Hidden fermionic structure of CFT

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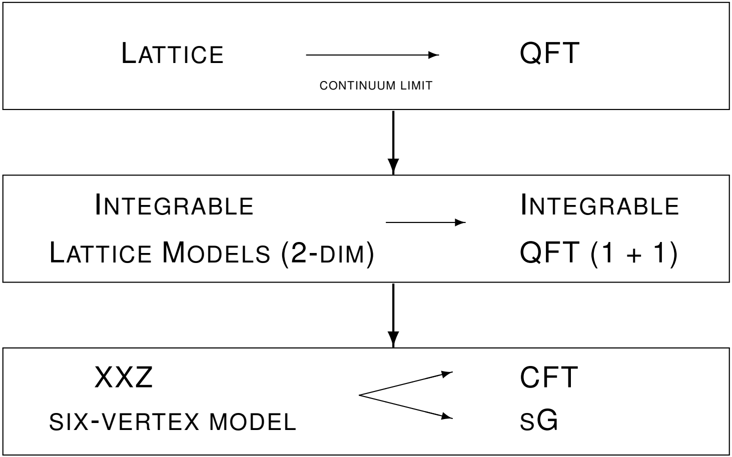
Plan

Basic motivation: to apply fermionic basis originated from the lattice to the CFT

- Introduction to hidden fermionic structure on the lattice
  - Partition function on cylinder
  - Fermionic operators and fermionic basis
  - Jimbo-Miwa-Smirnov theorem
  - Functions ρ and ω
- Application to CFT and further perspectives
  - Conjectures on scaling limit
  - Operators: local and non-local
  - Identification of fermionic basis
  - The "ρ-Problem" and reflection equation
  - OPE in fermionic basis and conformal blocks
  - Discussion of recursion relation for the conformal blocks
- Conclusions

Lattice–QFT correspondence



XXZ vs. CFT

- study of the low-lying excitations of the XXZ Hamiltonian and finite-size corrections

$$\mathcal{H}_{\mathrm{xxz}} = \frac{1}{2}\sum_{j=1}^{L}\Big(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \frac{1}{2}(q+q^{-1})\,\sigma_j^z\sigma_{j+1}^z\Big)$$

deformation parameter: $q = e^{\pi i \nu}, \quad \frac{1}{2} < \nu < 1$

twisted boundary conditions: $\sigma_{L+1}^{\pm} = e^{\pm i\Phi}\sigma_1^{\pm}, \quad \sigma_{L+1}^z = \sigma_1^z$

periodic b.c.: $\Phi = 0$

with the ground-state energy in the limit $L \to \infty$

$$E_0(L) = e_\infty L - \frac{\pi c}{6}L^{-1} + O(L^{-2}), \quad c = 1 - \frac{6\Phi^2}{\pi^2(1-\nu)}$$

XXZ vs. CFT

- Operator content: for each primary operator $\mathcal{O}_\alpha$ with scaling dimension Δ_α and spin s_α, there exists a tower of states in the spectrum of $\mathcal{H}_{\mathrm{XXZ}}$ with energies $E^\alpha_{j,j'}(L)$ and momenta $P^\alpha_{j,j'}(L)$

$$E^\alpha_{j,j'}(L) = E_0(L) + 2\pi(\Delta_\alpha + j + j')L^{-1} + O(L^{-2}),$$ Blöte, Cardy, Nightingale (85)

$$P^\alpha_{j,j'}(L) = 2\pi(s_\alpha + j - j')L^{-1}, \quad j, j' = 0, 1, \cdots$$ Affleck (85), Alcaraz, Martins (88)$\cdots$

- Effective Hamiltonian approach: the XXZ-Hamiltonian with periodic b.c.

$$\mathcal{H}_{\mathrm{XXZ}} = \mathcal{H}_{\mathrm{Gauss}} + \cdots$$ Lukyanov (98)

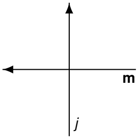
corresponds to $c = 1$ (up to irrelavant operators).

- Study of asymptotic behavior of vacuum spin-spin correlation functions confirm predictions via CFT Luther, Peschel (75), Lukyanov, Terras (02)

Kitanine, Kozlowsky, Maillet, Slavnov, Terras (08)

The six-vertex model, Matsubara space and monodromy matrix

It is well known that the XXZ model is related to the six-vertex model.
Integrable structure is generated by *R*-matrix or *L*-operator



$$\equiv \quad L_{j,\mathbf{m}}(\zeta) = q^{-\frac{1}{2}\sigma_j^z\sigma_{\mathbf{m}}^z} - \zeta^2 q^{\frac{1}{2}\sigma_j^z\sigma_{\mathbf{m}}^z} - \zeta(q-q^{-1})(\sigma_j^+\sigma_{\mathbf{m}}^- + \sigma_j^-\sigma_{\mathbf{m}}^+)$$

Consider the XXZ-model in infinite volume. Space of states is $\mathfrak{H}_{\mathrm{S}} = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2$

Also we introduce the Matsubara space: $\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{j}=\mathbf{1}}^{N} \mathbb{C}^2$

and monodromy matrix: $T_{\mathrm{S},\mathbf{M}} = \overset{\curvearrowright}{\prod_{j=-\infty}^{\infty}} T_{j,\mathbf{M}}\,,\ \ T_{j,\mathbf{M}} \equiv T_{j,\mathbf{M}}(1),\ \ T_{j,\mathbf{M}}(\zeta) = \overset{\curvearrowleft}{\prod_{\mathbf{m}=\mathbf{1}}^{N}} L_{j,\mathbf{m}}(\zeta/\tau_{\mathbf{m}})$

In homogeneous case we take $\tau_{\mathbf{m}} = q^{1/2}$.

