Four dimensional ambitwistor strings and form factors of local and Wilson line operators

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Outline

Scattering equations

$$f_i(z,p) \;=\; 0, \qquad i \in \{1, ..., n\}.$$

$$f_i(z,p) \;\equiv\; \sum_{j=1, j\neq i}^{n} \frac{2p_i \cdot p_j}{z_i - z_j}$$

$$p = (p_1, p_2, ..., p_n)\,, p_i^2 = 0\,, \quad z = (z_1, z_2, ..., z_n) \in \hat{\mathbb{C}}^n\,, \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\,.$$

Invariant under projective special linear group

$$\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2, \quad \mathbb{Z}_2 = \{\mathbf{1}, -\mathbf{1}\}\,, \mathbf{1} - \text{unity matrix}\,.$$

only $(n-3)$ equations are independent.

Cachazo, He, Yuan, 2014

Scattering equations

Scattering equations - alternative polynomial form:

$$h_m(z,p) = \sum_{S\subset I, |S|=m} p_S^2\, z_S\,, \quad I = \{1,2,...,n\}$$

$$p_S = \sum_{i\in S} p_i, \quad z_S = \prod_{i\in S} z_i, \quad p_\emptyset = 0, \quad z_\emptyset = 1, \quad h_0 = h_1 = h_{n-1} = h_n = 0.$$

$$h_m(z,p) \;=\; 0, \qquad 2\le m\le n-2.$$

Bezout theorem and moduli space of genus 0 curves:

$V_n(p) = \left\{ z\in C^{n-3} |\ h_m(z,p) = 0;\ 2\le m\le n-2 \right\}$ has $(n-3)!$ points

Solutions:

$$z^{(j)} = \left(z_1^{(j)}, ..., z_n^{(j)}\right) \in \mathcal{M}_{0,n}\,,$$

$$\mathcal{M}_{0,n} = \left\{ z\in \left(\mathbb{CP}^1\right)^n : z_i \ne z_j \right\} / \mathrm{PSL}\left(2,\mathbb{C}\right).$$

Multidimensional complex contour integral in $\hat{\mathbb{C}}^n$:

$$A_n^{(0)}(\sigma, p, \varepsilon) = i \oint_{\mathcal{C}} d\Omega_{\mathrm{CHY}}\ C(\sigma, z)\ E(z, p, \varepsilon),$$

where

$$d\Omega_{\mathrm{CHY}} = \frac{1}{(2\pi i)^{n-3}} \frac{d^n z}{d\omega} \prod{}' \frac{1}{f_a(z, p)},$$

$$\prod{}' \frac{1}{f_a(z, p)} = (-1)^{i+j+k} (z_i - z_j)(z_j - z_k)(z_k - z_i) \prod_{a \neq i,j,k} \frac{1}{f_a(z, p)},$$

$$d\omega = (-1)^{p+q+r} \frac{dz_p dz_q dz_r}{(z_p - z_q)(z_q - z_r)(z_r - z_q)}.$$

Sum over inequivalent zeros of scattering equations:

$$A_n^{(0)}(\sigma,p,\varepsilon) = i \sum_{\text{solutions } j} J\left(z^{(j)},p\right)\, C\left(\sigma,z^{(j)}\right)\, E\left(z^{(j)},p,\varepsilon\right).$$

Jacobian:

$$J(z,p) = \frac{1}{\det{}'\,\Phi}.$$

$$\det{}'\,\Phi = (-1)^{i+j+k+r+s+t}\,\frac{\left|\Phi^{ijk}_{rst}\right|}{(z_{ij}z_{jk}z_{ki})(z_{rs}z_{st}z_{tr})},$$

$$\Phi_{ab} = \frac{\partial f_a}{\partial z_b} = \begin{cases} \frac{2p_a\cdot p_b}{(z_a-z_b)^2} & a\neq b, \\ -\sum\limits_{j=1,j\neq a}^{n}\frac{2p_a\cdot p_j}{\left(z_a-z_j\right)^2} & a=b. \end{cases}$$

$(2n) \times (2n)$ anti-symmetric matrix:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad A_{ab} = \begin{cases} \frac{2p_a \cdot p_b}{z_a - z_b} & a \neq b, \\ 0 & a = b, \end{cases}$$

$$B_{ab} = \begin{cases} \frac{2\varepsilon_a \cdot \varepsilon_b}{z_a - z_b} & a \neq b, \\ 0 & a = b, \end{cases} \quad C_{ab} = \begin{cases} \frac{2\varepsilon_a \cdot p_b}{z_a - z_b} & a \neq b, \\ -\sum\limits_{j=1, j \neq a}^{n} \frac{2\varepsilon_a \cdot p_j}{z_a - z_j} & a = b. \end{cases}$$

Polarization factor:

$$E(p, \varepsilon, z) = \frac{(-1)^{i+j}}{2(z_i - z_j)} \mathrm{Pf}\ \Psi^{ij}_{ij},$$

where Ψ^{ij}_{ij} - $(n-2) \times (n-2)$ matrix with rows and columns i, j deleted.

