Fuzzy de Siter Space

Maja Burić
University of Belgrade
Faculty of Physics

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Outline

General motivation to study noncommutative spaces is the idea that spacetime at small scales might have structure different from that of a manifold; perhaps discrete.

Discreteness can be realized in various ways: e.g, as point-like (lattice, simplicial) structure of space, but also through representation of coordinates by operators with discrete spectrum. The latter has an advantage that one can easily introduce symmetries.

Discreteness of spacetime could solve the problem of singularities in general relativity in various ways: by modification of Einstein equations, by absence of singularities in the spectra of coordinates (metric, curvature) in physical states, by uncertainty relations which prevent localization.

The existence of minimal length could also give a natural upper limit for momentum integrations and resolve the UV divergences.

A discrete structure physicists are very familiar with is that of an algebra: algebra of operators in quantum mechanics, Lie algebras, matrix algebras. We shall assume that noncommutative space is an algebra of operators (+ further appropriate conditions).

As we wish, further, to describe physics on noncommutative or quantum spaces, we need to generalize the notion of smoothness i.e. differentiation. Furthermore, we would like to introduce gravity and identify it with geometry. Therefore we need noncommutative differential geometry.

There are several different approaches to this: we work within the noncommutative frame formalism of Madore (Cambridge UP 1995).

This formalism is defined as quantization of the moving frame description of geometry of Cartan. It has a natural, in-built commutative limit.

A paradigmatic example of noncommutative space in which all requirements can be met naturally is the fuzzy sphere.

We shall try to generalize the fuzzy sphere. The motivation is to obtain 4-dimensional noncommutative spacetimes with spherical symmetry and thus find examples of realistic noncommutative configurations of the gravitational field (cosmology, black holes).

We will first give an oveview of basic elements of the noncommutative frame formalism, and then show how can it be used to describe fuzzy de Sitter space using the algebra of de Sitter group SO(1,4) and its unitary irreducible representations.

The talk is based on the work with J. Madore (EPJC 2015) and D. Latas and L. Nenadović (in progress).

Basic ingredients of the NC frame formalism are:

- a set of coordinates x^μ which generate the noncommutative space $\mathcal{A}$

$$[x^\mu, x^\nu] = i\bar{k} J^{\mu\nu}(x)$$

- a set of vector fields e_α which give the free falling frame; they are usually defined by momenta p_α which may but need not belong to $\mathcal{A}$

$$e_\alpha f = [p_\alpha, f], \qquad \Delta f = [p_\alpha, [p^\alpha, f]]$$

- a set of dual 1-forms θ^α which define the differential structure

$$\theta^\alpha(e_\beta) = \delta^\alpha_\beta, \qquad df = (e_\alpha f)\theta^\alpha$$

In addition, we assume that all structures are linear.

On curved commutative manifold the moving frame has the form $e_\alpha f = e^\mu_\alpha\,(\partial_\mu f)$, that is

$$p_\alpha = e^\mu_\alpha\,\partial_\mu, \qquad e^\mu_\alpha = [p_\alpha, x^\mu]$$

which means that momenta are outside $\mathcal{A}$. The space of vector fields is linear and has dimension which is equal to the dimension of spacetime.

Differential calculus is de Rham differential calculus, 1-forms anticommute $\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha$, $d^2 = 0$, etc.

The free falling frame is locally orthonormal

$$g(\theta^\alpha,\theta^\beta) = g^{\alpha\beta} = \eta^{\alpha\beta}, \quad g(dx^\mu, dx^\nu) = g^{\mu\nu} = e^\mu_\alpha\, e^\nu_\beta\,\eta^{\alpha\beta}.$$

On noncommutative space the commutation rules change,

$$[x^\mu, x^\nu] \neq 0, \qquad \{\theta^\alpha, \theta^\beta\} \neq 0$$

but in order to keep the condition of orhonormality of the local frame, $g^{\alpha\beta}$=$\eta^{\alpha\beta}$=const, we assume

$$[f, \theta^\alpha] = 0.$$

Further we impose $d^2 = 0$. Consistency of these geometric relations with algebraic conditions $[x^\mu, x^\nu] = i\hbar J^{\mu\nu}$ gives a number of consistency constraints.

In short: coordinates and their relations define noncommutative space $\mathcal{A}$ and its algebraic properties, momenta and their relations define the differential and geometric properties.

A new element in the noncommutative setup is that momenta can belong to the initial algebra, $p_\alpha \in \mathcal{A}$: the partial derivatives and the de Rham calculus do not play a special role (or do not exist). On matrix spaces derivations are always inner, all p_α are in $\mathcal{A}$.