The local operators

Introduce a local operator $\mathcal{O}$ on $\mathfrak{H}_{\mathrm{S}}$ which acts non-trivially only on a finite segment of $\mathfrak{H}_{\mathrm{S}}$. We call quasi-local operator with tail α the following product

$$q^{2\alpha S(0)}\mathcal{O}$$

Here we defined

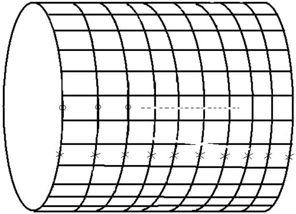
$$S(k)=\frac{1}{2}\sum_{j=-\infty}^{k}\sigma_j^z$$

So, $S(0)$ acts on the semi-infinite chain and $S=S(\infty)$ is the total spin.
We call

$$q^{2\alpha S(0)}$$

lattice 'primary field' and parameter α – disorder field.

Partition function on cylinder



Cut corresponds to

insertion of local operator $\mathcal{O}$

κ – “imaginary” magnetic field

α – disorder field

Matsubara expectation values:

$$Z^{\kappa}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}\ \ q^{2\kappa S+2\alpha S(0)}\mathcal{O}\right)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}\ \ q^{2\kappa S+2\alpha S(0)}\right)}$$

Fermionic operators

Describe the basis of quasi-local operators via certain creation operators. Jimbo, Miwa, Smirnov, Takeyama, HB (07–09)

Creation operators $\mathbf{t}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ together with annihilation operators $\mathbf{b}, \mathbf{c}$ are constructed with help of representation theory of quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and act in space

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s}$$

where $\mathcal{W}_{\alpha-s,s}$ is subspace of quasi-local operators of the spin s. They are formal power series of $\zeta^2 - 1$ and have the block structure

$$\begin{aligned}
\mathbf{t}^*(\zeta) \quad &: \ \mathcal{W}_{\alpha-s,s} \to \mathcal{W}_{\alpha-s,s}\,, \\
\mathbf{b}^*(\zeta), \mathbf{c}(\zeta) \ &: \ \mathcal{W}_{\alpha-s+1,s-1} \to \mathcal{W}_{\alpha-s,s}\,, \\
\mathbf{c}^*(\zeta), \mathbf{b}(\zeta) \ &: \ \mathcal{W}_{\alpha-s-1,s+1} \to \mathcal{W}_{\alpha-s,s}\,.
\end{aligned}$$

$\mathbf{t}^*(\zeta)$ is bosonic and generates commuting integrals of motion. It commutes with all fermionic operators $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$ and $\mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$.

Anti-commutation relations and fermionic basis

Fermionic operators satisfy canonical anti-commutation relations

$$\big[\mathbf{c}(\zeta),\mathbf{c}^*(\zeta')\big]_+ = \psi(\zeta/\zeta',\alpha), \quad \big[\mathbf{b}(\zeta),\mathbf{b}^*(\zeta')\big]_+ = -\psi(\zeta'/\zeta,\alpha)$$

$$\text{with} \quad \psi(\zeta,\alpha) = \frac{1}{2}\,\zeta^{\alpha}\,\frac{\zeta^2+1}{\zeta^2-1}.$$

Annihilation operators **b** and **c** “kill” lattice “primary field” $q^{2\alpha S(0)}$

$$\mathbf{b}(\zeta)\big(q^{2\alpha S(0)}\big) = 0, \quad \mathbf{c}(\zeta)\big(q^{2\alpha S(0)}\big) = 0, \quad S(k) = \frac{1}{2}\sum_{j=-\infty}^{k}\sigma_j^z.$$

Space of states is generated via multiple action of $\mathbf{t}^*(\zeta), \mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$ on “primary field” $q^{2\alpha S(0)}$. In this way we get fermionic basis.

Mode expansions and locality

Annihilation operators are singular at $\zeta^2 \to 1$ while creation operators are regular:

$$\mathbf{b}(\zeta) = \zeta^{-\alpha-\mathbb{S}} \sum_{p=0}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{b}_p, \quad \mathbf{c}(\zeta) = \zeta^{\alpha+\mathbb{S}} \sum_{p=0}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{c}_p$$

$$\mathbf{b}^*(\zeta) = \zeta^{\alpha+\mathbb{S}} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{b}^*_p, \quad \mathbf{c}^*(\zeta) = \zeta^{-\alpha-\mathbb{S}} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{c}^*_p$$

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}^*_p$$

Locality:

$$\mathbf{b}_p(X) = \mathbf{c}_p(X) = 0 \quad \text{for} \quad p > \text{length}(X)$$

$$\text{length}(\mathbf{b}^*_p(X)) \leq \text{length}(X) + p, \quad \text{length}(\mathbf{c}^*_p(X)) \leq \text{length}(X) + p$$

$$\text{length}(\mathbf{t}^*_p(X)) \leq \text{length}(X) + p$$

Relation to the correlation functions

- Any correlation function corresponding to any quasi-local operator $\mathcal{O}$ is generated by two transcendental functions ρ and ω. ρ is related to one-point function, ω is related to nearest neighbor correlators

$$\omega(\zeta,\zeta') = Z^{\kappa}\big(\mathbf{b}^*(\zeta)\mathbf{c}^*(\zeta')q^{2\alpha S(0)}\big)$$

  Both functions depend on temperature, disorder parameter and magnetic field, we call them physical part.

- In contrast to this, the basis is pure algebraic. It is built using representation theory of quantum group. We call it algebraic part.

