Non-perturbative superpotentials and discrete torsion

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It is well acknowledged that heterotic string (M-) theory on a Calabi-Yau manifold can lead to realistic low-energy physics

Even quite advanced questions like realistic Yukawa structure or proton stability can be taken into account

In my talk I will be interested in non-perturbative superpotentials in $D = 4$ effective field theory which are crucial for moduli stabilization

We start with heterotic string (M-) theory in $D = 10$ $(D = 11)$ and compactify to $D = 4$ on a CY manifold. To specify a $D = 4$ vacuum we need to specify: a CY manifold X $(g_{m\bar{n}})$, a vector bundle V on X $(A_m, A_{\bar{n}})$, a B-field:

$$F_{mn} = F_{\bar{m}\bar{n}} = 0 \,, \quad g^{m\bar{n}} F_{m\bar{n}} = 0 \,, \quad dB = 0$$

Moduli problem: integration constants $\Longrightarrow$ massless moduli fields which do not have potential energy and can take any values $\Longrightarrow$ importance of non-perturbative effects

Kahler moduli, complex structure moduli, vector bundle moduli

Dine, Seiberg, Wen, Witten'86: non-perturbative superpotentials for moduli fields can come from worldsheet instantons - Euclidean strings wrapping complex curves in X.

$$S_{cl} \sim \int_C d^2 z [\partial_z Y^m \partial_{\bar z} \bar{Y}^{\bar n} g_{m\bar n} + \partial_{\bar z} Y^m \partial_z \bar{Y}^{\bar n} g_{m\bar n}]$$

The action is minimized when one of the terms is zero:

$$\partial_{\bar z} Y^m = 0 \Longrightarrow Y^m = Y^m(z) \text{ holomorphic curve}$$

$$S_{cl} \sim \int_C d^2 z \partial_z Y^m \partial_{\bar z} \bar{Y}^{\bar n} g_{m\bar n} = \int_C \omega = A(C) \ - \ \text{finite}$$

Holomorphic curves preserve supersymmetry

Not all holomorphic curves contribute to the superpotential: they must have genus 0, they must be *isolated*. The general formula for the superpotential was derived by Witten'99:

$$W(C) = e^{i\int_C \omega_{\mathbf{C}}} \frac{\text{Pfaff}[\bar\partial_{V|_C \otimes \mathcal{S}}]}{\det[\bar\partial_{N_C}]}$$

The exponent depends on Kahler moduli:

$$i \int_C \omega_{\mathbf{C}} = -\frac{A(C)}{2\pi\alpha'} + i \int_C B = i\alpha_I(C)T^I(\mathrm{x})\,, \quad \alpha_I = \int_C \omega_I$$

ω_I is a basis of $(1,1)$-forms on X

Pfaff depends on bundle moduli and complex structure moduli

It's anomalous $\Longrightarrow$ it's not a function but a section of a line bundle on the moduli space (often a polynomial)

It is impossible to compute directly because no explicit solutions for the gauge connection are known

In a homology class there are many holomorphic, isolated, genus 0 curves. The number is referred to as Gromov-Witten invariants $n_{[C]}$

$$W([C]) = e^{i\alpha_I(C)T^I(x)} \sum_{i=1}^{n_{[c]}} \frac{\mathrm{Pfaff}[\bar{\partial}_{V|_{C_i}\otimes\mathcal{S}}]}{\det[\bar{\partial}_{N_{C_i}}]}$$

The full superpotential: $W = \sum_{[C]\in H_2(X,\mathbf{Z})} W([C])$

Beasley and Witten'04 showed that for a large class of CY compactifications $W([C]) = 0$. They called it a *residue theorem*. Define a CY manifold X as a complete intersection in $\mathcal{A} = \mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_a}$. X is defined as a set of m polynomial equations $\sum_{i=1}^{a} n_i - m = 3$

The bundle is taken as a restriction from $\mathcal{A}$: $V = \mathcal{V}|_X$.

Beasley and Witten constructed a top. sigma model on $\mathcal{A}$ with supersymmetric vacua: CY X, a holomorphic curve $C \in X$, a bundle $V = \mathcal{V}|_X$. They used standard arguments from topological field theory that the partition function is independent of the coupling. In one limit they find it to be the heterotic string theory superpotential, in another limit they find that it is zero because of unsaturated fermionic zero modes.