In fact, dimension of the space of all vector fields is infinite because of noncommutativity: the set of vector fields is not linear over $\mathcal{A}$ as

$$(he_\alpha)(fg) = h(e_\alpha f)g + hf(e_\alpha g) \neq (he_\alpha f)g + f(he_\alpha g)$$

By a specfic choice of p_α we define the tangent space and give it finite dimension, thereby defining also cotangent space and the differential.

To recall the fuzzy sphere. Cartesian coordinates x^a of the fuzzy sphere are the proportional to the generators of the SO(3) algebra,

$$[J^a, J^b] = i\epsilon^{ab}{}_c J^c, \qquad x^a = \frac{\lambda\!\!\!^-}{r}\, J^a$$

in a unitary irreducible representation. The radius of the sphere is related to the SO(3) Casimir.

Momenta p_a are proportional to J_a as well,

$$p_a = \frac{1}{i\lambda\!\!\!^-}\, x_a = \frac{1}{ir}\, J_a .$$

From $dx^a = [p_c, x^a]\,\theta^c$ and $ds^2 = \delta_{\alpha\beta}\theta^\alpha\theta^\beta = g_{ab}dx^a dx^b$, we find

$$g^{ab} = \frac{1}{r^2}\,(r^2\delta^{ab} - x^b x^a).$$

The scalar curvature is constant. Fuzzy sphere has spherical symmetry; cotangent space is 3-dimensional.

Similar construction can be carried over to other homogeneous spaces: we perform it for the 4-dimensional de Sitter space.

In commutative case, de Sitter space can be defined as imbedding

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \frac{3}{\Lambda}$$

in flat 5-dimensional space

$$ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2.$$

It has maximal symmetry, that is SO(1,4).

For fuzzy de Sitter we start with the algebra of the SO(1,4) group which has 10 generators $M_{\alpha\beta}$. Commutation relations are given by

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma} M_{\beta\delta} - \eta_{\alpha\delta} M_{\beta\gamma} - \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\beta\delta} M_{\alpha\gamma})$$

with $\eta_{\alpha\beta} = \mathrm{diag}(+ - - - -)$.

The group has two Casimir operators,

$$\mathcal{Q} = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta}, \qquad \mathcal{W} = -W_\alpha W^\alpha$$

where vector W_α is quadratic in the generators,

$$W_\alpha = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} M^{\beta\gamma} M^{\delta\eta}.$$

$SO(1,4)$ has a well known contraction to the Poincaré group defined by $M_{i4} \to \mu M_{i4}$, $i = 1, 2, 3$, $M_{04} \to \mu M_{04}$ for $\mu = 1/\sqrt{\Lambda} \to \infty$. In this limit W_α becomes the Pauli-Lubanski vector.

The second Casimir relation, $\mathcal{W}$=const, is analogous to the imbedding of the 4-dimensional commutative de Sitter space in 5 flat directions. We identify coordinates as

$$x^\alpha = \ell\, W^\alpha$$

and define fuzzy de Sitter space to be a unitary irreducible representation of the de Sitter algebra.

The quartic Casimir gives the value of the cosmological constant

$$\eta_{\alpha\beta}\, x^\alpha x^\beta = -\ell^2 \mathcal{W} = \frac{3}{\Lambda}\,.$$

There are two choices of momenta that give geometries with de Sitter space in the commutative limit.

One possibility is to choose all de Sitter generators as momenta

$$ip_A = \sqrt{\Lambda/3}\, M_{\alpha\beta}.$$

We fix ℓ accordingly as $\ell = \bar{k}\sqrt{\Lambda}$, $\mathcal{W} = -\frac{3}{\bar{k}^2\Lambda^2}$.

The tangent space is 10-dimensional. The frame components are

$$
\begin{array}{llll}
e^0_j = 0, & e^0_{j+3} = 0 & e^0_{j+6} = \sqrt{\Lambda/3}\, x^j & e^0_{10} = \sqrt{\Lambda/3}\, x \\
e^i_j = -\epsilon^i{}_{jk}\sqrt{\Lambda/3}\, x^k & e^i_{j+3} = \delta^i_j\sqrt{\Lambda/3}\, x^4 & e^i_{j+6} = \delta^i_j\sqrt{\Lambda/3}\, x^0 & e^i_{10} = 0 \\
e^4_j = 0 & e^4_{j+3} = -\sqrt{\Lambda/3}\, x^j & e^4_{j+6} = 0 & e^4_{10} = \sqrt{\Lambda/3}\, x
\end{array}
$$

so for the spacetime components of the metric we obtain

$$g^{\alpha\beta} = \eta^{\alpha\beta} - \frac{\Lambda}{3}\, x^\beta x^\alpha, \qquad R = 4\Lambda$$

