# Nested Bethe ansatz for RTT–algebra of $\mathrm{sp}(4)$ type

Čestmír Burdík

JINR, Dubna and ČVUT, Prague

*In collaboration with Ondřej Navrátil*

# Main result

We study the highest weight representations of RTT–algebras for R–matrix of $\mathrm{sp}(4)$ type by the nested algebraic Bethe ansatz. Such models were solved by Reshetikhin in $\mathrm{sp}(2n)$ but using a very special type of representation. The explicit construction of Bethe vectors is done in the general case and the explicit formulae for Bethe equations we obtained.

The formulation of the quantum inverse scattering method, or algebraic Bethe ansatz, by the Leningrad school provides eigenvectors and eigenvalues of the transfer matrix. The latter is the generating function of the conserved quantities of a large family of quantum integrable models. The transfer matrix eigenvectors are constructed from the representation theory of the RTT–algebras. In order to construct these eigenvectors, one should first prepare Bethe vectors, depending on a set of complex variables. The first formulation of Bethe vectors for $\mathrm{gl}(n)$-invariant models was given by P.P. Kulish and N.Yu. Reshetikhin in [4] where the nested algebraic Bethe ansatz was introduced. These vectors are given by recursion on the rank of the algebra, and the use of the embedding $\mathrm{gl}(n-1) \subset \mathrm{gl}(n)$ and for $\mathrm{gl}(2)$ case is well know. We will describe this construction for know case of RTT–algebra of $\mathrm{gl}(3)$ type. The crucial fact for this construction is that RTT–algebra of $\mathrm{gl}(2)$ type is RTT–subalgebra of RTT-algebra $\mathrm{gl}(3)$.

# Plan

- Introduction
- RTT–algebra of $\mathrm{gl}(3)$ type
- RTT–algebra os $\mathrm{sp}(4)$ type
- Conclusion

[4] P. P. Kulish, N. Yu. Reshetikhin, Diagonalization of GL(N) invariant transfer matrices and quantum N-wave system (Lee model), J. Phys. A: 16 (1983) L591  L596.

In our paper we consider the normalized R–matrix of $\mathrm{gl}(n)$ type

$$\mathbf{R}(x,y) = \frac{x-y}{x-y+1}\,\mathbf{I}\otimes\mathbf{I} + \frac{1}{x-y+1}\,\mathbf{P} = \frac{1}{f(x,y)}\,\mathbf{I}\otimes\mathbf{I} + \frac{g(x,y)}{f(x,y)}\,\mathbf{P}\,, \qquad (1)$$

where

$$\mathbf{I} = \sum_{i=1}^{n}\mathbf{E}_i^i\,, \qquad \mathbf{P} = \sum_{i,k=1}^{n}\mathbf{E}_k^i\otimes\mathbf{E}_i^k\,, \qquad g(x,y) = \frac{1}{x-y}\,, \qquad f(x,y) = \frac{x-y+1}{x-y}\,,$$

and then the R–matrix of $\mathrm{sp}(4)$ type,

$$\mathbf{R}(x,y) = \frac{1}{x-y+1}\left((x-y)\mathbf{I}\otimes\mathbf{I} + \mathbf{P} - \frac{x-y}{x-y+3}\,\mathbf{Q}\right) = \tag{2}$$
$$= \frac{1}{f(x,y)}\Big(\mathbf{I}\otimes\mathbf{I} + g(x,y)\mathbf{P} - h(x,y)\mathbf{Q}\Big),$$

where

$$\mathbf{I} = \sum_{k=-2}^{2}\mathbf{E}_k^k\,, \qquad \mathbf{P} = \sum_{i,k=-2}^{2}\mathbf{E}_k^i\otimes\mathbf{E}_i^k\,, \qquad \mathbf{Q} = \sum_{i,k=-2}^{2}\theta_{i,k}\mathbf{E}_k^i\otimes\mathbf{E}_{-k}^{-i},$$
$$h(x,y) = \frac{1}{x-y+3}\,, \qquad \theta_{i,k} = \mathrm{sgn}(i)\cdot\mathrm{sgn}(k)\,.$$

In these formulas $\mathbf{E}_i^k$ are elementary matrices with 1 in the $i$–th row and $k$-th collum and 0 elsewhere. So $\mathbf{I}$ is the unit matrix and the relations $\mathbf{E}_k^i\mathbf{E}_s^r = \delta_s^i\mathbf{E}_k^r$ are hold.

These R–matrices fulfill the Yang–Baxter equation

$$\mathbf{R}_{1,2}(x,y)\mathbf{R}_{1,3}(x,z)\mathbf{R}_{2,3}(y,z) = \mathbf{R}_{2,3}(y,z)\mathbf{R}_{1,3}(x,z)\mathbf{R}_{1,2}(x,y) \tag{3}$$

The RTT–algebra is an associative algebra $\mathcal{A}$ with unity, which is generated by the elements $T^i_k(x)$, where $T^i_k(x)$ are defined by means of the monodromy matrix

$$\mathbf{T}(x) = \sum_{i,k} \mathbf{E}^k_i \otimes T^i_k(x)\,,$$

for which the RTT–equation

$$\mathbf{R}_{1,2}(x,y)\mathbf{T}_1(x)\mathbf{T}_2(y) = \mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(x,y) \tag{4}$$

holds.

Next we define the elements

$$H(x) = \mathrm{Tr}\big(\mathbf{T}(x)\big) = \sum_i T^i_i(x)\,.$$

Since in both cases R–matrices are invertible, we obtain from the RTT–equation that the operators $H(x)$ and $H(y)$ commute for any $x$ and $y$, i.e. for any $x$ and $y$ the relation $H(x)H(y) = H(y)H(x)$ is valid.

We assume that in the vector space of the representation of the RTT–algebra $\mathcal{A}$ there is an element $\omega$, the vacuum vector for which is

$$T^i_k(x)\omega = 0 \quad \text{for} \quad i > k\,, \qquad \text{(or for } i < k\text{)}$$
$$T^i_i(x)\omega = \lambda_i(x)\omega\,.$$

In the vector space $\mathcal{W} = \mathcal{A}\omega$ we will search eigenvectors of the operators $H(x)$, i.e. non-zero elements $w \in \mathcal{W}$ which for any $x$ solution of the equation are.

$$H(x)w = E(x)w\,.$$

In the case of $\mathrm{gl}(n)$ relations the RTT–equations (4) give us

$$\begin{aligned} T^i_k(x)T^r_s(y) + g(x,y)T^r_k(x)T^i_s(y) &= T^r_s(y)T^i_k(x) + g(x,y)T^r_k(y)T^i_s(x) \\ T^i_k(x)T^r_s(y) + g(y,x)T^i_s(x)T^r_k(y) &= T^r_s(y)T^i_k(x) + g(y,x)T^i_s(y)T^r_k(x) \end{aligned} \tag{5}$$

It is easy to see that the RTT–algebra of $\mathrm{gl}(n-1)$ is a RTT–subalgebra of $\mathrm{gl}(n)$. This fact is used by Kulish and Reshetikhin for formulation of the Nested Bethe ansatz in the paper [4].

In the case of the R–matrix of $\mathrm{sp}(4)$ type,the RTT–algebra $\mathcal{A}$ is generated by the elements $T^i_k(x)$, where $i,\ k = \pm 1,\ \pm 2$ and RTT–equations (4) give us

$$T^i_k(x)T^r_s(y) + g(x,y)T^r_k(x)T^i_s(y) + \delta^{i,-r}h(x,y)\sum_{p=-2}^{2}\theta_{p,r}T^p_k(x)T^{-p}_s(y) =$$

$$= T^r_s(y)T^i_k(x) + g(x,y)T^r_k(y)T^i_s(x) + \delta_{k,-s}h(x,y)\sum_{p=-2}^{2}\theta_{k,p}T^r_p(y)T^i_{-p}(x)$$

$$T^i_k(x)T^r_s(y) + g(y,x)T^i_s(x)T^r_k(y) + \delta_{k,-s}h(y,x)\sum_{p=-2}^{2}\theta_{s,p}T^i_p(x)T^r_{-p}(y) =$$

$$= T^r_s(y)T^i_k(x) + g(y,x)T^i_s(y)T^r_k(x) + h(y,x)\delta^{i,-r}\sum_{p=-2}^{2}\theta_{i,p}T^p_s(y)T^{-p}_k(x).$$

For this RTT–algebra we do not see any RTT-subalgebra structure similar to the $\mathrm{gl}(n)$ case but as we show latter we can still formulate the nested Bethe ansatz.