Question: are there any examples where $W \neq 0$??

The aim of the talk is to present such examples.

There is an additional assumption in the analysis of BW: since top. field theory is defined on $\mathcal{A}$ it follows that the area of all holomorphic curves is measured using $\omega_{\mathcal{A}}|_X$. It is not necessarily the same as the Kahler form ω_X. The residue theorem is applicable only if $\omega_X = \omega_{\mathcal{A}}|_X \Longleftrightarrow h^{1,1}(X) = h^{1,1}(\mathcal{A})$. If $h^{1,1}(X) > h^{1,1}(\mathcal{A})$ there are classes in X which do not descend from the ambient space and the residue theorem is no longer valid

$\omega_X = \omega_{\mathcal{A}}|_X + \Delta\omega_X$ where $\Delta\omega_X$ is the contribution from extra classes

$$\int_C \omega_X = \int_C (\omega_{\mathcal{A}}|_X + \Delta\omega_X) \geq \int_C \omega_{\mathcal{A}}|_X$$

Two curves C_1 and C_2 satisfying

$$\int_{C_1} \omega_{\mathcal{A}}|_X = \int_{C_2} \omega_{\mathcal{A}}|_X$$

are treated as lying in the same homology class by BW

However, if $h^{1,1}(X) > h^{1,1}(\mathcal{A})$ these two curves might have a different area and lie in different homology classes. To say it differently: the cancelation followed from the residue theorem always happens in topological theory. But it does not necessarily imply that the same cancelation happens in heterotic string theory.

If $h^{1,1}(X) = h^{1,1}(\mathcal{A})$ then the same cancelation happens in heterotic string theory.

If $h^{1,1}(X) > h^{1,1}(\mathcal{A})$ nothing can be said in advance about the cancelation. The residue theorem is still indirectly applicable. If we measure the area using $\omega_{\mathcal{A}}|_X$ instead of ω_X then we formally can use the residue theorem which implies cancelation of certain one-loop determinants. They correspond to the curves lying in general in different homology classes but satisfying $\int_{C_1} \omega_{\mathcal{A}}|_X = \int_{C_2} \omega_{\mathcal{A}}|_X$. But this cancelation does not imply vanishing of the superpotential

There is another ingredient missing in the general formula for the superpotential called *discrete torsion*

An element of an Abelian group g is called a torsion element if $g + g + \ldots + g = 0$

$$H_2(X, \mathbf{Z}) = \mathbf{Z}^k \oplus G_{tor}$$

G_{tor} is a finite group called discrete torsion. If $G_{tor} \neq 0$ there are torsion holomorphic curves. It is not relevant for complete intersection CY and for the residue theorem. Let's assume that a complete intersection CY manifold $\tilde{X}$ has a freely acting discrete symmetry K. We can construct a different CY manifold $X = \tilde{X}/K$. It turns out X might have torsion. It affects the B-field and its field strength H. A heterotic $D = 4$ vacuum is specified by a choice of H which must vanish in $H^3(X, \mathbf{R})$. But it does not mean that it vanishes in $H^3(X, \mathbf{Z})$.

The universal coefficient theorem:

$$H_2(X,\mathbf{Z})_{tor} = H^3(X,\mathbf{Z})_{tor}$$

If H defines a non-trivial torsion element in $H^3(X,\mathbf{Z})$ its potential B is not globally defined and $\int_C B$ is also not defined. The solution was suggested by Aspinwall-Morrison'94. One can view the exponential prefactor as a map:

$$e^{-S_{cl}} : H_2(X,\mathbf{Z}) \to \mathbf{C}^*$$

This map has to be modified:

$$e^{-S_{cl}} : C \to e^{i\alpha_I(C)T^I} \prod_{\alpha=1}^{r} \chi_\alpha^{\beta_\alpha(C)}$$

$\beta_1,\ldots,\beta_r$ are generators of G_{tor}, $\beta_\alpha(C)$ is the value of the generator on the curve C, χ_α is a character of G_{tor}. The choice of the characters depends on the choice of H which is a (discrete) choice of a vacuum.

