Critical String from Non-Abelian Vortex in Four Dimensions

Mikhail Shifman and Alexei Yung
1 Introduction

It is believed that confinement in QCD is due to formation of confining vortex strings.

*Nambu, Mandelstam and ’t Hooft 1970’s:*

**Confinement** is a dual Meissner effect upon condensation of monopoles.

Monopoles condense $\rightarrow$ electric Abrikosov-Nielsen-Olesen flux tubes are formed $\rightarrow$ electric charges are confined

Higgs Phase for monopoles

charge

anticharge
Hadron spectrum is well described by linear Regge trajectories. However in all known examples the Regge trajectories show linear behavior only at asymptotically large spins.

Examples:

- Abrikosov-Nielsen-Olesen (ANO) vortex in weakly coupled Abelian-Higgs model
- Seiberg-Witten confinement in $\mathcal{N} = 2$ super-Yang-Mills theory

Length of the rotating string:

$$L^2 \sim \frac{J}{T}$$

Transverse size of the string is given by the inverse mass of the bulk fields forming the string:

$$m \sim g\sqrt{T}$$
String length $\gg$ its transverse size:

$$mL \gg 1, \quad J \gg \frac{1}{g^2}$$

We expect
In the real world Regge trajectories are linear at $J \sim 1$

Can we find any example of 4D theory where confining string remains thin at $J \sim 1$?

Thin string condition:

$$T \ll m^2$$
2 Thin string regime

How the problem of ”thick” string is seen in the world sheet effective theory?

ANO string: Nambu-Goto action

\[ S_{\text{NG}} = T \int d^2 \sigma \left\{ \sqrt{h} + O \left( \frac{\partial^n}{m^n} \right) \right\} \]

where

\[ h = \det(\partial_\alpha x^\mu \partial_\beta x_\mu) \]

Polchinski-Strominger, 1991: Without higher derivative terms
the world sheet theory is not UV complete
Higher derivative terms at weak coupling, $g \ll 1$

$$O \left( \frac{\partial^n}{m^n} \right), \quad m \sim g\sqrt{T}$$

At $J \sim 1$ \quad $\partial \to \sqrt{T}$

Thus higher derivative terms

$$\to \left( \frac{T}{m^2} \right)^n$$

blow up at weak coupling!

*Polyakov:* string surface become ”crumpled”.
We want to find a regime in which the string remains thin. This means that the higher derivative corrections should be parametrically small.

The low-energy world-sheet theory should be UV complete. This leads us to the following necessary conditions to have such a regime:

(i) The low-energy world-sheet theory on the string must be conformally invariant;

(ii) It must have the critical value of the Virasoro central charge.

- Bosonic string D=26
  Superstring D=10
- ANO string in D=4 is not critical
Non-Abelian vortex strings

Non-Abelian strings were suggested in $\mathcal{N} = 2$ U(N) QCD
Hanany, Tong 2003
Auzzi, Bolognesi, Evslin, Konishi, Yung 2003
Shifman Yung 2004
Hanany Tong 2004

$Z_N$ Abelian string: Flux directed in the Cartan subalgebra, say for $SO(3) = SU(2)/\mathbb{Z}_2$

$$flux \sim \tau_3$$

Non-Abelian string: Orientational zero modes
Rotation of color flux inside SU(N).
Idea:

Non-Abelian string has more moduli than ANO string. It has translational + orientational moduli.

We can fulfill the criticality condition: The solitonic non-Abelian vortex must have six orientational moduli, which, together with four translational moduli, will form a ten-dimensional space.
3 $\mathcal{N} = 2$ Supersymmetric QCD

$\mathcal{N} = 2$ QCD with gauge group $U(N) = SU(N) \times U(1)$ and $N_f$ flavors of fundamental matter – quarks