The other possibility is to introduce 4 momenta:

$$i\tilde{p}_0 = \sqrt{\Lambda}\, M_{04} \qquad i\tilde{p}_i = \sqrt{\Lambda}\,(M_{i4} + M_{0i}),$$

and denote the corresponding differential by $\tilde{d}$. We find the metric

$$\tilde{d}s^2 = -(\tilde{\theta}^0)^2 + (\tilde{\theta}^i)^2 = -\tilde{d}\tau^2 + e^{2\tau}\,\tilde{d}x^i\tilde{d}x^i$$

with time $\tau = -\log(x^0 + x^4)$

One finds that $R = 3\Lambda/4$.

The second fuzzy geometry does not have full de Sitter symmetry while the first does. The cosmological constant has values related to $\mathcal{W}$.

Conclusions 1

- It is possible to represent spaces of maximal symmetry by Lie algebras of the corresponding symmetry group. In the case of even-dimensional spaces, one uses imbedding in higher-dimensional flat space; coordinates are components of the W-symbol of the highest rank, vector $W^\alpha = \epsilon^{\alpha\cdots} M_{\alpha .} \dots M_{..}$
- Fuzzy frame and metric with the exact or limiting symmetry can be obtained. Physical quantities like radius or cosmological constant are related to the values of Casimir operators in the irreducible representations of the algebra. There are usually double-scaling limits as we typically have two dimensional constants, $\bar{k}$ (noncommutativity) and Λ (cosmological constant), $\hbar = 1$.
- Details about the spectra of coordinates depend on the representations.

Unitary irreducible representations of de Sitter group are known, found by Thomas (AM 1941), Newton (AM 1950), Dixmier (BSMF 1961) by induction from (k, k') representations of the SO(4) subgroup. They are all infinite-dimensional as SO(1,4) is noncompact.

They can be denoted by two quantum numbers (s and ρ (ν, q)) and fall into the following categories

- Principal continuous series: $\rho \geq 0$, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

  $\mathcal{Q} = -s(s+1) + \frac{9}{4} + \rho^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} + \rho^2)$

- Complementary continuous series: $\nu \in R, \ \ |\nu| < \frac{3}{2}, \ \ s = 0, 1, 2 \dots$

  $\mathcal{Q} = -s(s+1) + \frac{9}{4} - \nu^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} - \nu^2)$

- Discrete series: $s = \frac{1}{2}, 1, \frac{3}{2}, 2 \dots, \ \ q = s, s-1, \dots 0 \text{ or } \frac{1}{2}$

  $\mathcal{Q} = -s(s+1) - (q+1)(q-2), \quad \mathcal{W} = -s(s+1)q(q-1)$

$(\nu = i\rho$, $q = 1/2 + i\rho)$

In order to determine the spectrum of W^α (that is x^α) one could use known matrix elements of $M_{\alpha\beta}$ in specific UIR's in the SO(4) basis $f^{k,k'}_{m,m'}$ and solve the eigenvalue problem (on the computer). Even that is difficult,

$$W_0\, f^{k,k'}_{m,m'} = \Big(k'(k'+1) - k(k+1)\Big) f^{k,k'}_{m,m'}$$

$$W_4\, f^{k,k'}_{m,m'} =$$

$$-\frac{i}{2} A_{k,k'}(k-k')\Big(\sqrt{(k-m+1)(k'+m'+1)}\, f^{k+\frac{1}{2},k'+\frac{1}{2}}_{m-\frac{1}{2},m'+\frac{1}{2}} - \sqrt{(k+m+1)(k'-m'+1)}\, f^{k+\frac{1}{2},k'+\frac{1}{2}}_{m+\frac{1}{2},m'-\frac{1}{2}}$$

$$-\frac{i}{2} B_{k,k'}(k+k'+1)\Big(\sqrt{(k+m)(k'+m'+1)}\, f^{k-\frac{1}{2},k'+\frac{1}{2}}_{m-\frac{1}{2},m'+\frac{1}{2}} + \sqrt{(k-m)(k'-m'+1)}\, f^{k-\frac{1}{2},k'+\frac{1}{2}}_{m+\frac{1}{2},m'-\frac{1}{2}}\Big)$$