$\beta_\alpha(C)$ are discrete analogs of $\alpha_I(C) = \int_C \omega_I$
$\chi_\alpha = e^{is_\alpha}$ are discrete analogs of T^I which specify a heterotic string vacuum
Curves in the same homology class in $H_2(X, \mathbf{Z})$ pick up the same torsion factor, curves in different homology classes pick up different torsion factors
To find a more general expression for the superpotential, let $[C]$ a homology class of a curve C in $H_2(X, \mathbf{R})$ (ignoring torsion). All curves in $[C]$ have the same area but may belong to different torsion classes. The contribution to the superpotential from the curves in $[C]$ is then

$$W([C]) = e^{i\alpha_I(C)T^I(\mathrm{x})} \sum_{i=1}^{n_{[c]}} \frac{\mathrm{Pfaff}[\bar{\partial}_{V|_{C_i}\otimes\mathcal{S}}]}{\det[\bar{\partial}_{N_{C_i}}]} \prod_{\alpha=1}^{r} \chi_\alpha^{\beta_\alpha(C_i)}$$

The full superpotential is $W = \sum_{[C]\in H_2(X,\mathbf{R})} W([C])$
Only holomorphic isolated genus zero curves contribute

Consider the Schoen manifold. It is a complete intersection CY $\tilde{X} \subset \mathcal{A} = \mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ with coordinates $[t_0, t_1], [x_0, x_1, x_2], [y_0, y_1, y_2]$

$$F_1 = t_0(x_0^3 + x_1^3 + x_2^3) + t_1(x_0 x_1 x_2) = 0$$

$$F_2 = (\lambda_1 t_0 + t_1)(y_0^3 + y_1^3 + y_2^3) + (\lambda_2 t_0 + \lambda_3 t_1)(y_0 y_1 y_2) = 0$$

$\lambda_1, \lambda_2, \lambda_3$ are complex structure parameters

$19 = h^{1,1}(\tilde{X}) > h^{1,1}(\mathcal{A}) = 3$. There are 16 $(1,1)$ classes in $\tilde{X}$ which do not come from $\mathcal{A} \Longrightarrow \omega_{\tilde{X}} \neq \omega_{\mathcal{A}}|_{\tilde{X}}$

$H_2(\tilde{X}, \mathbf{Z}) = \mathbf{Z}^{19} \Longrightarrow$ no torsion

There is $\mathbf{Z}_3 \times \mathbf{Z}_3$ symmetry :

$$g_1 : [x_0, x_1, x_2] \rightarrow [x_0, e^{2\pi i/3} x_1, e^{4\pi i/3} x_2]$$

$$[y_0, y_1, y_2] \rightarrow [y_0, e^{2\pi i/3} y_1, e^{4\pi i/3} y_2]$$

$$g_2 : [x_0, x_1, x_2] \rightarrow [x_1, x_2, x_0], \quad [y_0, y_1, y_2] \rightarrow [y_1, y_2, y_0]$$

We can define $X = \tilde{X}/\mathbf{Z}_3 \times \mathbf{Z}_3$

$h^{1,1}(X) = h^{1,1}(\mathcal{A}) = 3$. The invariant classes of $\tilde{X}$ which become the 3 classes of X are precisely the restrictions of the classes of Kahler forms on $\mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^2$

$H_2(X, \mathbf{Z}) = \mathbf{Z}^3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3 \Longrightarrow$ there is torsion

$G_{tor} = \mathbf{Z}_3 \oplus \mathbf{Z}_3$

Below we will see that the superpotential is non-zero in theories on $\tilde{X}$ and X

Let us look at eq. $F_1 = 0 \subset \mathbf{P}^1 \times \mathbf{P}^2$. At each point $[t_0, t_1]$ it defines an elliptic curve. We have an elliptic fibration over $\mathbf{P}^1$ called a rational elliptic surface dP_9. Over some points on $\mathbf{P}^1$ the fiber becomes singular. Take a point $[t_0 = 0, t_1 = 1]$: the elliptic fiber above this point is $x_0 x_1 x_2 = 0$. It gives 3 spheres intersecting at a triangle with singularities at the intersection points. This type of singularity is called I_3 in Kodaira classification