$E(p, \varepsilon, z)$ is independent of the i, j choice on solutions of scattering equations!

Cyclic or Parke-Taylor factor:

$$C(\sigma, z) = \frac{1}{(z_{\sigma_1} - z_{\sigma_2})(z_{\sigma_2} - z_{\sigma_3}) \dots (z_{\sigma_r} - z_{\sigma_1})}.$$

under $\mathrm{PSL}(2,\mathbb{C})$ transformations

$$E(p, \varepsilon, g \cdot z) = \left(\prod_{j=1}^{n} (cz_j + d)^2 \right) E(p, \varepsilon, z),$$

$$C(\sigma, g \cdot z) = \left(\prod_{j=1}^{n} (cz_j + d)^2 \right) C(\sigma, z).$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}).$$

Scattering equations: $n = 3$, $\sigma = (1, 2, 3)$ example

One solution of scattering equations:

$$z_1^{(1)} = 0, \qquad z_2^{(1)} = 1, \qquad z_3^{(1)} = \infty.$$

Factors:

$$\begin{aligned} J(z, p) &= (z_1 - z_2)^2 (z_2 - z_3)^2 (z_3 - z_1)^2, \\ C(\sigma, z) &= \frac{1}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}, \\ E(z, p, \varepsilon) &= \frac{2\left[(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot p_1) + (\varepsilon_2 \cdot \varepsilon_3)(\varepsilon_1 \cdot p_2) + (\varepsilon_3 \cdot \varepsilon_1)(\varepsilon_2 \cdot p_3)\right]}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}. \end{aligned}$$

Amplitude:

$$\begin{aligned} A_3^{(0)}(\sigma, p, \varepsilon) &= iJ\left(z^{(1)}, p\right) C\left(\sigma, z^{(1)}\right) E\left(z^{(1)}, p, \varepsilon\right) \\ &= 2i\left[(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot p_1) + (\varepsilon_2 \cdot \varepsilon_3)(\varepsilon_1 \cdot p_2) + (\varepsilon_3 \cdot \varepsilon_1)(\varepsilon_2 \cdot p_3)\right]. \end{aligned}$$

Contour integral representation:

$$A_n^{(0)}(\sigma,p,\varepsilon) = i\frac{(-1)^{i+j+k}}{(2\pi i)^{n-3}}\oint_{\mathcal{C}}\frac{d^n z}{d\omega}\,\frac{z_{ij}z_{jk}z_{ki}}{\prod\limits_{a\neq i,j,k} f_a(z,p)}\;C(\sigma,z)\;E(z,p,\varepsilon)\,.$$

polynomial form

$$A_n^{(0)}(\sigma,p,\varepsilon) = i\frac{(-1)^{n}}{(2\pi i)^{n-3}}\oint_{\mathcal{C}}\frac{d^n z}{d\omega}\,\frac{\prod\limits_{i<j}(z_i-z_j)}{\prod\limits_{m=2}^{n-2} h_m(z,p)}\;C(\sigma,z)\;E(z,p,\varepsilon)\,.$$

fixing $\mathrm{PSL}(2,\mathbb{C})$ gauge: $z_1=0,\;\; z_{n-1}=1,\;\; z_n=\infty$

$$A_n^{(0)}(\sigma,p,\varepsilon) = -\frac{i}{(2\pi i)^{n-3}}\oint_{\mathcal{C}}\frac{R(z,\sigma,p,\varepsilon)\;dz_2\wedge\ldots\wedge dz_{n-2}}{h'_2(z,p)\ldots h'_{n-2}(z,p)},$$

$$R(z,\sigma,p,\varepsilon) = z_n^4\left(\prod_{i<j<n} z_{ij}\right)C(\sigma,z)\;E(z,p,\varepsilon)\,,\quad h'_m(z,p) = \left.\frac{dh_m(z,p)}{dz_n}\right|_{z_n=0}$$

Scattering equations: global residues

Local residue

$$\mathrm{Res}_{\{h_2',\ldots,h_{n-2}'\}}\left(f, z^{(j)}\right) = \frac{1}{(2\pi i)^{n-3}} \oint\limits_{\Gamma_\delta} \frac{f(z)\; dz_2 \wedge \ldots \wedge dz_{n-2}}{h_2'(z)\ldots h_{n-2}'(z)},$$

$$\Gamma_\delta = \left\{(z_2, \ldots, z_{n-2}) \in \mathbb{C}^{n-3} \,|\, |h_m'(z)| = \delta\right\}.$$

Global residue

$$\mathrm{Res}_{\{h_2',\ldots,h_{n-2}'\}}(f) = \sum_{\text{solutions } j} \mathrm{Res}_{\{h_2',\ldots,h_{n-2}'\}}\left(f, z^{(j)}\right).$$

$$A_n^{(0)}(\sigma, p, \varepsilon) = i\,\mathrm{Res}\left(\, R(z, \sigma, p, \varepsilon)\,\right).$$

There is algorithm to compute global residue using methods of computational algebraic geometry and computer algebra!