$$-\frac{i}{2} C_{k,k'}(k+k'+1)\Big(\sqrt{(k-m+1)(k'-m')}\, f^{k+\frac{1}{2},k'-\frac{1}{2}}_{m-\frac{1}{2},m'+\frac{1}{2}} + \sqrt{(k+m+1)(k'+m')}\, f^{k+\frac{1}{2},k'-\frac{1}{2}}_{m+\frac{1}{2},m'-\frac{1}{2}}\Big)$$

$$-\frac{i}{2} D_{k,k'}(k-k')\Big(\sqrt{(k+m)(k'-m')}\, f^{k-\frac{1}{2},k'-\frac{1}{2}}_{m-\frac{1}{2},m'+\frac{1}{2}} - \sqrt{(k-m)(k'+m')}\, f^{k-\frac{1}{2},k'-\frac{1}{2}}_{m+\frac{1}{2},m'-\frac{1}{2}}\Big)$$

Constants $A_{k,k'}$, $B_{k,k'}$, $C_{k,k'}$, $D_{k,k'}$ are known and depend on the representation.

For the Class I UIR's (principal and complementary series) there are Hilbert space representations, which were for example used to construct coherent states (Gelfand, Fizmatgiz 1958, Naimark, Fizmatgiz 1958, Vilenkin, Nauka 1965, Perelomov, Springer 1986). These representations are the simplest to work with. However, they have $W^0 = 0$, which implies $W^\alpha = 0$, that is $\mathcal{W} = 0$. From the point of view of our geometric representation, this property is unwanted.

Yet, starting from Class-I representations one can build higher spin UIR's. Moylan (JMP 1982, 1984) found a Hilbert space representation of all UIR's of the principal series, using the representation spaces of the UIR's of the Poincaré group. The representation is very interesting for itself and allowed us to solve the eigenvalue problem of coordinates in the simplest nontrivial case, s=1/2, and to obtain result: all W^α have discrete spectrum.

To represent the SO(1,4) Moylan uses the sum of two irreducible representations of the Poincaré group proposed by Bargmann and Wigner (PNAS 1948). We will describe one summand for cases s=0, 1/2.

The states for s=0 are the wave functions in momentum representation $\psi(\vec{p})$ with the scalar product given by

$$(\psi, \psi) = \int \frac{d^3 p}{2p_0} \, \psi^* \psi .$$

The group generators are

$$L_{ij} = i \left(p_i \frac{\partial}{\partial p^j} - p_j \frac{\partial}{\partial p^i} \right)$$

$$L_{0i} = i p_0 \frac{\partial}{\partial p^i} \quad \text{with} \quad p_0 = \sqrt{-p_i p^i + m^2}$$

$$L_{40} = -\frac{\rho}{m} p_0 + \frac{1}{2m} \{p^i, L_{0i}\}$$

$$L_{4k} = -\frac{\rho}{m} p_k - \frac{1}{2m} \{p^0, L_{0k}\} - \frac{1}{2m} \{p^i, L_{ik}\}$$

They are hermitian with respect to this scalar product and one can easily check that $W^\alpha = 0$: this is Class I representation.

The states for s=1/2 are Dirac bispinors in momentum representation $\psi(\vec{p})$, with the Bargmann-Wigner scalar product given by

$$(\psi, \psi) = \int \frac{d^3 p}{2p_0}\, \psi^\dagger \gamma^0 \psi.$$

This scalar product is positive on the subspace of solutions of the Dirac equation, i.e. in the Poincaré group UI representation space.

The group generators are

$$M_{ij} = L_{ij} + S_{ij}, \qquad S_{ij} = \frac{i}{4}\,[\gamma_i, \gamma_j]$$

$$M_{0i} = L_{0i} + S_{0i}, \qquad S_{0i} = \frac{i}{4}\,[\gamma_0, \gamma_i]$$

$$M_{40} = -\frac{\rho}{m}\, p_0 + \frac{1}{2m}\,\{p^i, M_{0i}\}$$

$$M_{4k} = -\frac{\rho}{m}\, p_k - \frac{1}{2m}\,\{p^0, M_{0k}\} - \frac{1}{2m}\,\{p^i, M_{ik}\}$$

Notice that the Bargmann-Wigner scalar product is counterintuitive as, with respect to it, both hermitian matrices S_{ij} and the antihermitian S_{0i} are hermitian operators. Thus the eigenvalues of an antihermitian matrix can be real (with respect to this scalar product).