By examining the eq. $F_1 = 0$ one can show there are 4 singular fibers of I_3 type. The Schoen manifold $\tilde{X}$ can be viewed as a double elliptic fibrations over $\mathbf{P}^1$. These extra spheres are the reason why $\tilde{X}$ has additional 16 classes which do not descend from the ambient space. They are also the reason for the torsion of the quotient CY manifold X
The classes which descend from $\tilde{X}$ to X are the invariant classes of $\tilde{X}$ represented by invariant differential forms. They are given by restrictions of the Kahler forms on $\mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^2$: $J_I = \mathcal{J}_I|_{\tilde{X}}$. The Kahler form on $\tilde{X}$ is

$$\omega_{\tilde{X}} = \omega_{\mathcal{A}}|_{\tilde{X}} + \Delta\omega_{\tilde{X}} = 2\pi\alpha' \sum_{I=1}^{3} \mathrm{Im}(T^I) J_I + \Delta\omega_{\tilde{X}}$$

The Kahler form on X is given in terms of the invariant classes:

$$\omega_X = 2\pi\alpha' \sum_{I=1}^{3} \mathrm{Im}(T^I) J_I$$

Gromov-Witten invariants of X

The Gromov-Witten invariants of X can be computed using the type II prepotential: Braun, Kreuzer, Ovrut, Scheidegger'07 $H_2(X,\mathbf{Z}) = \mathbf{Z}^3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3$. Let $[C_1], [C_2], [C_3]$ be invariant homology classes dual to J_1, J_2, J_3:

$$\int_{C_I} J_J = \delta_{IJ}$$

$$p = e^{-S_{cl}}([C_1]) = e^{iT^1}, \; q = e^{iT^2}, \; r = e^{iT^3}$$

In addition, there are torsion classes β_1, β_2 satisfying $3\beta_1 = 0, 3\beta_2 = 0$. Any curve can be labeled by the integers $\beta_1(C) = m_1, \beta_2(C) = m_2, \;\; m_1, m_2 = 0, 1, 2$

$$[C] = (n_1, n_2, n_3, m_1, m_2) \in \mathbf{Z}^3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3$$

$$e^{-S_{cl}}[C] = p^{n_1} q^{n_2} r^{n_3} \chi_1^{m_1} \chi_2^{m_2}$$

The type II prepotential:

$$\mathcal{F}_X = \sum_{[C]\in H_2(X,\mathbf{Z})} n_{[C]} Li_3(e^{-S_{cl}}[C])$$

$$= \sum_{[C]\in H_2(X,\mathbf{Z})} n_{[C]} Li_3(p^{n_1} q^{n_2} r^{n_3} \chi_1^{m_1} \chi_2^{m_2})$$

If we know $\mathcal{F}_X$ we can expand it and read off the Gromov-Witten invariants. The result to low order in p, q, r:

$$\mathcal{F}_X = p(1+\chi_1+\chi_1^2)(1+\chi_2+\chi_2^2)(1+\mathcal{O}(q)+\mathcal{O}(r))+\mathcal{O}(p^2)$$

No terms $\sim p^0 \Longrightarrow$ no isolated genus 0 curves in homology classes $(0, n_2, n_3, m_1, m_2)$

There is a term $\sim p \Longrightarrow$ there are isolated genus 0 curves in the classes $(1, n_2, n_3, m_1, m_2)$

The exponential part of superpotential from curves in these classes is

$$W \sim e^{iT^1 + in_2T^2 + in_3T^3}$$

The leading contribution is $W \sim e^{iT^1}$. All other terms are exponentially suppressed. We will concentrate on the leading contribution $\sim e^{iT^1}$

$$\mathcal{F}_X = p(1 + \chi_1 + \chi_1^2)(1 + \chi_2 + \chi_2^2)$$

There are 9 isolated genus 0 curves whose (complexified) area behaves as iT^1. They are all in 9 different homology classes once we take torsion into account

$n_{[C]} = 1$ for each class $[C] = (1, 0, 0, m_1, m_2)$

Overall we have 9 isolated curves in X of interest, each having the same area $\sim e^{iT^1}$ but each curve is in its own homology class with respect to torsion

Due to $\mathbf{Z}_3 \times \mathbf{Z}_3$-symmetry these 9 curves come from 81 curves on the covering manifold $\tilde{X}$. They will break into 9 orbits, 9 curves in each orbit and then an entire orbit becomes a single curve on X.