Ambitwistor strings in critical dimension

Ambitwistor space $\mathbb{A}$ is a space of complex super null geodesics

$$\mathbb{A} = \left\{(x^\mu, p_\mu, \psi_r^\mu) \in T_S^*M \,\middle|\, \mathcal{H} = g^{-1}(p,p) = 0,\, \mathcal{G}_r = -p\cdot\psi_r = 0\right\} \Big/ \{D_0, D_r\}$$

$$T_S^*M = (T^* \oplus \Pi T \oplus \Pi T)M$$

vector fields generating super null geodesics are given by

$$D_0 = p^\mu \nabla_\mu = p^\mu \left(\frac{\partial}{\partial x^\mu} + \Gamma^\rho_{\mu\nu} p_\rho \frac{\partial}{\partial p^\nu}\right),$$

$$D_r = \psi_r^\mu \frac{\partial}{\partial x^\mu} + p^\mu \frac{\partial}{\partial \psi_r^\mu}.$$

$$D_0 \lrcorner\, \omega + \mathrm{d}(\mathcal{H}) = 0, \qquad D_r \lrcorner\, \omega + d(\mathcal{G}_r) = 0$$

$$\omega = \mathrm{d}\theta, \quad \theta = p_\mu \mathrm{d}x^\mu + \frac{1}{2}\sum_{r=1}^{2} g_{\mu\nu}\psi_r^\mu \mathrm{d}\psi_r^\nu$$

Ambitwistor strings in critical dimension

String action in conformal gauge (pullback of contact structure θ)

$$S = \frac{1}{2\pi}\int_\Sigma P_\mu \bar{\partial} X^\mu + \frac{1}{2}\sum_{r=1}^{2}\Psi_{r\,\mu}\bar{\partial}\Psi_r^\mu - \frac{1}{2}\tilde{e}P^2 - \sum_{r=1}^{2}\chi_r \Psi_r^\mu P_\mu \,.$$

$$X^\mu \in \Omega^0(\Sigma), \quad P_\mu \in \Omega^0(\Sigma, K_\Sigma), \quad \Psi_r^\mu \in \Pi\Omega^0(\Sigma, K_\Sigma^{1/2}),$$
$$\tilde{e} \in \Omega^{0,1}(\Sigma, T_\Sigma), \quad \chi_r \in \Pi\Omega^{0,1}(\Sigma, T_\Sigma^{1/2})$$

Gauge symmetry (translations along null geodesics)

$$\delta X^\mu = \alpha P^\mu + \sum_r \eta_r \Psi_r \,, \qquad \alpha \in \Omega^0(\Sigma, T_\Sigma),\ \eta_r \in \Omega^0(\Sigma, T_\Sigma^{1/2})$$

$$\delta P_\mu = 0 \,, \qquad \delta\tilde{e} = \bar{\partial}\alpha \,,$$

$$\delta\Psi_r^\mu = \eta_r P^\mu \,, \qquad \delta\chi_r = \bar{\partial}\eta_r \,.$$

Mason, Skinner, 2013

Quantisation

$$b, \tilde{b} \in \Pi\Omega^0(\Sigma, K_\Sigma^2)\,, \qquad \beta_r \in \Omega^0(\Sigma, K_\Sigma^{3/2})\,,$$

$$c, \tilde{c} \in \Pi\Omega^0(\Sigma, T_\Sigma)\,, \qquad \gamma_r \in \Omega^0(\Sigma, T_\Sigma^{1/2})\,.$$

BRST operator

$$Q = \oint c\left(T_m + \frac{1}{2}T_{\rm gh}\right) + \tilde{c}\left(\mathcal{H}_m + \frac{1}{2}\mathcal{H}_{\rm gh}\right) + \sum_r \gamma_r\left(\mathcal{G}_{m,r} + \frac{1}{2}\mathcal{G}_{{\rm gh},r}\right)\,,$$

$$T_m = -P_\mu \partial X^\mu - \frac{1}{2}\sum_r \Psi_r \cdot \partial \Psi_r\,,$$

$$\mathcal{H}_m = -\frac{1}{2}P^2\,,$$

$$\mathcal{G}_{m,r} = -\Psi_r \cdot P\,.$$

Mason, Skinner, 2013

Adamo, Casali, Skinner, 2013

Ambitwistor strings in critical dimension

Gauge fixing - adding to lagrangian the term$\{Q, \int_\Sigma \tilde{b}\, F(\tilde{e})\}$

$$F(\tilde{e}) = \tilde{e} - \sum_{r=1}^{3g-3+n} s_r \mu_r \,,$$

where $\mu_r \in H^{0,1}(\Sigma, T_\Sigma(-\sigma_1 \cdots - \sigma_n))$ - basis of Beltrami differentials