The JMS-theorem allows to explicitly calculate Jimbo, Miwa, Smirnov (09)

$$Z^{\kappa}\Big\{\mathbf{t}^*(\zeta_1^0)\cdots\mathbf{t}^*(\zeta_p^0)\mathbf{b}^*(\zeta_1^+)\cdots\mathbf{b}^*(\zeta_q^+)\mathbf{c}^*(\zeta_q^-)\cdots\mathbf{c}^*(\zeta_1^-)\Big(q^{\alpha\sum_{j=-\infty}^{0}\sigma_j^z}\Big)\Big\} =$$

$$= \prod_{i=1}^{p} 2\rho(\zeta_i^0)\det\Big|\omega(\zeta_i^+,\zeta_j^-)\Big|_{i,j=1,\cdots q}$$ generating function for series in ζ^2-1

The function ω via function Φ

There are several equivalent definitions of ω

- via deformed Abelian integrals
- via solution of linear and non-linear integral equations that come from thermodynamical description of the six-vertex model Göhmann, HB (09-12)

Introduce function Φ:

$$\Phi(\zeta,\zeta') = \tilde{\Phi}(\zeta,\zeta') + \Delta_\zeta^{-1}\psi(\zeta/\zeta',\alpha), \quad \tilde{\Phi}(\zeta,\zeta') = \left(\frac{\zeta}{\zeta'}\right)^{\alpha} \frac{P(\zeta,\zeta')}{A_\kappa(\zeta)A_\kappa(\zeta')}$$

One must solve kind of ‘Riemann-Hilbert’ problem

$$\frac{\Delta_\zeta \Phi(\zeta,\zeta')}{\rho(\zeta)(1+\mathfrak{a}_\kappa(\zeta))} + \tilde{\Phi}(\zeta,\zeta') = r(\zeta,\zeta'), \quad (\Delta_\zeta f)(\zeta) := f(q\zeta) - f(q^{-1}\zeta)$$

where the remainder r is a ‘regular’ function of ζ.

- The function ρ : $$\rho(\zeta)=\frac{T_{\kappa'}(\zeta)}{T_\kappa(\zeta)}$$

- Baxter's *TQ*-relation: $T_\kappa(\zeta)Q_\kappa(\zeta)=d(\zeta)Q_\kappa(q\zeta)+a(\zeta)Q_\kappa(q^{-1}\zeta)$,

$$Q_\kappa(\zeta)=\zeta^{-\kappa}A_\kappa(\zeta),\qquad A_\kappa(\zeta)=\prod_{j=1}^{N/2}(\zeta/\zeta_j-\zeta_j/\zeta)$$

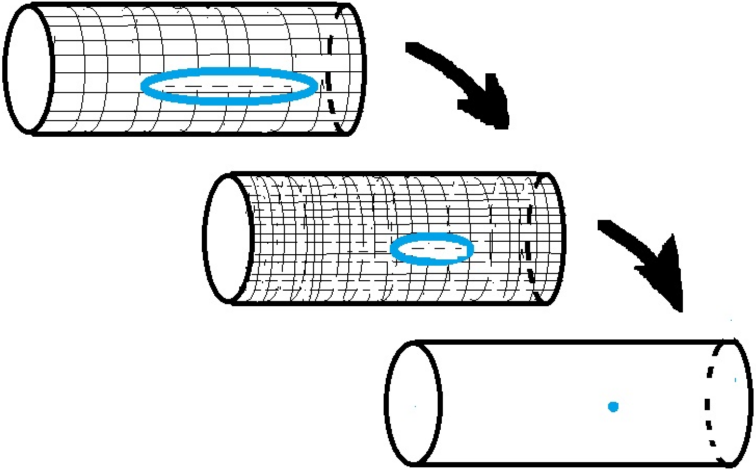
$$\mathfrak{a}_\kappa(\zeta):=\frac{d(\zeta)Q_\kappa(q\,\zeta)}{a(\zeta)Q_\kappa(q^{-1}\zeta)},\quad d(\zeta)=\prod_{i=1}^{N}(\zeta/\tau_j-\tau_j/\zeta),\quad a(\zeta)=d(q\zeta)$$

BAE: $\mathfrak{a}_\kappa(\zeta_j)=-1, j=1,\cdots,N/2$

- The function ω: $\frac{1}{4}\omega(\zeta,\zeta')=H_\zeta\,H_{\zeta'}\;\Phi(\zeta,\zeta')$

$$(H_\zeta f)(\zeta):=\frac{\mathfrak{a}_\kappa(\zeta)}{1+\mathfrak{a}_\kappa(\zeta)}f(q\zeta)\,+\,\frac{1}{1+\mathfrak{a}_\kappa(\zeta)}f(q^{-1}\zeta)\,-\,\rho(\zeta)f(\zeta)$$

Application to CFT: continuum limit



Scaling limit

Aim is two-fold :

- to obtain the CFT with non-trivial $c = 1 - 6\nu^2/(1-\nu)$, $q = e^{\pi i\nu}$, $1/2 < \nu < 1$
- to consider asymptotic series for $\kappa \to \infty$

Scaling or conformal limit: Introduce lattice spacing a and take

$\tau_j = q^{1/2}$, $N \to \infty$, $a \to 0$, $Na = 2\pi R$ with fixed radius of cylinder R

Lieb distribution gives: $\zeta_j \simeq (\pi j/N)^\nu$

Spectral parameter must be re-scaled: $\zeta = \lambda \bar{a}^\nu$, $\bar{a} = Ca$

Conjecture:

$$\rho^{\mathrm{sc}}(\lambda) = \lim_{\text{scaling}} \rho(\lambda\bar{a}^\nu),$$

$$\omega^{\mathrm{sc}}(\lambda,\mu) = \frac{1}{4} \lim_{\text{scaling}} \omega(\lambda\bar{a}^\nu, \mu\bar{a}^\nu)$$

Conjectures on operators in scaling:

- The creation operators are well-defined in the scaling limit for space direction when $ja = x$ is finite

$$\tau^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{t}^*(\lambda\bar{a}^{\nu}), \quad \beta^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{b}^*(\lambda\bar{a}^{\nu}), \quad \gamma^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{c}^*(\lambda\bar{a}^{\nu})$$

Asymptotic expansions at $\lambda \to \infty$ look

$$\log\bigl(\tau^*(\lambda)\bigr) \simeq \sum_{j=1}^{\infty}\tau^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}$$

$$\frac{1}{\sqrt{\tau^*(\lambda)}}\beta^*(\lambda) \simeq \sum_{j=1}^{\infty}\beta^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}, \quad \frac{1}{\sqrt{\tau^*(\lambda)}}\gamma^*(\lambda) \simeq \sum_{j=1}^{\infty}\gamma^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}.$$