These 81 curves in $\tilde{X}$ can be constructed very explicitly. All of them originate from $\mathbf{P}^1 \in \mathcal{A}$ parametrized by $[t_0, t_1]$. To find them we simply solve $F_1 = 0, F_2 = 0$ for arbitrary $[t_0, t_1]$:

$$x_0 x_1 x_2 = 0\,,\ x_0^3 + x_1^3 + x_2^3 = 0\,,\ y_0 y_1 y_2 = 0\,,\ y_0^3 + y_1^3 + y_2^3 = 0$$

This system has $3 \cdot 3 \cdot 3 \cdot 3 = 81$ solutions. Each solution defines a point in $\mathbf{P}^2 \times \mathbf{P}^2 : s_i$ and a curve of the form

$C_i = \mathbf{P}^1 \times s_i = [t_0, t_1] \times s_i$

All these curves are holomorphic genus zero and isolated

One particular solution is $s_1 = (1, -1, 0) \times (1, -1, 0)$.

Acting on s_1 with $\mathbf{Z}_3 \times \mathbf{Z}_3$ symmetry we obtain an orbit consisting with 9 curves. All of them will be identified on the quotient manifold. Similarly we can find the remaining curves and orbits. These 81 curves are precisely the ones whose 9 orbits give rise to the 9 curves counted by the leading prepotential

The superpotential on $\tilde{X}$

We consider heterotic string theory on the Schoen manifold $\tilde{X}$ with some vector bundle. The 81 curves form the full set of curves satisfying

$$\int_{C_i} \omega_{\mathcal{A}}|_{\tilde{X}} = \int_{C_j} \omega_{\mathcal{A}}|_{\tilde{X}} = 2\pi\alpha' \mathrm{Im}(T^1)$$

However, $\omega_{\tilde{X}} = \omega_{\mathcal{A}}|_{\tilde{X}} + \Delta\omega_{\tilde{X}}$ and it might happen that

$$\int_{C_i} \omega_{\tilde{X}} \neq \int_{C_j} \omega_{\tilde{X}}$$

Using mathematical properties of elliptic fibrations one can show that all these 81 curves lie in **different homology classes**. They all have different area. The consequence: the superpotential in this theory is non-zero for a generic vector bundle.

As long as $\mathrm{Pfaff}[\bar{\partial}_{V|_{C_i}\otimes\mathcal{S}}] \neq 0$ at least for one curve the superpotential is not zero because each curve is a unique isolated, genus 0 curve in its homology class.

The Schoen manifold avoids the Beasley-Witten no-go residue theorem because $h^{1,1}(\tilde{X}) > h^{(1,1)}(\mathcal{A})$

However, the residue theorem is still indirectly applicable

$$W(\{C_i\}) = \sum_{i=1}^{81} e^{i\alpha_I(C_i)T^I(\mathrm{x})} \frac{\mathrm{Pfaff}[\bar{\partial}_{V|_{C_i}\otimes\mathcal{S}}]}{\det[\bar{\partial}_{N_{C_i}}]}$$

The exponential prefactors $e^{i\alpha_I(C_i)T^I(x)} = e^{iT^1(x)+\dots}$ are different for all 81 terms so the sum is not zero. However, if we decide to measure the area using $\omega_{\mathcal{A}}\big|_{\tilde{X}}$ instead of $\omega_{\tilde{X}}$ the residue theorem is applicable $\Longrightarrow$

$$\sum_{i=1}^{81} \frac{\mathrm{Pfaff}[\bar{\partial}_{V|_{C_i}\otimes\mathcal{S}}]}{\det[\bar{\partial}_{N_{C_i}}]} = 0$$

Let us pick a toy model for a vector bundle $\mathcal{V}$ on $\mathcal{A}$ with $\tilde{V} = \mathcal{V}|_{\tilde{X}}$ on $\tilde{X}$. We are also interested in theory on $X = \tilde{X}/\mathbf{Z}_3 \times \mathbf{Z}_3 \Longrightarrow$ we choose $\tilde{V}$ to be *equivariant* under $\mathbf{Z}_3 \times \mathbf{Z}_3 \Longrightarrow V = \tilde{V}/\mathbf{Z}_3 \times \mathbf{Z}_3$.