Integrating over parameters introduced by the action of Q we get

$$\prod_{r=1}^{3g-3+n} \bar{\delta}\left(\int_\Sigma \mu_r P^2\right) \int_\Sigma \tilde{b}\, \mu_r$$

and

$$S_{\text{gauge-fixed}} = \frac{1}{2\pi} \int_\Sigma P_\mu \bar{\partial} X^\mu + \frac{1}{2} \sum_r \Psi_{r\mu} \bar{\partial} \Psi_r^\mu + b\bar{\partial} c + \tilde{b}\bar{\partial}\tilde{c} + \sum_r \beta_r \bar{\partial} \gamma_r \,.$$

Mason, Skinner, 2013

Adamo, Casali, Skinner, 2013

Ambitwistor strings in critical dimension

NS-NS fixed vertex operators:

$$U = c\tilde{c} \prod_r \delta(\gamma_r)\, \epsilon_r \cdot \Psi_r\, e^{ik\cdot X} \,,$$

$$c\tilde{c}\, V = c\tilde{c} \prod_r (\epsilon_r \cdot P + k \cdot \Psi_r\, \epsilon_r \cdot \Psi_r)\, e^{ik\cdot X} \,.$$

NS-NS integrated vertex operator:

$$\mathcal{V}_i = \bar{\delta}\left(\int_\Sigma \mu_i P^2\right) \int_\Sigma \tilde{b}\, \mu_r \int_\Sigma b\, \mu_i\, c\tilde{c} V \,.$$

Choosing convenient basis for Beltrami differentials:

$$\mathcal{V}_i = \int_\Sigma \bar{\delta}\left(\mathrm{Res}_i P^2\right)\, V \,.$$

Ambitwistor strings in critical dimension

Path integral localizes on the solution of

$$\bar{\partial} P = 2\pi i \sum_i k_i \,\bar{\delta}(\sigma - \sigma_i) d\sigma \,,$$

$$P(\sigma) = \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i}\, d\sigma \,.$$

so that

$$\mathcal{V}_i = \int_\Sigma \bar{\delta}\left(k_i \cdot P(\sigma_i)\right)\, V \,.$$

and amplitude is given by

$$\mathcal{M}_n^{(0)} = \left\langle U_1 U_2 c_3 \tilde{c}_3 V_3 \prod_{i=4}^{n} \mathcal{V}_i \right\rangle \,.$$

Four dimensional ambitwistor strings

Four dimensional projective ambitwistor space $\mathbb{PA}$:

$$\mathbb{PA} = \left\{(Z, W) \in \mathbb{T} \times \mathbb{T}^* |\, Z \cdot W = 0\right\} \Big/ \{Z \cdot \partial_Z - W \cdot \partial_W\} \,.$$

$$Z = (\lambda_\alpha, \mu^{\dot\alpha}, \chi^r) \in \mathbb{T} = \mathbb{C}^{4|\mathcal{N}}, \quad W = (\tilde\mu, \tilde\lambda, \tilde\chi) \in \mathbb{T}^*$$

The point $(x, \theta, \tilde\theta)$ in non-chiral super Minkowski space corresponds to a quardric $\mathbb{CP}^1 \times \mathbb{CP}^1$ parametrized by $(\lambda, \tilde\lambda)$:

$$\mu^{\dot\alpha} = i(x^{\alpha\dot\alpha} + i\theta^{r\alpha}\tilde\theta_r^{\dot\alpha})\lambda_\alpha \,, \qquad \chi^r = \theta^{r\alpha}\lambda_\alpha \,,$$

$$\tilde\mu^{\alpha} = -i(x^{\alpha\dot\alpha} - i\theta^{r\alpha}\tilde\theta_r^{\dot\alpha})\tilde\lambda_{\dot\alpha} \,, \qquad \tilde\chi_r = \tilde\theta_r^{\dot\alpha}\tilde\lambda_{\dot\alpha} \,,$$