The applications to integrable systems are connected with the highest weight representations of RTT–algebras. For the RTT–algebra of $\mathfrak{gl}(2)$ type the vacuum vector $\omega$ is fulfilled

$$T_1^2(x)\omega = 0\,, \qquad T_1^1(x)\omega = \lambda_1(x)\omega\,, \qquad T_2^2(x)\omega = \lambda_2(x)\omega\,.$$

We always denote the set of variables by a bar

$$\overline{u} = \{u_1, u_2, \ldots, u_N\}\,, \qquad \overline{u}_k = \overline{u} \setminus \{u_k\}\,.$$

For the highest weight representation the vector space of representation $\mathcal{W}$ is defined by the elements

$$\left|\overline{u}\right\rangle = T_2^1(\overline{u})\omega\,, \quad \text{where} \quad T_2^1(\overline{u}) = T_2^1(u_1)\,T_2^1(u_2)\ldots T_2^1(u_N)\,.$$

From the commutation relations (5) we will use

$$T_1^1(x)\,T_2^1(u) = f(u,x)\,T_2^1(u)\,T_1^1(x) - g(u,x)\,T_2^1(x)\,T_1^1(u)\,,$$
$$T_2^2(x)\,T_2^1(u) = f(x,u)\,T_2^1(u)\,T_2^2(x) - g(x,u)\,T_2^1(x)\,T_2^2(u)\,.$$

By induction it is possible to obtain formulas

$$T_1^1(x)\,T_2^1(\overline{u}) = F(\overline{u},x)\,T_2^1(\overline{u})\,T_1^1(x) - \sum_{u_k\in\overline{u}} g(u_k,x)F(\overline{u}_k,u_k)\,T_2^1\left(\{\overline{u}_k,x\}\right)T_1^1(u_k)\,,$$
$$T_2^2(x)\,T_2^1(\overline{u}) = F(x,\overline{u})\,T_2^1(\overline{u})\,T_2^2(x) - \sum_{u_k\in\overline{u}} g(x,u_k)F(u_k,\overline{u}_k)\,T_2^1\left(\{\overline{u}_k,x\}\right)T_2^2(u_k)\,,$$

where

$$F(\overline{u},x) = \prod_{u_k\in\overline{u}} f(u_k,x)\,, \qquad F(x,\overline{u}) = \prod_{u_k\in\overline{u}} f(x,u_k)\,.$$

If we apply these relations to the vacuum vector $\omega$, we get relationships

$$T_1^1(x)\big|\overline{u}\big\rangle = \lambda_1(x)F(\overline{u},x)\big|\overline{u}\big\rangle - \sum_{u_k\in\overline{u}}\lambda_1(u_k)g(u_k,x)F(\overline{u}_k,u_k)\big|\{\overline{u}_k,x\}\big\rangle\,,$$

$$T_2^2(x)\big|\overline{u}\big\rangle = \lambda_2(x)F(x,\overline{u})\big|\overline{u}\big\rangle - \sum_{u_k\in\overline{u}}\lambda_2(u_k)g(x,u_k)F(u_k,\overline{u}_k)\big|\{\overline{u}_k,x\}\big\rangle\,,$$

$$H(x)\big|\overline{u}\big\rangle = \Big(\lambda_1(x)F(\overline{u},x) + \lambda_2(x)F(x,\overline{u})\Big)\big|\overline{u}\big\rangle +$$
$$+\sum_{u_k\in\overline{u}} g(x,u_k)\Big(\lambda_1(u_k)F(\overline{u}_k,u_k) - \lambda_2(u_k)F(u_k,\overline{u}_k)\Big)\big|\{\overline{u}_k,x\}\big\rangle$$

**Theorem** Let the Bethe conditions

$$\lambda_1(u_k)F(\overline{u}_k, u_k) = \lambda_2(u_k)F(u_k, \overline{u}_k)$$

be fulfilled for any $u_k \in \overline{u}$. Then for any $x$ is vector $\left|\overline{u}\right\rangle$ eigenvectors of operators $H(x)$ with eigenvalue

$$E(x; \overline{u}) = \lambda_1(x)F(\overline{u}, x) + \lambda_2(x)F(x, \overline{u})\,.$$

To fix the notation, we will take the vacuum vector $\omega$ for highest weight representation of the RTT–algebra of $\mathrm{gl}(3)$ type as

$$T_1^2(x)\omega = 0\,, \qquad T_2^3(x)\omega = 0\,, \qquad T_1^3(x)\omega = 0\,,$$
$$T_1^1(x)\omega = \lambda_1(x)\omega\,, \qquad T_2^2(x)\omega = \lambda_2(x)\omega\,, \qquad T_3^3(x)\omega = \lambda_3(x)\omega\,.$$

For further calculation we denote the R–matrix of $\mathrm{gl}(3)$ restricted to indices 1, 2 as $\tilde{\mathbf{R}}(x, y)$. Namely,

$$\tilde{\mathbf{R}} = \frac{1}{f(x,y)}\left(\tilde{\mathbf{I}}\otimes\tilde{\mathbf{I}} + g(x,y)\tilde{\mathbf{P}}\right), \quad \text{where} \quad \tilde{\mathbf{I}} = \sum_{a=1}^{2}\mathbf{E}_a^a, \quad \tilde{\mathbf{P}} = \sum_{a,b=1}^{2}\mathbf{E}_b^a\otimes\mathbf{E}_a^b\,.$$

Similarly, $\tilde{\mathbf{T}}(x)$ denotes the monodromy matrix

$$\tilde{\mathbf{T}}(x) = \sum_{a,b=1}^{2}\mathbf{E}_a^b\otimes T_b^a(x)\,.$$

Since the R–matrix $\tilde{\mathbf{R}}(x, y)$ evidently fulfiels the Yang–Baxter equation we can define $\tilde{\mathbf{T}}(x)$ RTT–subalgebra $\tilde{\mathcal{A}}$ by means of RTT–equation. This subalgebra is the RTT–algebra of the $\mathrm{gl}(2)$ type.

The idea of nested Bethe ansatz formulated in [4] is to take the eigenvectors of operators $H(x) = T_1^1(x) + T_2^2(x) + T_3^3(x)$ in the form

$$\sum_{a_1,a_2,\dots,a_M=1}^{2} T_3^{a_1}(v_1)T_3^{a_2}(v_2)\dots T_3^{a_M}(v_M)\Phi_{a_1,a_2,\dots,a_M}, \qquad (6)$$

where $\vec{v} = (v_1, v_2, \dots, v_M)$ is the ordered set of different numbers and $\Phi_{a_1,a_2,\dots,a_M} \in \tilde{\mathcal{A}}\omega = \mathcal{W}_0$.