A toy $SU(3)$-model with E_6 low-energy gauge group: choose 3 equivariant line bundles on $\mathcal{A}$ $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ satisfying $\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 = \mathrm{trivial}$ and let $\mathcal{V}_0 = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$, $\tilde{V}_0 = \mathcal{V}_0|_{\tilde{X}}$. The structure group is $S(U(1)^3)$. This can be viewed as a special point in moduli space of low-energy field theory with enhanced symmetry $E_6 \to E_6 \times S(U(1)^3)$. We want to Higgs these extra symmetries by deforming $\mathcal{V}_0 \to \mathcal{V}, \tilde{V}_0 \to \tilde{V} = \mathcal{V}|_{\tilde{X}}$ so that $\mathcal{V}, \tilde{V}$ have structure group $SU(3)$. Mathematically this procedure is called *extension*

$$\mathcal{L}_1 = \mathcal{O}_{\mathcal{A}}(-2,2,1)\,,\ \mathcal{L}_2 = \mathcal{O}_{\mathcal{A}}(0,1,-1)\,,\ \mathcal{L}_3 = \mathcal{O}_{\mathcal{A}}(2,-3,0)$$

$$\mathcal{V}_0 \to \mathcal{V} \to \tilde{V} = \mathcal{V}|_{\tilde{X}} \to V \text{ on } X$$

The moduli space of V: $\mathcal{M}(V) = \mathbf{P}^{12}$

A point $v \in \mathbf{P}^{12}$ can be parameterized

$$v = t_0^2 f_1(\vec{x}, \vec{y}) + t_0 t_1 f_2(\vec{x}, \vec{y}) + t_1^2 f_3(\vec{x}, \vec{y})$$

Here f_1, f_2, f_3 are invariant polynomials on $\mathbf{P}^2 \times \mathbf{P}^2$ of degree $(5, 1)$:

$$f_1 = \sum_{\alpha=1}^{7} a_\alpha(\mathrm{x}) E_\alpha \, , \;\; f_2 = \sum_{\alpha=1}^{7} b_\alpha(\mathrm{x}) E_\alpha \, , \;\; f_3 = \sum_{\alpha=1}^{7} c_\alpha(\mathrm{x}) E_\alpha$$

$$E_1 = x_0^5 y_0 + x_1^5 y_1 + x_2^5 y_2 \, , E_2 = x_0^2 x_1^3 y_0 + x_1^2 x_2^3 y_1 + x_2^2 x_0^3 y_2 \, , \ldots$$

$a_\alpha, b_\alpha, c_\alpha$ are projective vector bundle moduli. Not all of the are independent: there are 8 linear relations meaning that only 21-8=13 are independent $\Longrightarrow$ 12 coordinates on $\mathbf{P}^{12}$

Let us compute the Pfaffians in our toy models for our 81 curves in $\tilde{X}$ and 9 curves in X.

How to compute $\mathrm{Pfaff}[\bar\partial_{V|_C\otimes\mathcal{S}}]$? No solutions for the gauge connection are known and we do not know the connection on our chosen vector bundle

Pfaff is holomorphic section on $\mathbf{P}^{12}$, that is a homogeneous polynomial which is uniquely determined (up to a coefficient) by its zeroes. The zero modes of $\bar\partial_{V|_C\otimes\mathcal{S}}$ are elements of the cohomology group $H^0(C, V|_C\otimes\mathcal{S})$

$$\mathrm{Pfaff}[\bar\partial_{V|_C\otimes\mathcal{S}}]=0 \quad \text{iff} \quad h^0(C, V|_C\otimes\mathcal{S})\neq 0$$

The crucial moment is that $h^0(C, V|_C\otimes\mathcal{S})$ is not a topological invariant. In depends on the point in moduli space $\mathcal{M}(V)$. For a generic point in $\mathbf{P}^{12}$ $h^0(C, V|_C\otimes\mathcal{S})=0$ and $\mathrm{Pfaff}[\bar\partial_{V|_C\otimes\mathcal{S}}]\neq 0$. But for points lying on a special co-dimension 1 subspace $h^0(C, V|_C\otimes\mathcal{S})\neq 0$ and $\mathrm{Pfaff}[\bar\partial_{V|_C\otimes\mathcal{S}}]=0$.