Four dimensional ambitwistor string action :

$$S = \frac{1}{2\pi}\int_{\Sigma} W\cdot\bar{\partial}Z - Z\cdot\bar{\partial}W + aZ\cdot W + S_J\,,$$

$$Z \in \Omega^0(\Sigma, K_{\Sigma}^{1/2}\otimes\mathbb{T})\,, \qquad a \in \Omega^{0,1}(\Sigma)\,,$$
$$W \in \Omega^0(\Sigma, K_{\Sigma}^{1/2}\otimes\mathbb{T}^*)\,, \qquad J \in \Omega^0(\Sigma, K_{\Sigma}\otimes\mathfrak{g}).$$

Gauge GL(1) symmetry :

$$Z^I \to e^{\gamma}Z^I\,, \quad W_I \to e^{-\gamma}W_I\,, \quad a \to a - 2\bar{\partial}\gamma\,.$$

Geyer, Lipstein, Mason, 2014

Four dimensional ambitwistor strings

Quantization :

$$c \in \Pi\Omega^0(\Sigma, T_\Sigma)\,, \qquad v \in \Pi\Omega^0(\Sigma)\,,$$
$$b \in \Pi\Omega^0(\Sigma, K_\Sigma^2)\,, \qquad u \in \Pi\Omega^0(\Sigma, K_\Sigma)\,.$$

Gauge fixed action:

$$S = \frac{1}{2\pi}\int_\Sigma W\cdot\bar\partial Z - Z\cdot\bar\partial W + b\bar\partial c + u\bar\partial v + S_J\,.$$

BRST operator:

$$Q = \oint cT + vZ\cdot W + Q_{\mathrm{gh}}\,,$$
$$T = W\cdot\partial Z - Z\cdot\partial W + T_J\,.$$

Geyer, Lipstein, Mason, 2014

On-shell string vertexes:

$$\mathcal{V}_a = \int \frac{ds_a}{s_a} \bar{\delta}^2(\lambda_a - s_a\lambda) e^{is_a\left([\mu\,\tilde{\lambda}_a] + \chi^r \tilde{\eta}_{ar}\right)} J \cdot T_a \,,$$

$$\widetilde{\mathcal{V}'}_a = \int \frac{ds_a}{s_a} \bar{\delta}^2(\tilde{\lambda}_a - s_a\tilde{\lambda}) e^{is_a(\langle\tilde{\mu}\,\lambda_a\rangle + \tilde{\chi}_r \eta^r_a)} J \cdot T_a \,,$$

$$\widetilde{\mathcal{V}}_a = \int \frac{ds_a}{s_a} \bar{\delta}^{2|\mathcal{N}}(\tilde{\lambda}_a - s_a\tilde{\lambda}|\tilde{\eta}_a - s_a\tilde{\chi}) e^{is_a\langle\tilde{\mu}\,\lambda_a\rangle} J \cdot T_a \,.$$

N^{k-2}MHV amplitudes:

$$\mathcal{A}_{n,k} = \left\langle \widetilde{\mathcal{V}}_1 \ldots \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} \ldots \mathcal{V}_n \right\rangle \,.$$

Geyer, Lipstein, Mason, 2014

Four dimensional ambitwistor strings

Take exponentials into the action as sources

$$\int_\Sigma \sum_{i=1}^{k} is_i\langle\tilde{\mu}\lambda_i\rangle\bar{\delta}(\sigma-\sigma_i) + \sum_{p=k+1}^{n} is_p([\mu\,\tilde{\lambda}_p] + \chi\tilde{\eta}_p)\bar{\delta}(\sigma-\sigma_p)\,.$$

New equations of motion:

$$\bar{\partial}_\sigma Z = \bar{\partial}\left(\lambda,\mu,\chi\right) = \sum_{i=1}^{k} s_i\left(\lambda_i,0,0\right)\bar{\delta}\left(\sigma-\sigma_i\right)\;,$$

$$\bar{\partial}_\sigma W = \bar{\partial}\left(\tilde{\mu},\tilde{\lambda},\tilde{\chi}\right) = \sum_{p=k+1}^{n} s_p\left(0,\tilde{\lambda}_p,\tilde{\eta}_p\right)\bar{\delta}(\sigma-\sigma_p)\,,$$

Solutions:

$$Z(\sigma) = (\lambda,\mu,\chi) = \sum_{i=1}^{k}\frac{s_i\left(\lambda_i,0,0\right)}{\sigma-\sigma_i}\;,$$

$$W(\sigma) = \left(\tilde{\mu},\tilde{\lambda},\tilde{\chi}\right) = \sum_{p=k+1}^{n}\frac{s_p\left(0,\tilde{\lambda}_p,\tilde{\eta}_p\right)}{\sigma-\sigma_p}\,.$$