Freedom in definition of operators

- "Gauge transform":

$$\mathbf{b}^* \to e^{-\tilde{\Omega}}\,\mathbf{b}^*\,e^{\tilde{\Omega}}, \quad \mathbf{c}^* \to e^{-\tilde{\Omega}}\,\mathbf{c}^*\,e^{\tilde{\Omega}}$$

$$\tilde{\Omega} = \frac{1}{(2\pi i)^2}\oint\limits_{\Gamma}\frac{d\zeta^2}{\zeta^2}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\tilde{\omega}(\zeta,\xi)\,\mathbf{c}(\xi)\mathbf{b}(\zeta), \quad \omega \to \omega + \tilde{\omega}$$

- We choose $\tilde{\omega}$ in such a way that:

$$Z_\infty(\mathcal{O}q^{\alpha S(0)}) = \frac{\langle \text{vac}|\mathcal{O}q^{\alpha S(0)}|\text{vac}\rangle}{\langle \text{vac}|q^{\alpha S(0)}|\text{vac}\rangle} = \begin{cases} 1, & \text{if } \mathcal{O} = \tau^m,\ m \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

$$\zeta^{-\alpha}\mathbf{b}^*(\zeta) \to 0, \quad \zeta^{\alpha}\mathbf{c}^*(\zeta) \to 0, \quad \zeta \to 0$$

Screening operators

- The other "gauge" choice:
  $\mathbf{b}^* \to \mathbf{b}_0^* = O(\zeta^\alpha), \quad \mathbf{c}^* \to \mathbf{c}_0^* = O(\zeta^{2-\alpha}), \quad \zeta \to 0$
- Acting in the subspace $\mathcal{W}_{\alpha,0}$

$$\mathbf{b}_0^*(\zeta) = \sum_{j=1}^{\infty} \zeta^{\alpha-2+2j} \mathbf{b}^*_{\text{screen},j}, \quad \mathbf{c}_0^*(\zeta) = \sum_{j=1}^{\infty} \zeta^{-\alpha+2j} \mathbf{c}^*_{\text{screen},j}$$

  $\mathbf{b}^*_{\text{screen},j}, \mathbf{c}^*_{\text{screen},j}$ are non-local.
- Scaling

$$\beta^*_{\text{screen}}(\lambda) = \lim_{a\to 0} \frac{1}{2} \mathbf{b}_0^*(\lambda \bar{a}^\nu), \quad \gamma^*_{\text{screen}}(\lambda) = \lim_{a\to 0} \frac{1}{2} \mathbf{c}_0^*(\lambda \bar{a}^\nu) \quad \text{for} \quad \lambda \to 0$$

  and for $\rho = 1$

$$\beta^*_{\text{screen}}(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{\alpha+2j-2} \beta^*_{\text{screen},j}, \quad \gamma^*_{\text{screen}}(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-\alpha+2j} \gamma^*_{\text{screen},j}$$

Integrals of motion

- In 1987 Alexander Zamolodchikov introduced local integrals of motion which act on local operators as

$$(\mathbf{i}_{2n-1}O)(w) = \int_{C_w} \frac{dz}{2\pi i} h_{2n}(z)O(w) \qquad (n \geq 1)$$

where the densities $h_{2n}(z)$ are certain descendants of the identity operator I. An important property is that

$$\langle \Delta_- | \mathbf{i}_{2n-1}\big(O(z)\big) | \Delta_+ \rangle = (I^+_{2n-1} - I^-_{2n-1}) \langle \Delta_- | O(z) | \Delta_+ \rangle$$
$$L_n | \Delta_+ \rangle = \delta_{n,0} \Delta_+ | \Delta_+ \rangle \quad n \geq 0, \quad \langle \Delta_- | L_n = \delta_{n,0} \Delta_- \langle \Delta_- | \quad n \leq 0$$

where $I^{\pm}_{2n-1}$ denote the vacuum eigenvalues of the local integrals of motion on the Verma module with conformal dimension $\Delta_\pm$. The Verma module is spanned by the elements

$$\mathbf{i}_{2k_1-1} \cdots \mathbf{i}_{2k_p-1} \mathbf{l}_{-2l_1} \cdots \mathbf{l}_{-2l_q}(\phi_\alpha(0)), \quad \Delta_\alpha = \frac{\nu^2 \alpha(\alpha-2)}{4(1-\nu)}$$

In case when $\Delta_+ = \Delta_-$ the space is spanned by the even Virasoro generators $\{\mathbf{l}_{-2n}\}_{n \geq 1}$.

Asymptotic expansions

- Using the result by Bazhanov, Lukyanov, Zamolodchikov (96-99), we get asymptotic expansion

$$\log\rho^{\mathrm{sc}}(\lambda)\simeq\sum_{j=1}^{\infty}\lambda^{-\frac{2j-1}{\nu}}C_j\left(I^+_{2j-1}-I^-_{2j-1}\right)\to\tau^*_{2j-1}=C_j\mathbf{i}_{2j-1}$$

$$\omega^{\mathrm{sc}}(\lambda,\mu)\simeq\sqrt{\rho^{\mathrm{sc}}(\lambda)\rho^{\mathrm{sc}}(\mu)}\sum_{i,j=1}^{\infty}\lambda^{-\frac{2i-1}{\nu}}\mu^{-\frac{2j-1}{\nu}}\omega_{i,j}$$

Scaling limit of the determinant formula

$$Z_R^{\kappa,\kappa'}\left\{\tau^*(\lambda_1^0)\cdots\tau^*(\lambda_p^0)\beta^*(\lambda_1^+)\cdots\beta^*(\lambda_r^+)\gamma^*(\lambda_r^-)\cdots\gamma^*(\lambda_1^-)\left(\Phi_\alpha(0)\right)\right\}$$
$$=\prod_{i=1}^{p}\rho^{\mathrm{sc}}(\lambda_i^0)\times\det\left(\omega^{\mathrm{sc}}(\lambda_i^+,\lambda_j^-)\right)_{i,j=1,\dots,r}.$$

Technical problem: We get coefficients $\omega_{i,j}$ by the Wiener-Hopf technique only for $\kappa=\kappa'$ when $\Delta_+=\Delta_-$ and $\rho^{\mathrm{sc}}(\zeta)=1$ i.e. modulo the integrals of motion.