For 81 curves of the type $[t_0, t_1] \times s_i$, $s_i \in \mathbf{P}^2 \times \mathbf{P}^2$ the equation for this co-dim. 1 subspace:

$$\mathcal{R}_i = (f_1 f_3 - f_2^2)(s_i) = \sum_{\alpha,\beta=1}^{7} (a_\alpha c_\beta - b_\alpha b_\beta) E_\alpha(s_i) E_\beta(s_i) = 0$$

$$\mathcal{R}_1 = -(2b_1 - b_2 - b_3)^2 + (2a_1 - a_2 - a_3)(2c_1 - c_2 - c_3)\,,$$

$$\mathcal{R}_2 = -(b_2 + e^{4\pi i/3} b_3 + e^{2\pi i/3} b_1)^2$$

$$+(a_2 + e^{4\pi i/3} a_3 + e^{2\pi i/3} a_1)(c_2 + e^{4\pi i/3} c_3 + e^{2\pi i/3} c_1)\,, \ldots$$

Since Pfaff is also a homogeneous polynomial with the same zeros we conclude:

$$\mathrm{Pfaff}[\bar{\partial}_{V|_{C_i} \otimes \mathcal{S}}] = A_i \mathcal{R}_i$$

A_i is a non-zero complex number which cannot be determined by this method.

The theory on $\tilde{X}$: the superpotential is non-zero because each curve is in its own homology class and the Pfaffians are not identically zero.

The consistency check from the residue theorem. Due to $\mathbf{Z}_3 \times \mathbf{Z}_3$ symmetry all curves in the orbit give the same contribution

$$9 \sum_{i=1}^{9} A_i \mathcal{R}_i = 0$$

Strong consistency check: 9 polynomials $\mathcal{R}_i$ must be linearly dependent. It turns out to be true and it constrains the coefficients A_i :

$$A_1 = -A_4 - A_5\,, \;\; A_2 = e^{i\pi/3} A_4 - A_7\,, \;\; A_3 = e^{i\pi/3} A_5 - e^{2i\pi/3} A_7$$

$$A_6 = -e^{-i\pi/3} A_5 + e^{-i\pi/3}(A_4 - A_7), \;\; A_8 = A_4 + A_5 - A_7$$

$$A_9 = A_4 + e^{-i\pi/3}(A_5 - A_7)$$

The theory on X: the leading superpotential (up to exponentially subleading terms)

$$W_X([C]) = e^{iT^1} \sum_{i=1}^{9} A_i \mathcal{R}_i \chi_1^{m_i} \chi_2^{n_i} \,, \quad m_i, n_i = 0, 1, 2$$

where $\mathcal{R}_i$ are as above, A_i are constrained as above and χ_1, χ_2 are characters of $\mathbf{Z}_3$ depending on the choice of the characteristic class of the H-field. In the absence of torsion the superpotential would be zero by the residue theorem. Assuming that at least one χ_1 or χ_2 is non-trivial the sum is no-longer zero because

$$\sum_{i=1}^{9} A_i \mathcal{R}_i = 0 \implies \sum_{i=1}^{9} A_i \mathcal{R}_i \chi_1^{m_i} \chi_2^{n_i} \neq 0$$

This gives an explicit proof that the leading superpotential is non-zero. The key reason is *discrete torsion*

Conclusion

- The aim of the talk is to present first ever examples of heterotic string theory where the non-perturbative superpotential can be proven to be non-zero avoiding the no-go residue theorem of Beasley-Witten
- The first example involves the Schoen manifold where the superpotential is non-zero because there are isolated, genus zero curves unique in their homology classes
- The second example involves the quotient of the Schoen manifold by a $\mathbf{Z}_3 \times \mathbf{Z}_3$. The superpotential is non-zero due to discrete torsion
- Further directions: find a systematic way to find heterotic compatifications with non-vanishing superpotential. Apply to heterotic MSSM (one of which exists on a quotient of the Schoen manifold)