N^{k-2}MHV amplitudes:

$$\mathcal{A}_{n,k} = \int \frac{1}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{a=1}^{n} \frac{\mathrm{d}s_a \mathrm{d}\sigma_a}{s_a(\sigma_a - \sigma_{a+1})} \prod_{p=k+1}^{n} \bar{\delta}^2(\lambda_p - s_p \lambda(\sigma_p))$$

$$\prod_{i=1}^{k} \bar{\delta}^{2|\mathcal{N}}(\tilde{\lambda}_i - s_i \tilde{\lambda}(\sigma_i), \tilde{\eta}_i - s_i \tilde{\chi}(\sigma_i)) \,.$$

in coordinates on Riemann sphere $\sigma_\alpha = \frac{1}{s}(1,\sigma)$, $\quad (i\,j) = \sigma_{i\alpha}\sigma_j^\alpha$:

$$\mathcal{A}_{n,k} = \int \frac{1}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \prod_{p=k+1}^{n} \bar{\delta}^2(\lambda_p - \lambda(\sigma_p))$$

$$\prod_{i=1}^{k} \bar{\delta}^{2|\mathcal{N}}(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i), \tilde{\eta}_i - \tilde{\chi}(\sigma_i)) \,.$$

Scattering equations from the support of delta functions:

$$k_a \cdot P(\sigma_a) = \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}} P_{\alpha\dot{\alpha}}(\sigma_a) = \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}} \lambda_\alpha(\sigma_a) \tilde{\lambda}_{\dot{\alpha}}(\sigma_a) = 0 \,.$$

Wilson line matrix element:

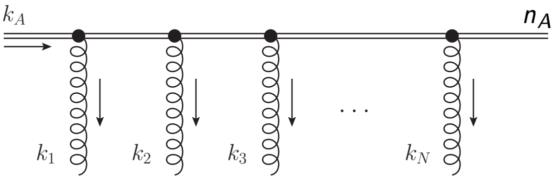
$$\mathfrak{M}\left(n, \varepsilon_1, \ldots, \varepsilon_N\right) = \langle k_1, \varepsilon_1, c_1; \ldots; k_N, \varepsilon_N, c_N | \mathfrak{R}_n^c(k) | 0 \rangle \,,$$

where

$$\mathfrak{R}_n^c(k) = \int d^4x \, e^{ix\cdot k} \mathrm{Tr}\left\{\frac{1}{\pi g} t^c [x]_n\right\},$$

$$[x]_n \equiv \mathcal{P}\exp\left\{ig \int_{-\infty}^{\infty} ds \, n \cdot A_b(x+sn)\, t^b\right\}$$

Vertex from expansion of Wilson line with direction n_A



Kotko, 2014

Off-shell momentum decomposition

$$k_T^\mu(q) = k^\mu - x(q)p^\mu \quad \text{with} \quad x(q) \equiv \frac{q\cdot k}{q\cdot p}.$$

$$k_T\cdot p = 0, \quad k_T\cdot q = 0.$$

$$k_T^\mu(q) = -\frac{\kappa}{2}\frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle}\ , \quad \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}.$$

$$k^2 = -\kappa\kappa^*\ .$$

p is the particle direction and q is an auxiliary four-vector.

Both κ and κ^* are independent of the auxiliary vector q!

Off-shell (reggeon) string vertex:

$$\mathcal{V}^{\mathrm{WL}}_{n,n+1} = \int \prod_{i=n}^{n+1} \frac{\mathrm{d}^2\lambda_i \mathrm{d}^2\tilde{\lambda}_i}{\mathrm{Vol}[\mathrm{GL}(1)]} \mathrm{d}^4\tilde{\eta}_i A^*_{2,2+1}(p^*, n, n+1)\Big|_{\lambda\to-\lambda}$$

$$\times \mathcal{V}_n(\sigma_n)\mathcal{V}_{n+1}(\sigma_{n+1})\Big|_{T^aT^b\to if^{abc}T^c\to T^c},$$

Bork, Onishchenko, 2017

Minimal off-shell vertex:

$$A^*_{2,2+1}(p^*, n, n+1) =$$

$$\frac{1}{\kappa^*} \prod_{A=1}^{4} \frac{\partial}{\partial \tilde{\eta}_p^A} \left[\frac{\delta^4(k + \lambda_n\tilde{\lambda}_n + \lambda_{n+1}\tilde{\lambda}_{n+1})\delta^8(\lambda_p\tilde{\eta}_p + \lambda_n\tilde{\eta}_n + \lambda_{n+1}\tilde{\eta}_{n+1})}{\langle p\, n\rangle\langle n\, n+1\rangle\langle n+1\, p\rangle} \right].$$