The matrices $\mathbf{E}^a_b$ can be identified with operators on vector space $\mathcal{V}$ using the relation

$$\mathbf{E}^a_b\mathbf{e}_c = \mathbf{E}^a_b\mathbf{E}^3_c = \delta^a_c\mathbf{E}^3_b = \delta^a_c\mathbf{e}_b$$

For these operators the relations $\mathbf{E}^a_b\mathbf{E}^c_d = \delta^a_d\mathbf{E}^c_d$ are hold.
As usual, we define in the dual space $\mathcal{V}^*$ linear operators $\left(\mathbf{E}^a_b\right)^* = \mathbf{F}^b_a$ using the relationship

$$\left\langle \mathbf{E}^b_a\mathbf{e}_c, \mathbf{f}^d \right\rangle = \mathrm{Tr}\left(\mathbf{E}^b_a\mathbf{E}^3_c\mathbf{E}^d_3\right) = \mathrm{Tr}\Big(\mathbf{E}^3_c\mathbf{E}^d_3\mathbf{E}^b_a\Big) = \left\langle \mathbf{e}_c, \mathbf{F}^b_a\mathbf{f}^d \right\rangle,$$

i.e. $\mathbf{F}^b_a\mathbf{f}^d = \delta^d_a\mathbf{f}^b$. Contrary to relation for operators $\mathbf{E}^a_b$, the operators $\mathbf{F}^a_b$ fulfil relations

$$\mathbf{F}^b_a\mathbf{F}^d_c = \delta^d_a\mathbf{F}^b_c\,.$$

The shape (6) for the eigenvectors can then be written as

$$\left\langle \mathbf{B}_{1,\ldots,M}(\vec{v}), \Phi \right\rangle,$$

where $\mathbf{B}(v) = \sum_{a=1}^{2} \mathbf{e}_a \otimes T_3^a(v)$ and

$$\mathbf{B}_{1,\ldots,M}(\vec{v}) = \mathbf{B}_1(v_1)\mathbf{B}_2(v_2)\ldots\mathbf{B}_M(v_M) =$$

$$= \sum_{a_1,\ldots,a_M=1}^{2} \mathbf{e}_{a_1} \otimes \ldots \otimes \mathbf{e}_{a_M} \otimes T_3^{a_1}(v_1)\ldots T_3^{a_M}(v_M)$$

$$\Phi = \sum_{b_1,\ldots,b_M=1}^{2} \mathbf{f}^{b_1} \otimes \mathbf{f}^{b_2} \otimes \ldots \otimes \mathbf{f}^{b_M} \otimes \Phi_{b_1,\ldots,b_M} .$$

What we will do now. We will apply to the vector(6) the elements $T_3^3(x)$, $T_2^2(x)$ and $T_1^1(x)$. To do this we use the commutation relations (5). So let me prepare some systematic way. For calculation it is easy to show that the next lemma is true.

**Lemma**
(a) For all $\Phi \in \mathcal{W}_0$ the relation

$$T_1^3(x)\Phi = T_2^3(x)\Phi = 0\,, \qquad T_3^3(x)\Phi = \lambda_3(x)\Phi$$

is holds.

(b) The following relations are true:

$$\mathbf{B}_1(x)\mathbf{B}_2(y) = \tilde{\mathbf{R}}_{2,1}(y,x)\mathbf{B}_2(y)\mathbf{B}_1(x)\,, \qquad \mathbf{B}_1(x)\mathbf{B}_2(y) = \tilde{\mathbb{R}}_{1,2}\mathbf{B}_2(x)\mathbf{B}_1(y)\,,$$

where

$$\tilde{\mathbb{R}}_{1,2} = \tilde{\mathbf{R}}_{1,2}(x,x) = \tilde{\mathbf{P}}_{1,2} = \sum_{a,b=1}^{2} \mathbf{E}_b^a \otimes \mathbf{E}_a^b\,.$$

We can write the first relation as

$$\Big\langle \mathbf{B}_1(x)\mathbf{B}_2(y), \mathbf{f}^{b_1} \otimes \mathbf{f}^{b_2} \Big\rangle = \Big\langle \mathbf{B}_2(y)\mathbf{B}_1(x), \tilde{\mathbf{R}}^*_{2^*,1^*}(y,x)\big(\mathbf{f}^{b_1} \otimes \mathbf{f}^{b_2}\big) \Big\rangle,$$

where

$$\tilde{\mathbf{R}}^*_{2^*,1^*}(y,x) = \frac{1}{f(y,x)}\Big(\mathbf{I}^* \otimes \mathbf{I}^* + g(y,x)\sum_{a,b=1}^{2} \mathbf{F}_b^a \otimes \mathbf{F}_a^b\Big), \qquad \mathbf{I}^* = \sum_{a=1}^{2} \mathbf{F}_a^a\,.$$

(c) The relations

$$T_3^3(x)\mathbf{B}(v) = f(x,v)\mathbf{B}(v)\,T_3^3(x) - g(x,v)\mathbf{B}(x)\,T_3^3(v)\,,$$

$$\tilde{\mathbf{T}}_0(x)\Big\langle \mathbf{B}_1(v), \mathbf{I}\otimes \mathbf{f}^b\Big\rangle_1 = f(v,x)\Big\langle \mathbf{B}_1(v), \widehat{\mathbf{T}}_{0,1}(x;v)\big(\mathbf{I}\otimes \mathbf{f}^b\big)\Big\rangle_1 - \\ - g(v,x)\Big\langle \mathbf{B}_1(x), \widehat{\mathbb{T}}_{0,1}(v)\big(\mathbf{I}\otimes \mathbf{f}^b\big)\Big\rangle_1\,,$$

where

$$\widehat{\mathbf{T}}_{0,1}(x;v) = \widehat{\mathbf{R}}_{0,1^*}(x,v)\tilde{\mathbf{T}}_0(x)\,,$$

$$\widehat{\mathbf{R}}_{0,1^*}(x;v) = \frac{1}{f(v,x)}\Big(\mathbf{I}\otimes\mathbf{I}^* + g(v,x)\sum_{a,b=1}^{2}\mathbf{E}_b^a\otimes\mathbf{F}_a^b\Big)$$

$$\widehat{\mathbb{T}}_{0,1}(v) = \widehat{\mathbf{T}}_{0,1}(v;v) = \widehat{\mathbb{R}}_{0,1^*}\tilde{\mathbf{T}}_0(v)\,, \qquad \widehat{\mathbb{R}}_{0,1^*} = \widehat{\mathbf{R}}_{0,1^*}(v,v) = \sum_{a,b=1}^{2}\mathbf{E}_b^a\otimes\mathbf{F}_a^b\,,$$

are valid.

(d) If we denote

$$\mathbf{B}_{k;1,\dots,M}(x,\vec{v}_k)=\mathbf{B}_k(x)\mathbf{B}_1(v_1)\dots\mathbf{B}_{k-1}(v_{k-1})\mathbf{B}_{k+1}(v_{k+1})\dots\mathbf{B}_N(v_M)\,,$$
$$\tilde{\mathbf{R}}^*_{k;1,\dots,k}(\vec{v})=\tilde{\mathbf{R}}^*_{k,1}(v_k,v_1)\dots\tilde{\mathbf{R}}^*_{k,k-1}(v_k,v_{k-1})\,,\qquad \tilde{\mathbf{R}}_{1;1\dots,1}(\vec{v})=\mathbf{I}^*$$

we have

$$\left\langle \mathbf{B}_{1,\dots,M}(\vec{v}),\mathbf{f}^{b_1}\otimes\dots\otimes\mathbf{f}^{b_M}\right\rangle=\left\langle \mathbf{B}_{k;1,\dots,M}(v_k,\vec{v}_k),\tilde{\mathbf{R}}^*_{k;1,\dots,k}(\vec{v})\big(\mathbf{f}^{b_1}\otimes\dots\otimes\mathbf{f}^{b_M}\big)\right\rangle.$$

Further we define

$$\widehat{\mathbf{T}}_{0,1,\dots,M}(x;\vec{v}) = \widehat{\mathbf{R}}_{0,1}(x,v_1)\widehat{\mathbf{R}}_{0,2}(x,v_2)\dots\widehat{\mathbf{R}}_{0,M}(x,v_M)\tilde{\mathbf{T}}_0(x)\,,$$

$$\widehat{\mathbb{T}}_{k;0,1,\dots,M}(\vec{v}) = \widehat{\mathbf{T}}_{0,1,\dots,M}(v_k;\vec{v})\,.$$

Now we will formulate the theorem which can be found in the original work by Kulish and Reshetikhin [4].