Correspondence to CFT 3-point correlator

- Important conjecture: it is possible to state the correspondence

$$\frac{\langle\Delta_-|P_\alpha(\{\mathbf{I}_{-k}\})\phi_\alpha(0)|\Delta_+\rangle}{\langle\Delta_-|\phi_\alpha(0)|\Delta_+\rangle} = \lim_{\mathbf{n}\to\infty, a\to 0, \mathbf{n}a=2\pi R} Z^\kappa\left\{q^{2\alpha S(0)}\mathcal{O}\right\}$$

between a polynomial $P_\alpha(\{\mathbf{I}_{-k}\})$ and some combinations of $\beta^*_{2j-1}, \gamma^*_{2j-1}$.

Introduce $\beta^*_{2m-1} = D_{2m-1}(\alpha)\beta^{\mathrm{CFT}*}_{2m-1}, \qquad \gamma^*_{2m-1} = D_{2m-1}(2-\alpha)\gamma^{\mathrm{CFT}*}_{2m-1}$

Even and odd bilinear combinations

$$\phi^{\mathrm{even}}_{2m-1,2n-1} = (m+n-1)\frac{1}{2}\left(\beta^{\mathrm{CFT}*}_{2m-1}\gamma^{\mathrm{CFT}*}_{2n-1} + \beta^{\mathrm{CFT}*}_{2n-1}\gamma^{\mathrm{CFT}*}_{2m-1}\right),$$

$$\phi^{\mathrm{odd}}_{2m-1,2n-1} = d_\alpha^{-1}(m+n-1)\frac{1}{2}\left(\beta^{\mathrm{CFT}*}_{2n-1}\gamma^{\mathrm{CFT}*}_{2m-1} - \beta^{\mathrm{CFT}*}_{2m-1}\gamma^{\mathrm{CFT}*}_{2n-1}\right),$$

$$d_\alpha = \frac{\nu(\nu-2)}{\nu-1}(\alpha-1)$$

Identification with Virasoro Verma-module

- If we accept an equivalence of the spaces spanned by

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}\mathbf{l}_{-2l_1}\cdots\mathbf{l}_{-2l_q}(\phi_\alpha(0)) \qquad \text{and}$$

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}\phi^{\text{even}}_{2m_1-1,2n_1-1}\cdots\phi^{\text{even}}_{2m_r-1,2n_r-1}\phi^{\text{odd}}_{2\bar{m}_1-1,2\bar{n}_1-1}\phi^{\text{odd}}_{2\bar{m}_{\bar{r}}-1,2\bar{n}_{\bar{r}}-1}(\Phi_\alpha(0))$$

we can identify modulo integrals of motion ($\Delta\equiv\Delta_\alpha$)

Jimbo, Miwa, Smirnov, HB (10), HB (11)

$$\phi^{\text{even}}_{1,1}\cong\mathbf{l}_{-2},\qquad \phi^{\text{even}}_{1,3}\cong\mathbf{l}^2_{-2}+\frac{2c-32}{9}\mathbf{l}_{-4},\qquad \phi^{\text{odd}}_{1,3}\cong\frac{2}{3}\,\mathbf{l}_{-4}$$

$$\phi^{\text{even}}_{1,5}\cong\mathbf{l}^3_{-2}+\frac{c+2-20\Delta+2c\Delta}{3(\Delta+2)}\mathbf{l}_{-4}\mathbf{l}_{-2}\;+\;\cdots\;\mathbf{l}_{-6}$$

$$\phi^{\text{odd}}_{1,5}\cong\frac{2\Delta}{\Delta+2}\mathbf{l}_{-4}\mathbf{l}_{-2}+\frac{56-52\Delta-2c+4c\Delta}{5(\Delta+2)}\,\mathbf{l}_{-6}$$

$$\phi^{\text{even}}_{3,3}\cong\mathbf{l}^3_{-2}+\frac{6+3c-76\Delta+4c\Delta}{6(\Delta+2)}\mathbf{l}_{-4}\mathbf{l}_{-2}\;+\;\cdots\;\mathbf{l}_{-6}$$

Fermionic construction of primary field

Let $\mathcal{V}_\alpha$ be the subspace obtained by acting $\beta^*_{2j-1}, \gamma^*_{2j-1}$ and integrals of motion $\mathbf{i}_{2k-1}$. In case $\kappa = \kappa'$ we factor out the integrals of motion

$$\mathcal{V}_\alpha^{\text{quo}} = \mathcal{V}_\alpha / \sum \mathbf{i}_{2k-1} \mathcal{V}_\alpha$$

The basis of $\mathcal{V}_\alpha^{\text{quo}}$: $\quad \beta^*_{I^+}, \gamma^*_{I^-} \Phi_\alpha(0)$

$$\beta^*_{I^+} = \beta^*_{2k_1-1} \cdots \beta^*_{2k_n-1}, \quad \gamma^*_{I^-} = \gamma^*_{2j_n-1} \cdots \gamma^*_{2j_1-1}.$$

Acting on $\Phi_\alpha(0)$ by $\beta^*_{2j-1}, \gamma^*_{2j-1}, \beta^*_{\text{screen},j}, \gamma^*_{\text{screen},j}$, one gets a space $\mathcal{H}_\alpha \supset \mathcal{V}_\alpha^{\text{quo}}$

The claim is: $\quad \mathcal{V}^{\text{quo}}_{\alpha+2m\frac{(1-\nu)}{\nu}} \subset \mathcal{H}_\alpha, \quad m \in \mathbb{Z}_{\geq 0}$ Jimbo, Miwa, Smirnov (11)