Bork, Onishchenko, 2016

Amplitude with one leg off-shell:

$$A^*_{k,n+1} = \left\langle \widetilde{\mathcal{V}}_1 \ldots \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} \ldots \mathcal{V}_n \mathcal{V}^{\mathrm{WL}}_{n+1,n+2} \right\rangle .$$

$$A^*_{k,n+1} = \frac{\langle \xi p \rangle}{\kappa^*} \int \frac{d\beta_2}{\beta_2} \int \frac{d\beta_1}{\beta_1} \frac{1}{\beta_1^2 \beta_2} \frac{1}{\mathrm{Vol\,GL}(2,\mathbb{C})}$$
$$\times \prod_{a=1}^{n+2} \frac{ds_a d\sigma_a}{s_a(\sigma_a - \sigma_{a+1})} \prod_{p=k+1}^{n+2} \bar{\delta}^2(\lambda_p - s_p \lambda(\sigma_p)) \prod_{i=1}^{k} \bar{\delta}^{2|\mathcal{N}}(\tilde{\lambda}_i - s_i \tilde{\lambda}(\sigma_i), \tilde{\eta}_i - s_i \tilde{\chi}(\sigma_i)) .$$

where

$$\lambda_n = \lambda_p + \beta_2 \lambda_\xi , \qquad \tilde{\lambda}_n = \frac{1}{\langle \xi p \rangle} \left(\beta_1 \langle \xi | k + \frac{(1+\beta_1)}{\beta_2} \langle p | k \right) , \quad \tilde{\eta}_n = -\beta_1 \tilde{\eta}_p ,$$

$$\lambda_{n+1} = \lambda_\xi + \frac{(1+\beta_1)}{\beta_1 \beta_2} \lambda_p , \quad \tilde{\lambda}_{n+1} = -\frac{\beta_1}{\langle \xi p \rangle} \left(\langle p | k + \beta_2 \langle \xi | k \right) , \qquad \tilde{\eta}_{n+1} = \beta_1 \beta_2 \tilde{\eta}_p .$$

Transition to Grassmannian representation

$$1 = \frac{1}{\mathrm{Vol\,GL}(k)} \int \mathrm{d}^{k\times(n+2)} C \, \mathrm{d}^{k\times k} L \, (\det L)^{n+2} \delta^{k\times(n+2)} \left(C - L \cdot C^V[s,\sigma] \right) .$$

Veronese map from $(\mathbb{C}^2)^{n+2}/\mathrm{GL}(2)$ to $G(k,n+2)$ Grassmannian:

$$C^V[s,\sigma] = \begin{pmatrix} \vdots & \cdots & \vdots \\ \sigma^V[s_1,\sigma_1] & \cdots & \sigma^V[s_{n+2},\sigma_{n+2}] \\ \vdots & \cdots & \vdots \end{pmatrix}, \quad \sigma^V[s,\sigma] \equiv \begin{pmatrix} \xi \\ \xi\sigma \\ \vdots \\ \xi\sigma^{k-1} \end{pmatrix},$$

$$\xi_i = s_i^{-1} \prod_{j=1, j\neq i}^{k} (\sigma_j - \sigma_i)^{-1}, \qquad i \in (1,k)$$

$$\xi_i = s_i \prod_{j=1}^{k} (\sigma_j - \sigma_i)^{-1}, \qquad i \in (k+1, n+2)$$

Transition to Grassmannian representation

$$A^*_{k,n+1} = \frac{\langle \xi p \rangle}{\kappa^*} \int \frac{\mathrm{d}\beta_2}{\beta_2} \int \frac{\mathrm{d}\beta_1}{\beta_1} \frac{1}{\beta_1^2 \beta_2} \frac{1}{\mathrm{Vol}\,\mathrm{GL}(k)}$$
$$\times \int \mathrm{d}^{k\times(n+2)} C \, F(C) \, \delta^{k\times 2}(C \cdot \tilde{\lambda}) \delta^{k\times\mathcal{N}}(C \cdot \tilde{\eta}) \delta^{(n+2-k)\times 2}(C^{\perp} \cdot \lambda) \,,$$

$$F(C) = \frac{1}{(1 \cdots k)(2 \cdots k+1) \cdots (n+2 \cdots k-1)}$$

and

$$A^*_{k,n+1} = \frac{\langle \xi p \rangle}{\kappa^*} \int_{\Gamma} \frac{d^{k\times(n+2)} C'}{\mathrm{Vol}[GL(k)]} \frac{\delta^{k\times 2}\left(C' \cdot \underline{\underline{\tilde{\lambda}}}\right) \delta^{k\times 4}\left(C' \cdot \underline{\underline{\tilde{\eta}}}\right) \delta^{(n+2-k)\times 2}\left(C'^{\perp} \cdot \underline{\underline{\lambda}}\right)}{(1 \cdots k) \cdots (n+1 \cdots k-2)(n+1\; 1 \cdots k-1)} .$$