<u>**Theorem 4.1.**</u> Let $\Phi$ be an eigenvector of the operator

$$\widehat{H}_{1,\ldots,M}(x;\vec{v}) = \big(\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\big)_1^1 + \big(\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\big)_2^2\,,$$

i.e. $\widehat{H}_{1,\ldots,M}(x;\vec{v})\Phi = \mu(x;\vec{v})\Phi$ is valid. If for any $v_k \in \overline{v}$ the relations

$$\lambda_3(v_k)F(v_k,\overline{v}_k) = \mu(v_k;\vec{v})F(\vec{v}_k,v_k)\,,$$

hold, the vector $\langle \mathbf{B}_{1,\ldots,M}(\vec{v}),\Phi\rangle$ is an eigenvector of the operator $H(x) = T_1^1(x) + T_2^2(x) + T_3^3(x)$ with eigenvalue

$$E(x;\vec{v},\mu) = \lambda_3(x)F(x,\overline{v}) + \mu(x;\vec{v})F(\overline{v},x)\,.$$

The proof of the theorem follows from following lemma

**Lemma 4.2.** For any $\Phi = \sum_{b_1,\dots,b_M=1}^{2} \mathbf{f}^{b_1} \otimes \dots \otimes \mathbf{f}^{b_M} \otimes \Phi_{b_1,\dots,b_M}$, where $\Phi_{b_1,\dots,b_M} \in \mathcal{W}_0$, it is true

$$\text{(a)} \qquad T_3^3(x)\Big\langle \mathbf{B}_{1,\dots,M}(\vec{v}), \Phi \Big\rangle = \lambda_3(x)F(x,\overline{v})\Big\langle \mathbf{B}_{1,\dots,M}(\vec{v}), \Phi \Big\rangle - \\ - \sum_{v_k \in \overline{v}} \lambda_3(v_k) g(x,v_k) F(v_k,\overline{v}_k)\Big\langle \mathbf{B}_{k;1,\dots,M}(x,\vec{v}_k), \tilde{\mathbf{R}}^*_{k;1,\dots,k}(\vec{v})\Phi \Big\rangle,$$

$$\text{(b)} \qquad \tilde{\mathbf{T}}_0(x)\Big\langle \mathbf{B}_{1,\dots,M}(\vec{v}), \Phi \Big\rangle = F(\overline{v},x)\Big\langle \mathbf{B}_{1,\dots,M}(\vec{v}), \widehat{\mathbf{T}}_{0,1,\dots,M}(x;\vec{v})\Phi \Big\rangle - \\ - \sum_{v_k \in \overline{v}} g(v_k,x) F(\overline{v}_k,v_k)\Big\langle \mathbf{B}_{k;1,\dots,M}(x,\vec{v}_k), \tilde{\mathbf{R}}^*_{k;1,\dots,k}(\vec{v})\widehat{\mathbb{T}}_{k;0,1,\dots,M}(\vec{v})\Phi \Big\rangle,$$

Now we will study the eigenvectors of $\widehat{H}_{1,\ldots,M}(x;\vec{v})$. The following lemma converts this problem to the $\mathrm{gl}(2)$ case.

**Lemma 4.3.** The operators $\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})$ satisfy the RTT–equation

$$\tilde{\mathbf{R}}_{0,0'}(x,y)\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\widehat{\mathbf{T}}_{0',1,\ldots,M}(y;\vec{v}) = \widehat{\mathbf{T}}_{0',1,\ldots,M}(y;\vec{v})\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\tilde{\mathbf{R}}_{0,0'}(x,y)\,.$$

PROOF. From the definition of the operators $\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})$ we obtain

$$\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})\widehat{\mathbf{T}}_{0',1,\ldots,M}(y;\vec{v}) = \widehat{\mathbf{R}}_{0,1}(x;v_1)\ldots\widehat{\mathbf{R}}_{0,M}(x;v_M)\tilde{\mathbf{T}}_0(x)$$
$$\widehat{\mathbf{R}}_{0',1}(y;v_1)\ldots\widehat{\mathbf{R}}_{0',M}(y;v_M)\tilde{\mathbf{T}}_{0'}(y) =$$
$$= \left(\widehat{\mathbf{R}}_{0,1}(x;v_1)\widehat{\mathbf{R}}_{0',1}(y;v_1)\right)\ldots\left(\widehat{\mathbf{R}}_{0,M}(x;v_M)\widehat{\mathbf{R}}_{0',M}(y;v_M)\right)\left(\tilde{\mathbf{T}}_0(x)\tilde{\mathbf{T}}_{0'}(y)\right)$$

Since for the operators $\tilde{\mathbf{T}}_0(x)$ the RTT–equation

$$\tilde{\mathbf{R}}_{0,0'}(x,y)\tilde{\mathbf{T}}_0(x)\tilde{\mathbf{T}}_{0'}(y) = \tilde{\mathbf{T}}_{0'}(y)\tilde{\mathbf{T}}_0(x)\tilde{\mathbf{R}}_{0,0'}(x,y)$$

are valid, it is enough to show that for each $v_k \in \overline{v}$ the relationship holds

$$\tilde{\mathbf{R}}_{0,0'}(x,y)\widehat{\mathbf{R}}_{0,k}(x;v_k)\widehat{\mathbf{R}}_{0',k}(y;v_k) = \widehat{\mathbf{R}}_{0',k}(y;v_k)\widehat{\mathbf{R}}_{0,k}(x;v_k)\tilde{\mathbf{R}}_{0,0'}(x,y)\,,$$

which can be verified by direct calculation.

Now we must only find the vacuum vectors of $\widehat{H}_{1,\dots,M}(x;\vec{v})$. The following lemma will give us the answer.

**Lemma 4.4.** For the vector

$$\widehat{\Omega} = \underbrace{\mathbf{f}^2 \otimes \ldots \otimes \mathbf{f}^2}_{M\times} \otimes\, \omega$$

we have

$$\widehat{T}_1^2(x;\vec{v})\widehat{\Omega} = 0\,, \qquad \widehat{T}_1^1(x;\vec{v})\widehat{\Omega} = \mu_1(x;\vec{v})\widehat{\Omega}\,, \qquad \widehat{T}_2^2(x;\vec{v})\widehat{\Omega} = \mu_2(x;\vec{v})\widehat{\Omega}\,,$$

where

$$\mu_1(x;\vec{v}) = \frac{\lambda_1(x)}{F(\overline{v},x)}\,, \qquad \mu_2(x;\vec{v}) = \lambda_2(x)\,.$$

The proof of this lemma will omit.

So we can now formulate the final theorem for the $\mathrm{gl}(3)$ case.

**Theorem 4.2.** Let for any $u_i \in \overline{u}$ and $v_k \in \overline{v}$ the Bethe conditions

$$\lambda_1(u_i)F(\overline{u}_i, u_i) = \lambda_2(u_i)F(\overline{v}, u_i)F(u_i, \overline{u}_i)\,,$$
$$\lambda_3(v_k)F(v_k, \overline{v}_k) = \lambda_2(v_k)F(\overline{v}_k, v_k)F(v_k, \overline{u})$$

be valid. Then $\left|\vec{v}, \overline{u}\right\rangle = \left\langle \mathbf{B}_{1,\dots,M}(\vec{v}); \Phi(\overline{u}; \vec{v})\right\rangle$ is eigenvector of $H(x)$ with eigenvalue

$$E(x; \overline{u}; \vec{v}) = \lambda_1(x)F(\overline{u}, x)F(\overline{v}, x) + \lambda_2(x)F(x, \overline{u})F(\overline{v}, x) + \lambda_3(x)F(x, \overline{v})$$

