$$

There exists a Lax representation for $\mathbb{Z}_m$-graded models with the action [Young, '06]

$$\mathcal{S} = \int\limits_{\mathscr{C}} d^2z \, \|\partial X\|^2 + \int\limits_{\mathscr{C}} X^* \widetilde{\omega}, \tag{15}$$

where the B-field is expressed in terms of the $\mathbb{Z}_m$-graded components $J^{(k)}$ of the current:

$$\widetilde{\omega} = \frac{1}{2} \sum_{k=0}^{m} \frac{(m-k)-k}{m} \mathrm{tr}(J^{(k)} \wedge J^{(m-k)}) \tag{16}$$

$\mathbb{Z}_m$-graded spaces. 2.

But: In general, there are many $\mathbb{Z}_m$-gradings on a given Lie algebra $\mathfrak{g}$.

Example: $su(3)$

$$\mathbb{Z}_2: \quad \begin{pmatrix} 0 & \mathbf{0} & 1 \\ 0 & 0 & \mathbf{1} \\ \mathbf{1} & 1 & 0 \end{pmatrix}, \quad \mathbb{Z}_3: \quad \begin{pmatrix} 0 & \mathbf{1} & 2 \\ 2 & 0 & \mathbf{1} \\ \mathbf{1} & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{0} & 1 \\ 0 & 0 & \mathbf{1} \\ \mathbf{2} & 2 & 0 \end{pmatrix},$$

$$\mathbb{Z}_4: \quad \begin{pmatrix} 0 & \mathbf{1} & 2 \\ 3 & 0 & \mathbf{1} \\ \mathbf{2} & 3 & 0 \end{pmatrix}, \quad \mathbb{Z}_5: \quad \begin{pmatrix} 0 & \mathbf{1} & 3 \\ 4 & 0 & \mathbf{2} \\ \mathbf{2} & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1} & 2 \\ 4 & 0 & \mathbf{1} \\ \mathbf{3} & 4 & 0 \end{pmatrix},$$

$$\mathbb{Z}_6: \quad \begin{pmatrix} 0 & \mathbf{1} & 3 \\ 5 & 0 & \mathbf{2} \\ \mathbf{3} & 4 & 0 \end{pmatrix}, \quad \mathbb{Z}_7: \quad \begin{pmatrix} 0 & \mathbf{1} & 3 \\ 6 & 0 & \mathbf{2} \\ \mathbf{4} & 5 & 0 \end{pmatrix}$$

$\mathbb{Z}_m$-graded spaces. 3.

A question arises: Are the models different for different choices of gradings?

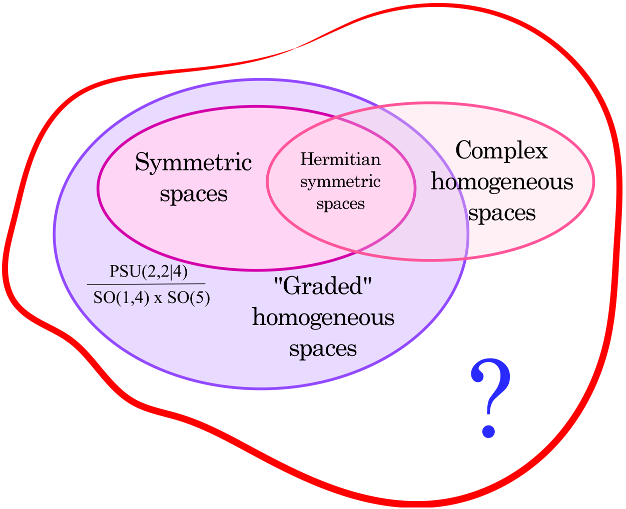
Answer: At least in the present case $[A^{(1)}_{N-1}$ gradings on $\mathfrak{g} = \mathfrak{su}_N]$ they can all be reduced to our model, with an appropriate choice of complex structure (up to a topological term). [DB, '16]

In general, the two classes of models are different: for instance, consider the target space $S^6 = \frac{G_2}{SU(3)}$, which is 3-symmetric but carries no integrable homogeneous complex structure.

Outlook

- Zero-curvature representations were known for σ-models with symmetric target spaces
- We have considered modified σ-models with complex homogeneous target spaces, for which there exist Lax pairs
- A concrete example of such model has been put forward, when the target space is the flag manifold $\frac{U(3)}{U(1)^3}$. When the worldsheet is a sphere $\mathbb{CP}^1$, all solutions of the e.o.m. have been constructed [DB, '15-'16]
- Crucial test of integrability: construct solutions, when the worldsheet is a torus $S^1 \times S^1$ (as in Hitchin ('90) for $\mathcal{M} = SU(2)$)

What is the space of integrable σ-models?



An example: the flag manifold

We will consider the simplest homogeneous, but non-symmetric target space – the flag manifold

$$\mathcal{F}_3 = \frac{U(3)}{U(1)^3} \tag{17}$$

It is the space of ordered triples of lines through the origin in $\mathbb{C}^3$, and can be parametrized by three orthonormal vectors

$u_i, \quad i = 1, 2, 3$
$\bar{u}_i \circ u_j = \delta_{ij}$, modulo phase rotations: $u_k \sim e^{i\alpha_k} u_k$.

Complex structures on the flag manifold

To formulate the model, we need to pick a particular complex structure on $\mathcal{F}_3$. The (co)tangent space to $\mathcal{F}_3$ is spanned at each point by the one-forms

$$J_{ij} := u_i \circ d\bar{u}_j, \quad i \neq j \tag{18}$$

One can pick any three non-mutually conjugate one-forms and *define* the action of the complex structure operator I on them:

$$I \circ J_{12} = \pm i J_{12}, \quad I \circ J_{23} = \pm i J_{23}, \quad I \circ J_{31} = \pm i J_{31} \tag{19}$$

Altogether there are $2^3 = 8$ possible choices, so that there are 8 invariant almost complex structures. However, only 6 of them are *integrable*.