In particular: $\Phi_{\alpha+2m\frac{(1-\nu)}{\nu}}(0) \cong \beta^*_{I_{\text{odd(m)}}} \gamma^*_{\text{screen},I(m)} \Phi_\alpha(0)$

$$\gamma^*_{\text{screen},I(m)} = \gamma^*_{\text{screen},m} \cdots \gamma^*_{\text{screen},1}, \quad I(m) = (1,2,\cdots,m), I_{\text{odd(m)}} = (1,3,\cdots,2m-1)$$

Conformal dimensions of operators:

$$\beta^*_{2j-1}, \gamma^*_{2j-1} : \quad 2j-1, \quad \beta^*_{\text{screen},j} : \quad \nu(2-\alpha-2j), \quad \gamma^*_{\text{screen},j} : \quad \nu(\alpha-2j)$$

The ρ-problem and reflection equation

So far, we could identify our fermionic basis with the Verma module of the Virasoro algebra only modulo integrals of motion when $\rho = 1$. In particular, we used the Wiener-Hopf factorization technique. A direct generalization of this technique to the case $\rho \neq 1$ seems to be extremely hard problem. One of the difficulties is that the function ρ is involved into the integration measure.
But recently Negro and Smirnov (13) reproduced our results using the reflection equation found by A. Zamolodchikov, Al. Zamolodchikov (96).
This idea can be extended to solve the identification problem when $\rho \neq 1$.
One starts with the chiral CFT of one bosonic field (like BLZ)

$$\varphi(z) = \varphi_0 - 2i\pi_0 \log(z) + i \sum_{k\neq 0} \frac{a_k}{k} z^{-k}$$

with canonical 0-mode: $\pi_0 = \frac{\partial}{i\partial\varphi_0}$ and Heisenberg algebra: $[a_k, a_l] = 2k\delta_{k,-l}$

The primary field: $\phi_a(0) := e^{a\varphi(0)}$ $\qquad \nu\alpha = 2ab, \quad b^2 = \nu - 1$

is ideintified with highest vector of the Hisenberg algebra

$$P\big(\{\partial_z^k \varphi(0)\}\big)e^{a\varphi(0)} \Longleftrightarrow P(\{i(k-1)!a_{-k}\})\Phi_a, \quad \Phi_a = e^{a\varphi_0}|0\rangle$$

Virasoro generators via Heisenberg algebra and symmetries σ_1, σ_2

$$\mathbf{l}_k = \frac{1}{4}\sum_{j\neq 0,k} a_j a_{k-j} + \big(i(k+1)Q/2 + \pi_0\big) a_k, \quad k \neq 0, \quad Q = b + b^{-1}$$

$$\mathbf{l}_0 = \frac{1}{2}\sum_{j\geq 1} a_{-j} a_j + \pi_0(\pi_0 + iQ), \quad \mathbf{l}_0 \phi_a = \Delta \phi_a, \quad \Delta = a(Q-a)$$

with the integrals of motion $\quad \mathbf{i}_1 = \mathbf{l}_{-1}, \quad \mathbf{i}_3 = 2\sum_{k=-1}^{\infty} \mathbf{l}_{-3-k}\mathbf{l}_k, \quad \cdots$
The following observation comes from the sine-Gordon model considered as integrable perturbation of the Liouville model that one-point functions

$$\frac{\left\langle P\big(\{\partial_z^k \varphi(0)\}\big)\, e^{a\varphi(0)}\right\rangle}{\left\langle e^{a\varphi(0)}\right\rangle}$$

must be invariant under $\quad \sigma_1: \quad a \to -a, \qquad \sigma_2: \quad a \to Q - a$

Reflection equation and fermions

Combining the above expectation values into the vector $V_N(a)$ where N is the index counting the descendants one finds the reflection equation

$$V(Q-a)=V(a), \quad V(a+Q)=S(a)V(a), \quad S(a)=U(-a)U(a)^{-1}$$

which is matrix Riemann-Hilbert problem V. Fateev, D. Fradkin, S. Lukyanov, A. Zamolodchikov, Al. Zamolodchikov (99)

The matrix $U(a)$ is block-diagonal: for every level k the $p(k)$-dimensional block $U^{(k)}$ where $p(k)$ is number of partitions of k relates the Virasoro basis at level k with the Heisenberg one:

$$v_i^{(k)}=\sum_{j=1}^{p(k)} U^{(k)}(a)_{i,j}h_j^{(k)}$$

Negro and Smirnov claimed that under both transformations $\sigma_{1,2}$ the fermionic operators $\beta^*_{2m-1},\gamma^*_{2m-1}$ interchange, more precisely:

$$\sigma_2:\ \beta^{\mathrm{CFT}*}_{2m-1}\to\gamma^{\mathrm{CFT}*}_{2m-1}, \quad \gamma^{\mathrm{CFT}*}_{2m-1}\to\beta^{\mathrm{CFT}*}_{2m-1}$$

$$\sigma_1:\ \gamma^{\mathrm{CFT}*}_{2m-1}\to\left(\frac{2a-(2m-1)b}{2a+(2m-1)b^{-1}}\right)\beta^{\mathrm{CFT}*}_{2m-1},\ \beta^{\mathrm{CFT}*}_{2m-1}\to\left(\frac{2a-(2m-1)b^{-1}}{2a+(2m-1)b}\right)\gamma^{\mathrm{CFT}*}_{2m-1}$$

Contribution of integrals of motion

Another claim by Negro and Smirnov is that the fermionic basis diagonalize the matrix Riemann-Hilbert problem. They showed it for the quotient $\mathcal{V}_\alpha^{\mathrm{quo}} = \mathcal{V}_\alpha / \sum \mathbf{i}_{2k-1}\mathcal{V}_\alpha$. We have succeeded in generalizing this for the space $\mathcal{V}_\alpha$ when the level is 2 and 4. Smirnov, HB (16)