PROOF: According to lemma 4.3, apply to operators $\widehat{\mathbf{T}}_{0,1,\ldots,M}(x;\vec{v})$ of the RTT–algebra of $\mathrm{gl}(2)$ type. The vector $\widehat{\Omega}$ is the vacuum vector with weights

$$\mu_1(x;\vec{v}) = \frac{\lambda_1(x)}{F(\overline{v},x)}\,, \qquad \mu_2(x;\vec{v}) = \lambda_2(x)$$

According of Theorem 3.1, if for any $u_i \in \overline{u}$ the relation

$$\mu_1(u_i;\vec{v})F(\overline{u}_i,u_i) = \mu_2(u_i;\vec{v})F(u_i,\overline{u}_i)$$

is true, the vector $\Phi(\overline{u};\vec{v}) = \widehat{T}_1^2(\overline{u};\vec{v})\widehat{\Omega}$ is an eigenvector of the operators

$$\mathrm{Tr}_0\big(\widehat{\mathbf{T}}_{0,1,\ldots}(x;\vec{v})\big) = \widehat{H}_{1,\ldots,M}(x;\vec{v})$$

with eigenvalue

$$\widehat{E}(x;\overline{u};\vec{v}) = \mu_1(x;\vec{v})F(\overline{u},x) + \mu_2(x;\vec{v})F(x,\overline{u})\,,$$

Further, according to Theorem 4.1, the vector $\big|\vec{v},\overline{u}\big\rangle$ is an eigenvector of the operator $H(x)$, if the condition

$$\lambda_3(v_k)F(v_k,\overline{v}_k) = \mu(v_k;\vec{v})F(\vec{v}_k,v_k) = \lambda_2(v_k)F(v_k,\overline{u})F(\vec{v}_k,v_k)$$

is met.

Let me start with the $\mathrm{sp}(4)$ case, for which we take the R–matrix

$$\mathbf{R}(x,y) = \frac{1}{x-y+1}\left((x-y)\mathbf{I}\otimes\mathbf{I} + \mathbf{P} - \frac{x-y}{x-y+3}\,\mathbf{Q}\right) = \\ = \frac{1}{f(x,y)}\Big(\mathbf{I}\otimes\mathbf{I} + g(x,y)\mathbf{P} - h(x,y)\mathbf{Q}\Big),$$

where

$$\mathbf{I} = \sum_{k=-2}^{2}\mathbf{E}_k^k\,,\qquad \mathbf{P} = \sum_{i,k=-2}^{2}\mathbf{E}_k^i\otimes\mathbf{E}_i^k\,,\qquad \mathbf{Q} = \sum_{i,k=-2}^{2}\theta_{i,k}\mathbf{E}_k^i\otimes\mathbf{E}_{-k}^{-i},$$

$$g(x,y) = \frac{1}{x-y}\,,\qquad f(x,y) = \frac{x-y+1}{x-y}\,,\qquad h(x,y) = \frac{1}{x-y+3}\,,$$

$$\theta_{i,k} = \mathrm{sgn}(i)\cdot\mathrm{sgn}(k)\,.$$

It is easy to obtain the commutation relation for this RTT algebra

$$T_k^i(x)T_s^r(y) + g(x,y)T_k^r(x)T_s^i(y) + \delta^{i,-r}h(x,y)\sum_{p=-2}^{2}\theta_{p,r}T_k^p(x)T_s^{-p}(y) =$$

$$= T_s^r(y)T_k^i(x) + g(x,y)T_k^r(y)T_s^i(x) + \delta_{k,-s}h(x,y)\sum_{p=-2}^{2}\theta_{k,p}T_p^r(y)T_{-p}^i(x)$$

$$T_k^i(x)T_s^r(y) + g(y,x)T_s^i(x)T_k^r(y) + \delta_{k,-s}h(y,x)\sum_{p=-2}^{2}\theta_{s,p}T_p^i(x)T_{-p}^r(y) =$$

$$= T_s^r(y)T_k^i(x) + g(y,x)T_s^i(y)T_k^r(x) + h(y,x)\delta^{i,-r}\sum_{p=-2}^{2}\theta_{i,p}T_s^p(y)T_k^{-p}(x).$$

from which we easy see that for any $i$, $k$ and for any $x$, $y$ is valid
$T_k^i(x)T_k^i(y) = T_k^i(y)T_k^i(x)$.

By calculation we will use the representation of the RTT–algebra $\mathcal{A}$ on vector space, for which there exist vacuum vector $\omega$, for which

$$T^i_k(x)\omega = 0 \quad \text{pro} \quad i < k\,, \qquad\qquad T^i_i(x)\omega = \lambda_i(x)\omega \quad \text{pro} \quad i = \pm 1,\, \pm 2\,,$$

and we denote $\mathcal{W} = \mathcal{A}\omega$.

# RTT–algebra with R–matrix $\tilde{\mathbf{R}}(x, y)$

First, we study the eigenvalues of the type

$$\left|0; \overline{v}; \overline{w}\right\rangle = T_1^2(\overline{v})\, T_{-2}^{-1}(\overline{w})\omega\,,$$

where $\overline{v} = \{v_1, \ldots, v_P\}$, $\overline{w} = \{w_1, \ldots, w_Q\}$ are the set of numbers

$$T_1^2(\overline{v}) = T_1^2(v_1)\ldots T_1^2(v_P)\,, \qquad T_{-2}^{-1}(\overline{w}) = T_{-2}^{-1}(w_1)\ldots T_{-2}^{-1}(w_Q)\,.$$

We denote $\mathcal{W}_0$ the vector space generated by $\left|0; \overline{v}; \overline{w}\right\rangle$.

**Lemma** If we denote

$$\mathbf{T}^{(+)}(x) = \sum_{i,k=1}^{2} \mathbf{E}_i^k \otimes T_k^i(x)\,, \qquad \mathbf{T}^{(-)}(x) = \sum_{i,k=1}^{2} \mathbf{E}_{-i}^{-k} \otimes T_{-k}^{-i}(x)\,,$$

$$\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(x,y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{T}_2^{(\epsilon_2)}(y) = \mathbf{T}_2^{(\epsilon_2)}(y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(x,y)$$

is valid on the space $\mathcal{W}_0$ for any where

$$\begin{aligned}
\mathbf{R}^{(+,+)}(x,y) &= \frac{1}{f(x,y)}\Big(\mathbf{I}_+ \otimes \mathbf{I}_+ + g(x,y)\mathbf{P}_+\Big), \\
\mathbf{R}^{(-,-)}(x,y) &= \frac{1}{f(x,y)}\Big(\mathbf{I}_- \otimes \mathbf{I}_- + g(x,y)\mathbf{P}_-\Big), \\
\mathbf{R}^{(+,-)}(x,y) &= \mathbf{I}_+ \otimes \mathbf{I}_- - k(x,y)\mathbf{Q}^{(+,-)}\,, \\
\mathbf{R}^{(-,+)}(x,y) &= \mathbf{I}_- \otimes \mathbf{I}_+ - h(x,y)\mathbf{Q}^{(-,+)}
\end{aligned}$$

and

$$\mathbf{I}_+ = \sum_{i=1}^{2} \mathbf{E}_i^i\,, \qquad \mathbf{P}_+ = \sum_{i,k=1}^{2} \mathbf{E}_k^i \otimes \mathbf{E}_i^k\,, \qquad \mathbf{Q}^{(+,-)} = \sum_{i,k=1}^{2} \mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i}\,,$$

$$\mathbf{I}_- = \sum_{i=1}^{2} \mathbf{E}_{-i}^{-i}\,, \qquad \mathbf{P}_- = \sum_{i,k=1}^{n} \mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_{-i}^{-k}\,, \qquad \mathbf{Q}^{(-,+)} = \sum_{i,k=1}^{2} \mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_k^i\,,$$

$$k(x,y) = \frac{1}{x-y-1}\,.$$

**Lemma** If we define

$$\tilde{\mathbf{R}}(x,y) = \mathbf{R}^{(+,+)}(x,y) + \mathbf{R}^{(+,-)}(x,y) + \mathbf{R}^{(-,+)}(x,y) + \mathbf{R}^{(-,-)}(x,y)$$
$$\tilde{\mathbf{T}}(x) = \mathbf{T}^{(+)}(x) + \mathbf{T}^{(-)}(x)$$

on the space $\mathcal{W}_0$, RTT–equation is

$$\tilde{\mathbf{R}}_{1,2}(x,y)\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{T}}_2(y) = \tilde{\mathbf{T}}_2(y)\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{R}}_{1,2}(x,y)\,.$$

Futher

**Lemma** R–matrix $\tilde{\mathbf{R}}(x,y)$ fulfills the Yang–Baxter equations

$$\tilde{\mathbf{R}}_{1,2}(x,y)\tilde{\mathbf{R}}_{1,3}(x,z)\tilde{\mathbf{R}}_{2,3}(y,z) = \tilde{\mathbf{R}}_{2,3}(y,z)\tilde{\mathbf{R}}_{1,3}(x,z)\tilde{\mathbf{R}}_{1,2}(x,y)$$

This RTT–algebra, which is defined by this R–matrix will be denoted by $\tilde{\mathcal{A}}$.