The action simplified

Pick the integrable complex structure $\mathscr{I}$, in which J_{12}, J_{13}, J_{23} are holomorphic one-forms. Then the action can be written as (DB '14)

$$\mathcal{S} = \int d^2z \left(|(J_{12})_{\bar{z}}|^2 + |(J_{13})_{\bar{z}}|^2 + |(J_{23})_{\bar{z}}|^2 \right) \tag{20}$$

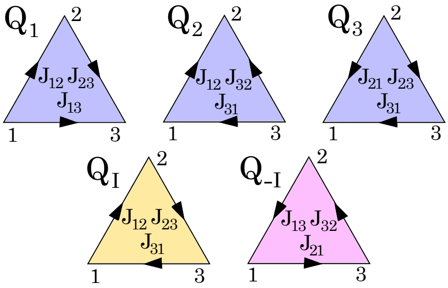
The e.o.m. are:

$$\mathscr{D}_z (J_{12})_{\bar{z}} = 0, \qquad \mathscr{D}_z (J_{31})_{\bar{z}} = 0, \qquad \mathscr{D}_z (J_{23})_{\bar{z}} = 0 \tag{21}$$

From the action (20) it is clear that the holomorphic curves defined by $(J_{12})_{\bar{z}} = (J_{13})_{\bar{z}} = (J_{23})_{\bar{z}} = 0$ minimize the action, hence are solutions of the e.o.m. From (21) it follows that $(J_{12})_{\bar{z}} = (J_{31})_{\bar{z}} = (J_{23})_{\bar{z}} = 0$ is a solution as well. This defines a curve, holomorphic in a different, non-integrable almost complex structure I.

We have seen that the curves, holomorphic in at least two different almost complex structures, satisfy the e.o.m. As we discussed, there are 8 almost complex structures on the flag manifold. Are there any other holomorphic curves that still solve the e.o.m.?

The answer is YES. The relevant complex structures are:



Holomorphic curves. 2

We have already discussed why the $\mathcal{Q}_I$-holomorphic curves and $\mathcal{Q}_1$-holomorphic curves satisfy the e.o.m.

To see why the $\mathcal{Q}_2$- and $\mathcal{Q}_3$-holomorphic curves satisfy the e.o.m., one should note that the differences between the respective Kähler forms are closed forms, i.e. for example $\omega_1 - \omega_2 = \Omega_{top}$ with $d\Omega_{top} = 0$. Therefore the two actions $\mathcal{S}_1$ and $\mathcal{S}_2$ differ by a topological term:

$$\mathcal{S}_1 - \mathcal{S}_2 = \int\limits_{\mathscr{C}} \Omega_{top} \tag{22}$$

This leads to an interesting bound on the instanton numbers of the holomorphic curves. To see this, note that the flag manifold may be embedded as

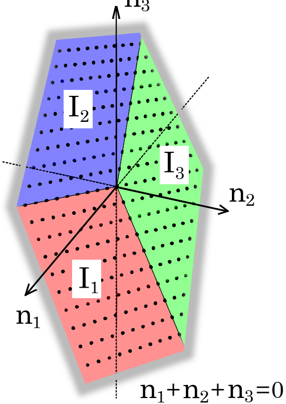
$$i : \mathcal{F}_3 \hookrightarrow \mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2 \tag{23}$$

The second cohomology $H^2(\mathcal{F}_3, \mathbb{R}) = \mathbb{R}^2$ can be described via the pull-backs of the Fubini-Study forms of the $\mathbb{CP}^2$'s, and the corresponding instanton numbers are $n_i = \int\limits_{\mathscr{C}} i^*(\Omega_{FS}^{(i)}),\ i = 1, 2, 3$.

These are subject to the condition

$$n_1 + n_2 + n_3 = 0. \tag{24}$$

The bounds on the topological numbers n_i for the holomorphic curves, which follow from the non-negativity of the actions $\mathcal{S}_i$, are:



Solutions for $\mathscr{C} = \mathbb{CP}^1$

The main point of introducing the action (20) is that, as it turns out, the corresponding Noether current is flat, in full analogy with what happens for σ-models with *symmetric* target-spaces.

The full consequences of this fact still remain to be investigated, but for the moment we can provide a complete description of the solutions of the e.o.m. for the case when the worldsheet $\mathscr{C} = \mathbb{CP}^1$. To describe these solutions, one should recall that there exist three fibrations

$$\pi_i : \mathcal{F}_3 \to (\mathbb{CP}^2)_i, \quad i = 1, 2, 3, \tag{25}$$

each with fiber $\mathbb{CP}^1$.

Solutions for $\mathscr{C} = \mathbb{CP}^1$. 2

All solutions to the e.o.m. are parametrized by the following data:

- One of the projections $\pi_i : \mathcal{F}_3 \to (\mathbb{CP}^2)_i, \quad i = 1, 2, 3$
- A harmonic map $v_{har} : \mathbb{CP}^1 \to (\mathbb{CP}^2)_i$ to the base of the projection
- A holomorphic map $w_{hol} : \mathbb{CP}^1 \to \mathbb{CP}^1$ to the fiber of the projection,

.

For every triple (i, v_{har}, w_{hol}) there exists a solution of the e.o.m., and all solutions are obtained in this way. (DB '15)

The crucial point is that the harmonic maps to the base manifold $\mathbb{CP}^2$ are known explicitly (Din, Zakrzewski '80) (and the holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ are just rational functions).