$$\underline{\underline{\lambda}}_i = \lambda_i, \qquad i = 1, \ldots n, \qquad \underline{\underline{\lambda}}_{n+1} = \lambda_p, \qquad \underline{\underline{\lambda}}_{n+2} = \xi$$
$$\underline{\underline{\tilde{\lambda}}}_i = \tilde{\lambda}_i, \qquad i = 1, \ldots n, \qquad \underline{\underline{\tilde{\lambda}}}_{n+1} = \frac{\langle \xi | k}{\langle \xi p \rangle}, \qquad \underline{\underline{\tilde{\lambda}}}_{n+2} = -\frac{\langle p | k}{\langle \xi p \rangle},$$
$$\underline{\underline{\tilde{\eta}}}_i = \tilde{\eta}_i, \qquad i = 1, \ldots n, \qquad \underline{\underline{\tilde{\eta}}}_{n+1} = \tilde{\eta}_p, \qquad \underline{\underline{\tilde{\eta}}}_{n+2} = 0.$$

Transition to scattering equation representation

$$A^*_{k,n+1} = \frac{1}{\text{Vol GL}(2,\mathbb{C})}$$

$$\times \int \prod_{a=1}^{n+2} \frac{d^2\sigma_a}{(a\,a+1)}\, Reg. \prod_{p=k+1}^{n+2} \bar{\delta}^2(\underline{\underline{\lambda}}_p - \underline{\underline{\lambda}}(\sigma_p)) \prod_{i=1}^{k} \bar{\delta}^{2|\mathcal{N}}(\underline{\underline{\tilde{\lambda}}}_i - \underline{\underline{\tilde{\lambda}}}(\sigma_i), \underline{\underline{\eta}}_i - \underline{\underline{\tilde{\chi}}}(\sigma_i)),$$

$$Reg. = \frac{\langle \xi p \rangle}{\kappa^*} \frac{(k\,n+1)}{(k\,n+2)}$$

$$\left(\underline{\underline{\lambda}}, \underline{\underline{\mu}}, \underline{\underline{\chi}}\right) = \sum_{i=1}^{k} \frac{(\underline{\underline{\lambda}}_i, 0, 0)}{(\sigma\,\sigma_i)}, \qquad \left(\underline{\underline{\tilde{\mu}}}, \underline{\underline{\tilde{\lambda}}}, \underline{\underline{\tilde{\chi}}}\right) = \sum_{p=k+1}^{n+2} \frac{(0, \underline{\underline{\tilde{\lambda}}}_p, \underline{\underline{\eta}}_p)}{(\sigma\,\sigma_p)}.$$

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Multiple off-shell gluons (m on-shell and n off-shell)

$$\int \prod_{a=1}^{m+2n} \frac{d^2\sigma_a}{(a\, a+1)} \frac{Reg.^V(m+1,\ldots,m+n)}{\text{Vol GL}(2,\mathbb{C})} \prod_{p=k+1}^{m+2n} \bar{\delta}^2(\underline{\lambda}_p - \underline{\lambda}(\sigma_p)) \prod_{i=1}^{k} \bar{\delta}^{2|4}(\underline{\tilde{\lambda}}_i - \underline{\tilde{\lambda}}(\sigma_i), \underline{\eta}_i - \underline{\tilde{\chi}}(\sigma_i)),$$

where

$$Reg.^V(m+1,\ldots,m+n) = \prod_{j=1}^{n} Reg^V.(j+m), \quad Reg.^V(j+m) = \frac{\langle \xi_j p_j \rangle}{\kappa_j^*} \frac{(k\, 2j-1+m)}{(k\, 2j+m)}$$

and

$$\underline{\lambda}_i = \lambda_i, \quad i=1,\ldots m, \quad \underline{\lambda}_{m+2j-1} = \lambda_{p_j}, \quad \underline{\lambda}_{m+2j} = \xi_j, \quad j=1,\ldots n,$$

$$\underline{\tilde{\lambda}}_i = \tilde{\lambda}_i, \quad i=1,\ldots m, \quad \underline{\tilde{\lambda}}_{m+2j-1} = \frac{\langle \xi_j | k_{m+j}}{\langle \xi_j p_j \rangle}, \quad \underline{\tilde{\lambda}}_{m+2j} = -\frac{\langle p_j | k_{m+j}}{\langle \xi_j p_j \rangle}, \quad j=1,\ldots n,$$

$$\underline{\tilde{\eta}}_i = \tilde{\eta}_i, \quad i=1,\ldots m, \quad \underline{\tilde{\eta}}_{m+2j-1} = \tilde{\eta}_{p_j}, \quad \underline{\tilde{\eta}}_{m+2j} = 0, \quad j=1,\ldots n.$$

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Conclusion and future directions

- Supersymmetrization of off-shell legs
- Amplituhedron for off-shell amplitudes
- Loop corrections for off-shell amplitudes
- Extension to gravity and supergravity
- Integrability and statistical model partition functions

Thank you for your attention!