If the operator is represented by the operator-valued matrix M, the condition of its hermiticity is

$$\gamma_0\, p_0^{-1} M = M^\dagger p_0^{-1} \gamma_0 .$$

We use the Dirac representation of γ-matrices and obtain the $M_{\alpha\beta}$'s by tensoring them with the momentum-space representation for s=0.

We calculate coordinates:

$$W^0 = -\frac{1}{2m}\begin{pmatrix} (\rho-\frac{i}{2})p_i\sigma^i + i\,p_0^2\,\frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_0 p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2}p_0 \\ \epsilon^{ijk}p_0 p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2}p_0 & (\rho-\frac{i}{2})p_i\sigma^i + ip_0^2\,\frac{\partial}{\partial p^i}\,\sigma^i \end{pmatrix}$$

$$W^4 = -\frac{1}{2}\begin{pmatrix} ip_0\,\frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2} \\ \epsilon^{ijk}p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2} & ip_0\,\frac{\partial}{\partial p^i}\sigma^i \end{pmatrix}$$

$$W^i = \begin{pmatrix} W^i_{11} & W^i_{12} \\ W^i_{12} & W^i_{11} \end{pmatrix},$$

$$W^i_{11} = \frac{p_0}{2m}\left(-ip^i\frac{\partial}{\partial p_k}\,\sigma_k + (\rho-\frac{i}{2})\,\sigma^i\right)$$

$$W^i_{12} = \frac{1}{2m}\left(-2ip^i + i\epsilon^{ijk}p_m\sigma^m M_{jk} + \epsilon^{ijk}(i\rho p_j - p_m p^m\frac{\partial}{\partial p^j} - p_j p_m\frac{\partial}{\partial p_m})\sigma_k\right).$$

In order to find the spectra we need to solve the eigenvalue problems. From the other representation we know that W^0 has discrete spectrum in all UIR's. On the other hand, because of de Sitter symmetry, the spectra of all space-like components of W^α are the same, therefore we need to solve the eigenvalue problem of W^4, which is the simplest, only. It would be also nice to find eigenvalues of $W^0 + W^4$ which was, in our second geometry, singled out as time ore more precisely, $e^{-\tau}$.

We proceed in several steps. First, by a unitary transformation we transform W_4 to a block-diagonal form. We choose the eigenvectors in every block as spinors $\psi(\vec{p}) = \frac{f(p)}{p}\,\varphi_{jm} + \frac{h(p)}{p}\,\chi_{jm}$, with

$$\varphi_{jm} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}\, Y^{m-1/2}_{j-1/2} \\ \sqrt{\frac{j-m}{2j}}\, Y^{m+1/2}_{j-1/2} \end{pmatrix}, \quad \chi_{jm} = \begin{pmatrix} \sqrt{\frac{j+1-m}{2(j+1)}}\, Y^{m-1/2}_{j+1/2} \\ -\sqrt{\frac{j+1+m}{2(j+1)}}\, Y^{m+1/2}_{j+1/2} \end{pmatrix}$$

Y^m_l are spherical harmonics in momentum space.

After the change of variables

$$f(p) = p_0^{-i\rho-1/2}\, p^{j+1/2}\tilde{f}(p), \quad h(p) = p_0^{-i\rho-1/2}\, p^{-j-1/2}\tilde{h}(p)$$

we obtain hypergeometric equations

$$p^2\frac{d^2\tilde{f}}{dp_0^2} + 2(j+1)p_0\frac{d\tilde{f}}{dp_0} - (2i\lambda + j)(2i\lambda - j - 1)\tilde{f} = 0$$

$$p^2\frac{d^2\tilde{h}}{dp_0^2} - 2jp_0\frac{d\tilde{h}}{dp_0} - (2i\lambda + j)(2i\lambda - j - 1)\tilde{h} = 0$$

The solutions are, e.g.

$$\tilde{f} = F\left(j + 2i\lambda, j + 1 - 2i\lambda; j + 1; \frac{p_0 + m}{2m}\right).$$

This function reduces to a polynomial for integer or half-integer values of $2i\lambda$, which is, because of hermiticity, real: $i\lambda$ is quantized. Integrability of the solution gives the range of possible values.

Conclusions 2

- The spectra of W^0, W^i and W^4 are discrete, and therefore this representation gives fuzzy de Sitter space.
- One should check the spectrum of time τ. The corresponding differential equations have also three singular points, of which one is irregular, so by know we were not able to find solutions.
- Principal series representations of higher spin s should behave similarly: they can also be obtained explicitely using the Poincaré algebra representations.