We used slightly different normalization of the operators. Our notations:

$$C_{2n-1}(\nu) = -\frac{\sqrt{\pi(1-\nu)}}{\nu n!}\frac{\Gamma\left(\frac{2n-1}{2\nu}\right)}{\Gamma\left(1+\frac{(2n-1)(1-\nu)}{2\nu}\right)}, \quad D_{2n-1}(\alpha,\nu) = \sqrt{\frac{i}{\nu}}\frac{1}{(n-1)!}\frac{\Gamma\left(\frac{\alpha}{2}+\frac{2n-1}{2\nu}\right)}{\Gamma\left(\frac{\alpha}{2}+\frac{(2n-1)(1-\nu)}{2\nu}\right)}.$$

Level 2: $\beta_1^*\gamma_1^* = D_1(\alpha,\nu)D_1(2-\alpha,\nu)\left(\mathbf{l}_{-2} - \frac{\alpha+1}{\alpha}\mathbf{i}_1^2\right) + \Big(A_{1,1}(\alpha,\nu) + B_{1,1}(\nu)\Big)\mathbf{i}_1^2$

$$A_{1,1}(\alpha,\nu) = \sin\pi\left(\frac{1-\nu}{2\nu}+\frac{\alpha}{2}\right)\sin\pi\left(\frac{1-\nu}{2\nu}-\frac{\alpha}{2}\right)$$

$$\times \frac{i}{2\pi^2\nu}\int_{-\infty}^{\infty}\tanh\frac{\pi}{2}(t+i\alpha)\left|\Gamma\left(\frac{2\nu-1}{2\nu}+\frac{it}{2}\right)\Gamma\left(\frac{\nu+1}{2\nu}+\frac{it}{2}\right)\right|^2\frac{t}{t^2+1}dt,$$

$$B_{1,1}(\nu) = \frac{i}{8\nu}\operatorname{ctg}\pi\left(\frac{1-\nu}{2\nu}\right)C_1(\nu)^2.$$

For $\alpha = (1-\nu)/\nu$ it must be singular vector, hence $A_{1,1}((1-\nu)/\nu,\nu) = 0$

OPE in fermionic basis

Here we take parameterization used in Liouville CFT

$$\nu = 1 + b^2, \quad c = 1 + 6Q^2, \quad Q = b + b^{-1}$$

$$\alpha = \frac{2a}{Q}, \quad \kappa = \frac{2a_1}{Q} - 1, \quad \kappa' = \frac{2a_2}{Q} - 1, \quad \Delta \to \Delta_a = a(Q - a)$$

- The OPE in the fermionic basis looks Smirnov,HB (15)

$$\Phi_{a_1}(z)\Phi_{a_2}(0) \cong$$

$$\sum_a C^a_{a_1,a_2} \; z^{\Delta_a - \Delta_{a_1} - \Delta_{a_2}} \sum_{\#(I^+)=\#(I^-)} z^{|I^+|+|I^-|} \Omega_{I^+,I^-}(a_1, a_2, a) \beta^*_{I^+} \gamma^*_{I^-} \Phi_a(0)$$

where $\cong$ means modulo integrals of motion,

$$\Omega_{I^+,I^-}(a_1, a_2, a) \equiv \Omega_{I^+,I^-}(a) = \frac{R_{I^+,I^-}(a_1, a_2, a)}{\prod_{l \in I^+} D_l(a) \prod_{l \in I^-} D_l(Q - a)}$$

$R_{I^+,I^-}(a_1, a_2, a)$ is a rational function.

Pole structure

The poles of $R_{l^+,l^-}(a_1, a_2, a)$ are at the points

$$a = a_{m,n}, \quad a_{m,n} = -\frac{n-1}{2}b - \frac{m-1}{2}b^{-1}$$

where $mn \equiv 0 \pmod 2$. We consider only the case $n \equiv 0 \pmod 2$, the case $m \equiv 0 \pmod 2$ is obtained by duality $b \to 1/b$.

- "Resonance" poles: $a = a_{m,n}$ and $a = a_{-m,-n} = Q - a_{m,n}$ with $m, n \geq 1$
- "Unwanted" poles: $a = a_{m,n}$ with $m = 0$ or $n = 0$ or $m > 0, n < 0$ or $m < 0, n > 0$

Null-Vectors

Introduce formally annihilation operators $\beta_j = t_j(a)^{-1}\gamma^*_{-j},\ \gamma_j = t_j(Q-a)^{-1}\beta^*_{-j}$ so that

$$[\beta_j,\beta^*_k]_+ = \delta_{j,k}, \quad [\gamma_j,\gamma^*_k]_+ = \delta_{j,k}$$

For $n \equiv 0(\mathrm{mod} 2)$ the null-vectors are constructed as

$$\left(C_{m,n}\right)^{\frac{n}{2}}\beta^*_{I^+}\gamma^*_{I^-}, \quad \#(I^-) = \#(I^+) + n$$

Jimbo, Miwa, Smirnov (11)

$$C_{m,n} = \sum_{j=1}^{\infty}\beta^*_{2j-1}\gamma_{2j-1+2(n-m)} + \sum_{j=1}^{\left[\frac{n-m}{2}\right]} t_{2j-1}(a_{m,n})\gamma_{2j-1}\gamma_{2(n-m)-2j+1}, \quad n \geq m$$

$$C_{m,n} = \sum_{j=1}^{\infty}\beta^*_{2j+2(m-n)-1}\gamma_{2j-1} - \sum_{j=1}^{\left[\frac{m-n}{2}\right]} \frac{1}{t_{-2j+1}(a_{m,n})}\beta^*_{2j-1}\beta^*_{2(m-n)-2j+1}, \quad n < m$$

$$t_p(a) = \frac{1}{2\sin\frac{\pi}{Q}(p\,b^{-1} - 2a)}$$

Recursion relations

In 1984 Alexei Zamolodchikov found recursion relations for the residues of conformal blocks. Here we have a similar relation.