## Bethe vectors for RTT–algebra $\tilde{\mathcal{A}}$

Vacuum vector $\omega$ is defined by

$$T_2^1(x)\omega = T_{-1}^{-2}(x)\omega = 0\,, \qquad T_i^i(x)\omega = \lambda_i(x)\omega\,, \qquad i = \pm 1,\, \pm 2\,.$$

For elements

$$\tilde{H}^{(+)}(x) = T_1^1(x) + T_2^2(x)\,, \qquad \tilde{H}^{(-)}(x) = T_{-1}^{-1}(x) + T_{-2}^{-2}(x)$$

in the RTT–algebra $\tilde{\mathcal{A}}$ the relation

$$\left[\tilde{H}^{(+)}(x), \tilde{H}^{(-)}(y)\right] = 0$$

is true.

**Theorem** If for any $v_r \in \overline{v}$ and $w_s \in \overline{w}$ the conditions

$$\lambda_1(v_r)F(v_r,\overline{v}_r)F(v_r-2,\overline{w}) = \lambda_2(v_r)F(\overline{v}_r,v_r)F(\overline{w},v_r-2)$$

$$\lambda_{-1}(w_s)F(\overline{v},w_s+2)F(\overline{w}_s,w_s) = \lambda_{-2}(w_s)F(w_s+2,\overline{v})F(w_s,\overline{w}_s)$$

Are valid. Then $\left|0;\overline{v};\overline{w}\right\rangle = T_1^2(\overline{v})T_{-2}^{-1}(\overline{w})\omega$ eigenvector of $\tilde{H}^{(+)}(x)$ and $\tilde{H}^{(-)}(x)$ with eigenvalues

$$\tilde{E}^{(+)}(x;0,\overline{v},\overline{w}) = \lambda_1(x)F(x,\overline{v})F(x-2,\overline{w}) + \lambda_2(x)F(\overline{v},x)F(\overline{w},x-2)$$

$$\tilde{E}^{(-)}(x;0,\overline{v},\overline{w}) = \lambda_{-1}(x)F(\overline{v},x+2)F(\overline{w},x) + \lambda_{-2}(x)F(x+2,\overline{v})F(x,\overline{w})\,.$$

## General forms of Bethe vectors of the RTT–algebra of $\mathrm{sp}(4)$ type

Let us denote by a $\vec{u}$ the ordered set of mutually different numbers, $\vec{u} = (u_1, u_2, \ldots, u_N)$.Eigenvectors of the vectors RTT-algebra we will take in shape

$$\sum_{i_1,\ldots,i_N,k_1,\ldots,k_N=1}^{2} T^{i_1}_{-k_1}(u_1) T^{i_2}_{-k_2}(u_2) \ldots T^{i_N}_{-k_N}(u_N) \Phi^{-k_1,-k_2,\ldots,-k_N}_{i_1,i_2,\ldots,i_N}$$

where $\Phi^{-k_1,-k_2,\ldots,-k_N}_{i_1,i_2,\ldots,i_N} \in \mathcal{W}_0$.

Let's say

$$\mathbf{B}_1(u) = \sum_{i,k=1}^{2} \mathbf{e}_i \otimes \mathbf{f}^{-k} \otimes T^{i}_{-k}(u) \in \mathcal{V}_{1_+} \otimes \mathcal{V}^*_{1_-} \otimes \mathcal{A}$$

and define

$$\mathbf{B}_{1,\ldots,N}(\vec{u}) = \mathbf{B}_1(u_1)\mathbf{B}_2(u_2)\ldots\mathbf{B}_N(u_N) \in \mathcal{V}_+ \otimes \mathcal{V}^*_- \otimes \mathcal{A}\,,$$

where

$$\mathcal{V}_+ = \mathcal{V}_{1_+} \otimes \mathcal{V}_{2_+} \otimes \ldots \otimes \mathcal{V}_{N_+}\,, \qquad \mathcal{V}^*_- = \mathcal{V}^*_{1_-} \otimes \mathcal{V}^*_{2_-} \otimes \ldots \otimes \mathcal{V}^*_{N_-}\,.$$

We have

$$\mathbf{B}_{1,\dots,N}(\vec{u}) = \sum_{\vec{i},\vec{k}} \mathbf{e}_{\vec{i}} \otimes \mathbf{f}^{-\vec{k}} \otimes T^{\vec{i}}_{-\vec{k}}(\vec{u})$$

where

$$\mathbf{e}_{\vec{i}} = \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_N} \in \mathcal{V}_{1_+} \otimes \dots \otimes \mathcal{V}_{N_+}$$

$$\mathbf{f}^{-\vec{k}} = \mathbf{f}^{-k_1} \otimes \dots \otimes \mathbf{f}^{-k_N} \in \mathcal{V}^*_{1_-} \otimes \dots \otimes \mathcal{V}^*_{N_-}$$

$$T^{\vec{i}}_{-\vec{k}}(\vec{u}) = T^{i_1}_{-k_1}(u_1) \dots T^{i_N}_{-k_N}(u_N) \in \mathcal{A}\,.$$

The general shape of the custom vector can then be written as $\left\langle \mathbf{B}_{1,\dots,N}(\vec{u}), \Phi \right\rangle$, where

$$\Phi = \mathbf{f}^{(\vec{s})} \otimes \mathbf{e}_{-\vec{r}} \otimes \Phi^{-\vec{r}}_{\vec{s}} \in \mathcal{V}^*_+ \otimes \mathcal{V}_- \otimes \mathcal{W}_0\,,$$

$$\mathbf{f}^{(\vec{s})} = \mathbf{f}^{s_1} \otimes \dots \otimes \mathbf{f}^{s_N} \in \mathcal{V}^*_{1_+} \otimes \dots \otimes \mathcal{V}^*_{N_+} = \mathcal{V}^*_+\,,$$

$$\mathbf{e}_{-\vec{r}} = \mathbf{e}_{-r_1} \otimes \dots \otimes \mathbf{e}_{-r_N} \in \mathcal{V}_{1_-} \otimes \mathcal{V}_{N_-} = \mathcal{V}_-\,,$$

$$\Phi^{-\vec{r}}_{\vec{s}} = \Phi^{-r_1,\dots,-r_N}_{s_1,\dots,s_N} \in \mathcal{W}_0\,.$$

**Lemma** If we denote by $\overline{u}$ the set of elements $\vec{u}$, i.e. $\overline{u} = \{u_1, u_2, \ldots, u_N\}$, then we have

$$\mathbf{T}_0^{(+)}(x)\Big\langle \mathbf{B}_{1,\ldots,N}(\vec{u}), \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}} \Big\rangle = F(\overline{u}, x)\Big\langle \mathbf{B}_{1,\ldots,N}(\vec{u}), \widehat{\mathbf{T}}_{0;1,\ldots,N}^{(+)}(x; \vec{u})\big(\mathbf{I} \otimes \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}}\big) \Big\rangle -$$
$$- \sum_{u_k \in \overline{u}} g(u_k, x) F(\overline{u}_k, u_k)\Big\langle \mathbf{B}_{k;1,\ldots,N}(x, \vec{u}_k), \widehat{\mathbb{Z}}_{k;0,1,\ldots,N}^{(+)}(u_k; \vec{u}_k)\big(\mathbf{I} \otimes \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}}\big) \Big\rangle$$
$$\mathbf{T}_0^{(-)}(x)\Big\langle \mathbf{B}_{1,\ldots,N}(\vec{u}), \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}} \Big\rangle = F(x, \overline{u})\Big\langle \mathbf{B}_{1,\ldots,N}(\vec{u}), \widehat{\mathbf{T}}_{0;1,\ldots,N}^{(-)}(x; \vec{u})\big(\mathbf{I} \otimes \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}}\big) \Big\rangle -$$
$$- \sum_{u_k \in \overline{u}} g(x, u_k) F(u_k, \overline{u}_k)\Big\langle \mathbf{B}_{k;1,\ldots,N}(x, \vec{u}_k), \widehat{\mathbb{Z}}_{k;0,1,\ldots,N}^{(-)}(u_k; \vec{u}_k)\big(\mathbf{I} \otimes \mathbf{f}^{\vec{r}} \otimes \mathbf{e}_{-\vec{s}}\big) \Big\rangle$$

where

$$\widehat{\mathbf{T}}^{(+)}_{0;1,\ldots,N}(x;\vec{u}) = \widehat{\mathbf{R}}^{(+,+)}_{0;1^*_+,\ldots,N^*_+}(x;\vec{u})\mathbf{T}^{(+)}_0(x)\mathbf{R}^{(+,-)}_{0;1_-,\ldots,N_-}(x;\vec{u})$$

$$\widehat{\mathbf{T}}^{(-)}_{0;1,\ldots,N}(x;\vec{u}) = \widehat{\mathbf{R}}^{(-,+)}_{0;1^*_+,\ldots,N^*_+}(x;\vec{u})\mathbf{T}^{(-)}_0(x)\mathbf{R}^{(-,-)}_{0;1_-,\ldots,N_-}(x;\vec{u})$$

$$\widehat{\mathbf{R}}^{(+,+)}_{0;1^*_+,\ldots,N^*_+}(x;\vec{u}) = \widehat{\mathbf{R}}^{(+,+)}_{0,1^*_+}(x,u_1)\widehat{\mathbf{R}}^{(+,+)}_{0,2^*_+}(x,u_2)\ldots\widehat{\mathbf{R}}^{(+,+)}_{0,N^*_+}(x,u_N)$$

$$\mathbf{R}^{(+,-)}_{0;1_-,\ldots,N_-}(x;\vec{u}) = \mathbf{R}^{(+,-)}_{0,N_-}(x,u_N)\ldots\mathbf{R}^{(+,-)}_{0,2_-}(x,u_2)\mathbf{R}^{(+,-)}_{0,1_-}(x,u_1)$$

$$\widehat{\mathbf{R}}^{(-,+)}_{0;1^*_+,\ldots,N^*_+}(x;\vec{u}) = \widehat{\mathbf{R}}^{(-,+)}_{0,1^*_+}(x,u_1)\widehat{\mathbf{R}}^{(-,+)}_{0,2^*_+}(x,u_2)\ldots\widehat{\mathbf{R}}^{(+-,+)}_{0,N^*_+}(x,u_N)$$

$$\mathbf{R}^{(-,-)}_{0;1_-,\ldots,N_-}(x;\vec{u}) = \mathbf{R}^{(-,-)}_{0,N_-}(x,u_N)\ldots\mathbf{R}^{(-,-)}_{0,2_-}(x,u_2)\mathbf{R}^{(-,-)}_{0,1_-}(x,u_1)$$

$$\mathbf{B}_{k;1,\ldots,N}(x,\vec{u}_k) = \mathbf{B}_k(x)\mathbf{B}_1(u_1)\ldots\mathbf{B}_{k-1}(u_{k-1})\mathbf{B}_{k+1}(u_{k+1})\ldots\mathbf{B}_N(u_N)$$

$$\widehat{\mathbb{Z}}^{(+)}_{k;0,1,\ldots,N}(u_k;\vec{u}_k) = \big(\mathbf{R}^*\big)^{(+,+)}_{1,\ldots,k}(\vec{u})\mathbf{R}^{(-,-)}_{1,\ldots,k}(\vec{u})\widehat{\mathbb{T}}^{(+)}_{k;0,1,\ldots,N}(\vec{u})$$

$$\widehat{\mathbb{Z}}^{(-)}_{k;0,1,\ldots,N}(u_k;\vec{u}_k) = \big(\mathbf{R}^*\big)^{(+,+)}_{1,\ldots,k}(\vec{u})\mathbf{R}^{(-,-)}_{1,\ldots,k}(\vec{u})\widehat{\mathbb{T}}^{(-)}_{k;0,1,\ldots,N}(\vec{u})$$

$$\big(\mathbf{R}^*\big)^{(+,+)}_{1,\ldots,k}(\vec{u}) = \big(\mathbf{R}^*\big)^{(+,+)}_{k^*_+,1^*_+}(u_k,u_1)\big(\mathbf{R}^*\big)^{(+,+)}_{k^*_+,2^*_+}(u_k,u_2)\ldots\big(\mathbf{R}^*\big)^{(+,+)}_{k^*_+,(k-1)^*_+}(u_k,u_{k-1})$$

$$\mathbf{R}^{(-,-)}_{1,\ldots,k}(\vec{u}) = \mathbf{R}^{(-,-)}_{1_-,k_-}(u_1,u_k)\mathbf{R}^{(-,-)}_{2_-,k_-}(u_2,u_k)\ldots\mathbf{R}^{(-,-)}_{(k-1)_-,k_-}(u_{k-1},u_k)$$

$$\widehat{\mathbb{T}}^{(+)}_{k;0,1,\ldots,N}(\vec{u}) = \widehat{\mathbf{T}}^{(+)}_{0;1,\ldots,N}(u_k;\vec{u}) = \widehat{\mathbf{R}}^{(+,+)}_{0;1^*_+,\ldots,N^*_+}(u_k;\vec{u})\mathbf{T}^{(+)}_0(u_k)\mathbf{R}^{(+,-)}_{0;1_-,\ldots,N_-}(u_k;\vec{u})$$

$$\widehat{\mathbb{T}}^{(-)}_{k;0,1,\ldots,N}(\vec{u}) = \widehat{\mathbf{T}}^{(-)}_{0;1,\ldots,N}(u_k;\vec{u}) = \widehat{\mathbf{R}}^{(-,+)}_{0;1^*_+,\ldots,N^*_+}(u_k;\vec{u})\mathbf{T}^{(-)}_0(u_k)\mathbf{R}^{(-,-)}_{0;1_-,\ldots,N_-}(u_k;\vec{u})$$

**Lemma** Let $\Phi$ be eigenvectors of the operators

$$\widehat{H}^{(+)}_{1,\ldots,N}(x;\vec{u}) = \left(\widehat{\mathbf{T}}^{(+)}_{0,1,\ldots,N}(x;\vec{u})\right)^1_1 + \left(\widehat{\mathbf{T}}^{(+)}_{0,1,\ldots,N}(x;\vec{u})\right)^2_2$$
$$\widehat{H}^{(-)}_{1,\ldots,N}(x;\vec{u}) = \left(\widehat{\mathbf{T}}^{(-)}_{0,1,\ldots,N}(x;\vec{u})\right)^{-1}_{-1} + \left(\widehat{\mathbf{T}}^{(-)}_{0,1,\ldots,N}(x;\vec{u})\right)^{-2}_{-2}$$

with eigenvalues $\widehat{E}^{(+)}(x;\vec{u})$ and $\widehat{E}^{(-)}(x,\vec{u})$. If for $u_k \in \overline{u}$ is true

$$\widehat{E}^{(+)}(u_k;\vec{u}_k)F(\overline{u}_k,u_k) = \widehat{E}^{(-)}(u_k,\vec{u}_k)F(u_k,\overline{u}_k)\,,$$

then $\left\langle \mathbf{B}_{1,\ldots,N}(\vec{u}),\Phi\right\rangle$ is eigenvector of $H(x)$ with eigenvalue

$$E(x;\vec{u}) = F(\overline{u},x)\widehat{E}^{(+)}(x;\vec{u}) + F(x,\overline{u})\widehat{E}^{(-)}(x;\vec{u})\,.$$