- If $n-m$ is not odd positive (otherwise a bit more complicated), we have

$$\mathrm{res}_{a=a_{m,n}} \sum_{\#(I^+)=\#(I^-)} z^{|I^+|+|I^-|}\,\Omega_{I^+,I^-}(a)\;\beta^*_{I^+}\gamma^*_{I^-}\Phi_a$$

$$= W_{m,n} \sum_{\#(I^-)=\#(I^+)} z^{nm+|I^+|+|I^-|}\Omega_{I^+,I^-}(a_{m,n}+nb)\bigl(C_{m,n}\bigr)^{\frac{n}{2}}\beta^*_{I^+-2n}\gamma^*_{I^-+2n}\gamma^*_{I_{\mathrm{odd}(n)}}\Phi_a$$

where $\gamma^*_{I_{\mathrm{odd}(n)}}=\gamma^*_{2n-1}\cdots\gamma^*_1$ and $W_{m,n}$ contains

$$P_{m,n}=\prod_{j=1}^{n}\prod_{k=1}^{m}\Bigl(a_1+a_2-\frac{m-2k+3}{2}b^{-1}-\frac{n-2j+3}{2}b\Bigr)$$
$$\times\Bigl(a_1-a_2-\frac{m-2k+1}{2}b^{-1}-\frac{n-2j+1}{2}b\Bigr)$$

and a multiplier depending on b only. $W_{m,n}$ is related to 3-point function of Liouville CFT.

Conformal block

On a cylinder of radius $R = 1$ we consider the CFT with boundary conditions at $\pm\infty$ given by Φ_{a_2}, Φ_{Q-a_2}.
We know that the expectation values of the fermionic descendants of Φ_a on a cylinder are expressed in terms of function ω^{sc} or, more precisely, in terms of coefficients in its asymptotic expansion at $\lambda,\mu \to \infty$:

$$\omega^{\mathrm{sc}}(\lambda,\mu) \equiv \omega^{\mathrm{sc}}(\lambda,\mu|a_2,a) \simeq \sum_{i,j=1}^{\infty} \lambda^{-\frac{2i-1}{\nu}} \mu^{-\frac{2j-1}{\nu}} \omega_{2i-1,2j-1}(a_2,a)\,.$$

We know from the TBA-like approach

$$\omega_{2i-1,2j-1}(a_2,a) = D_{2i-1}(a) D_{2j-1}(Q-a) \Theta_{2i-1,2j-1}(a_2,a)\,,$$

where $\Theta_{2i-1,2j-1}(a_2,a)$ depends on a_2 as a polynomial in Δ_{a_2}, and on a as a polynomial in Δ_a with additional linear dependence on

$$d(a) = (b^{-1} - b)(Q - 2a)$$

The conformal block on the cylinder is obtained by computing the expectation values for the OPE. Introduce $\omega_{I^+,I^-}(a_2,a)$ by

$$\exp\Big\{\sum_{i,j=1}^{\infty}\omega_{2i-1,2j-1}(a_2,a)\gamma_{2i-1}\beta_{2j-1}\Big\}=\sum_{I^+,I^-}\omega_{I^+,I^-}(a_2,a)\gamma_{I^-}\beta_{I^+}$$

Then the conformal block is

$$\mathcal{F}(a_1,a_2,a,z)=z^{-2\Delta_{a_1}+\Delta_a}\sum_{I^+,I^-}z^{|I^+|+|I^-|}\Omega_{I^+,I^-}(a_1,a)\omega_{I^+,I^-}(a_2,a)$$

where $\Omega_{I^+,I^-}(a_1,a)=\Omega_{I^+,I^-}(a_1,a_1,a)$ are the above OPE-coefficients. Via the conformal mapping of the cylinder to the sphere, one can get symmetric under $a_1\leftrightarrow a_2$ combination

$$\mathcal{F}_{\mathrm{sym}}(a_1,a_2,a,z)=\Big(2\sinh(z/2)\Big)^{2\Delta_{a_1}}\mathcal{F}(a_1,a_2,a,z)$$

- Remark 1 In the conformal block $\mathcal{F}(a_1, a_2, a, z)$ or $\mathcal{F}_{\rm sym}(a_1, a_2, a, z)$ the unwanted poles cancel. So, we observe a nice duality between ω and Ω.

- Remark 2 The behaviour at $a \to \infty$ is described by

$$\log\big(\mathcal{F}_{\rm sym}(a_1, a_2, a, z)\big) =$$

$$+\Delta_a\big(\log z - \tfrac{1}{192}z^2 + \tfrac{73}{1474560}z^4 - \tfrac{1069}{1486356480}z^6 + \tfrac{250993}{20293720473600}z^8 + \cdots\big)$$

$$+(\Delta_{a_1} + \Delta_{a_2})\big(\tfrac{1}{16}z^2 - \tfrac{5}{12288}z^4 + \tfrac{17}{2949120}z^6 - \tfrac{1705}{16911433728}z^8 + \cdots\big)$$

$$+(c-1)\big(-\tfrac{1}{512}z^2 + \tfrac{7}{786432}z^4 - \tfrac{19}{188743680}z^6 + \tfrac{16019}{10823317585920}z^8 + \cdots\big) + O(a^{-1})$$

This is in full agreement with the formula by Alexei Zamolodchikov

Application to QFT, further perspectives

Further applications to QFT:

- Jimbo, Miwa and Smirnov (10-13) succeeded in applying fermionic structure for computation of one-point functions of the sine-Gordon model on cylinder
- also for sinh-Gordon model at finite temperature Smirnov, Negro (13)

Some problems are still open:

- Complete solution of the "ρ-problem"
- Generalization to higher spin and higher rank case. Some preliminary steps were undertaken Göhmann, Klümper, Nirov, Razumov, HB (10-14) Nirov, Hutsalyuk, HB (17)

Conclusions

- The factorization of static correlation functions of the XXZ spin chain was origially observed as a factorization of corresponding multiple integrals
- Factorized form is represented through polynomials of two transcendental functions ρ and ω.
- There is reach algebraic structure behind this factorization – hidden fermionic structure.
- This structure holds in conformal limit and for sine(sinh)-Gordon model.