## Eigenvectors of operators $\widehat{H}^{(\pm)}_{0,1,\ldots,N}(x;\vec{u})$

**Lemma** For any $\Phi = \mathbf{f}^{\vec{s}} \otimes \mathbf{e}_{-\vec{r}} \otimes \Phi_{\vec{s}}^{-\vec{r}}$, where $\Phi_{\vec{s}}^{-\vec{r}} \in \mathcal{W}_0$, and $\epsilon_0,\ \epsilon_{0'} = \pm$, is valid

$$\mathbf{R}_{0,0'}^{(\epsilon_0,\epsilon_{0'})}(x,y)\widehat{\mathbf{T}}^{(\epsilon_0)}_{0,1,\ldots,N}(x,\vec{u})\widehat{\mathbf{T}}^{(\epsilon_{0'})}_{0',1,\ldots,N}(y;\vec{u})\breve{} = \widehat{\mathbf{T}}^{(\epsilon_{0'})}_{0',1,\ldots,N}(y;\vec{u})\widehat{\mathbf{T}}^{(\epsilon_0)}_{0,1,\ldots,N}(x,\vec{u})\mathbf{R}_{0,0'}^{(\epsilon_0,\epsilon_{0'})}(x,y)\Phi,.$$

**Lemma** For

$$\widehat{\Omega} = \underbrace{\mathbf{f}^1 \otimes \ldots \otimes \mathbf{f}^1}_{N\times} \otimes \underbrace{\mathbf{e}_{-1} \otimes \ldots \otimes \mathbf{e}_{-1}}_{N\times} \otimes \omega$$

is valid

$$\widehat{T}_2^1(x;\vec{u})\widehat{\Omega} = 0\,, \qquad \widehat{T}_1^1(x;\vec{u})\widehat{\Omega} = \mu_1(x,\overline{u})\widehat{\Omega}\,, \qquad \widehat{T}_2^2(x;\vec{u})\widehat{\Omega} = \mu_2(x,\overline{u})\widehat{\Omega}\,,$$

$$\widehat{T}_{-1}^{-2}(x;\vec{u})\widehat{\Omega} = 0\,, \qquad \widehat{T}_{-1}^{-1}(x;\vec{u})\widehat{\Omega} = \mu_{-1}(x,\overline{u})\widehat{\Omega}\,, \qquad \widehat{T}_{-2}^{-2}(x;\vec{u})\widehat{\Omega} = \mu_{-2}(x,\overline{u})\widehat{\Omega}$$

where

$$\mu_1(x;\overline{u}) = \lambda_1(x)F(\overline{u},x-1)\,, \qquad \mu_2(x;\overline{u}) = \lambda_2(x)F(x-1,\overline{u})\,,$$

$$\mu_{-1}(x;\overline{u}) = \lambda_{-1}(x)F(x+1,\overline{u})\,, \qquad \mu_{-2}(x;\overline{u}) = \lambda_{-2}(x)F(\overline{u},x+1).$$

**Theorem** If for any $v_r \in \overline{v}$ and $w_s \in \overline{w}$ the conditions

$$\lambda_1(v_r)F(\overline{u}, v_r - 1)F(\overline{u}, v_r)F(v_r, \overline{v}_r)F(v_r - 2, \overline{w}) = \lambda_2(v_r)F(\overline{v}_r, v_r)F(\overline{w}, v_r - 2)$$

$$\lambda_{-1}(w_s)F(w_s + 1, \overline{u})F(w_s, \overline{u})F(\overline{v}, w_s + 2)F(\overline{w}_s, w_s) = \lambda_{-2}(w_s)F(w_s + 2, \overline{v})F(w_s, \overline{w}_s)$$

are fulfilled, then $\left|\vec{u}; \overline{v}; \overline{w}\right\rangle = \widehat{T}_1^2(\overline{v}; \vec{u})\widehat{T}_{-2}^{-1}(\overline{w}; \vec{u})\omega$ is eigenvector of $\tilde{H}^{(+)}(x; \vec{u})$ a $\tilde{H}^{(-)}(x, \vec{u})$ with eigenvalue

$$\begin{aligned}
\tilde{E}^{(+)}(x; \vec{u}, \overline{v}, \overline{w}) &= \lambda_1(x)F(\overline{u}, x - 1)F(x, \overline{v})F(x - 2, \overline{w}) + \\
&+ \lambda_2(x)F(x - 1, \overline{u})F(\overline{v}, x)F(\overline{w}, x - 2)
\end{aligned}$$

$$\begin{aligned}
\tilde{E}^{(-)}(x; \vec{u}, \overline{v}, \overline{w}) &= \lambda_{-1}(x)F(x + 1, \overline{u})F(\overline{v}, x + 2)F(\overline{w}, x) + \\
&+ \lambda_{-2}(x)F(\overline{u}, x + 1)F(x + 2, \overline{v})F(x, \overline{w})\,.
\end{aligned}$$

**Theorem** Let for any $u_k \in \overline{u}$, $v_r \in \overline{v}$ a $w_s \in \overline{w}$ the following conditions be fulfilled

$$\begin{aligned}
&\lambda_1(u_k)F(\overline{u}_k, u_k-1)F(\overline{u}_k, u_k)F(u_k, \overline{v})F(u_k-2, \overline{w}) = \\
&\qquad = \lambda_{-1}(u_k)F(u_k+1, \overline{u}_k)F(u_k, \overline{u}_k)F(\overline{v}, u_k+2)F(\overline{w}, u_k) \\
&\lambda_1(v_r)F(\overline{u}, v_r-1)F(v_r, \overline{v}_r)F(v_r-2, \overline{w}) = \\
&\qquad = \lambda_2(v_r)F(v_r-1, \overline{u})F(\overline{v}_r, v_r)F(\overline{w}, v_r-2) \\
&\lambda_{-1}(w_s)F(w_s+1, \overline{u})F(\overline{v}, w_s+2)F(\overline{w}_s, w_s) = \\
&\qquad = \lambda_{-2}(w_s)F(\overline{u}, w_s+1)F(w_s+2, \overline{v})F(w_s, \overline{w}_s)\,.
\end{aligned}$$

Then we obtain

$$\big|\vec{u};\overline{v};\overline{w}\big\rangle = \Big\langle \mathbf{B}_{1,\ldots,N}(\vec{u}), \Phi(\overline{v};\overline{w};\vec{v})\Big\rangle$$

where

$$\Phi(\overline{v};\overline{w};\vec{v}) = \widehat{T}_1^2(\overline{v},\vec{u})\widehat{T}_{-2}^{-1}(\overline{w};\vec{u})\widehat{\Omega}$$

is eigenvector $H(x)$ with eigenvalue

$$\begin{aligned}
E(x;\vec{u},\overline{v},\overline{w}) = {}&\lambda_1(x)F(\overline{u},x)F(\overline{u},x-1)F(x,\overline{v})F(x-2,\overline{w}) + \\
&+\lambda_{-1}(x)F(x,\overline{u})F(x+1,\overline{u})F(\overline{v},x+2)F(\overline{w},x) + \\
&+\lambda_2(x)F(\overline{v},x)F(\overline{w},x-2) + \lambda_{-2}(x)F(x+2,\overline{v})F(x,\overline{w})
\end{aligned}$$

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