The field content:
U(1) gauge field $A_\mu$
SU(N) gauge field $A^a_\mu$, $a = 1, \ldots, N^2 - 1$
complex scalar fields $a$, and $a^a$
+ fermions
Complex scalar fields $q^{kA}$ and $\tilde{q}_{Ak}$ (squarks) + fermions
$k = 1, \ldots, N$ is the color index, $A$ is the flavor index, $A = 1, \ldots, N_f$
Squark VEV’s

\[ \langle q^{kA} \rangle = \sqrt{\xi} \begin{pmatrix} 1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 1 \end{pmatrix}, \]

\[ k = 1, \ldots, N, \quad A = 1, \ldots, N_f = N, \]

Color-flavor locking

Both gauge \( U(N) \) and flavor \( SU(N) \) are broken, however diagonal \( SU(N)_{C+F} \) is unbroken

\[ \langle q \rangle \rightarrow U\langle q \rangle U^{-1} \]
4 Non-Abelian vortex strings

Example in $U(2) = U(1) \times SU(2)$

Abrikosov-Nielsen-Olesen string:

$$q|_{r \to \infty} \sim \sqrt{\xi} e^{i\alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T = 4\pi \xi$$

Non-Abelian string:

$$q|_{r \to \infty} \sim \sqrt{\xi} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

$$T = 2\pi \xi$$

Here $r$ and $\alpha$ are polar coordinates in the plane orthogonal to the string axis.
String solution breaks $SU(2)_{C+F} \rightarrow 2$ orientational zero modes.

$$\frac{SU(2)_{C+F}}{U(1)} = CP(1) = O(3)$$

We have two dimensional $O(3)$ sigma model living on the string world sheet.

$$S_{(1+1)} = \frac{\beta}{2} \int dt \; dz \; (\partial_k \vec{S})^2, \quad \vec{S}^2 = 1$$
For $U(N)$ gauge group in the bulk we have 2D $CP(N - 1)$ model on the string

$CP(N - 1) = U(1)$ gauge theory in the strong coupling limit

$$S_{CP(N-1)} = \int d^2x \left\{ |\nabla_\alpha n^P|^2 + \frac{e^2}{2} \left( |n^P|^2 - 2\beta \right)^2 \right\},$$

where $n^P$ are complex fields $P = 1, \ldots, N$,

Condition

$$|n^P|^2 = 2\beta = \frac{4\pi}{g^2},$$

imposed in the limit $e^2 \rightarrow \infty$
More flavors ⇒ semilocal non-Abelian string

The orientational moduli described by a complex vector $n^P$ (here $P = 1, \ldots, N$),

$\tilde{N} = (N_f - N)$ size moduli are parametrized by a complex vector $\rho^K$ ($K = N + 1, \ldots, N_f$).

The effective two-dimensional theory is the $\mathcal{N} = (2, 2)$ weighted CP model

$$S_{\text{WCP}} = \int d^2x \left\{ |\nabla_\alpha n^P|^2 + |\tilde{\nabla}_\alpha \rho^K|^2 + \frac{e^2}{2} \left( |n^P|^2 - |\rho^K|^2 - 2\beta \right)^2 \right\},$$

where $P = 1, \ldots, N$, $K = N + 1, \ldots, N_f$.

The fields $n^P$ and $\rho^K$ have charges $+1$ and $-1$ with respect to the auxiliary U(1) gauge field

$$e^2 \to \infty$$
5 From non-Abelian vortices to critical strings

String theory

\[ S = \frac{T}{2} \int d^2 \sigma \sqrt{h} h^{\alpha\beta} \partial_{\alpha} x^\mu \partial_{\beta} x_\mu \]

\[ + \int d^2 \sigma \sqrt{h} \left\{ h^{\alpha\beta} \left( \tilde{\nabla}_\alpha \tilde{n}_P \nabla_\beta n^P + \nabla_\alpha \tilde{\rho}_K \tilde{\nabla}_\beta \rho^K \right) + \frac{e^2}{2} \left( |n^P|^2 - |\rho^K|^2 - 2\beta \right)^2 \right\} + \text{fermions}, \]

where \( h^{\alpha\beta} \) is the world sheet metric. It is independent variable in the Polyakov formulation.
What about necessary conditions for thin string?

- Conformal invariance

\[ b_{WCP} = N - \tilde{N} = 0 \Rightarrow N = \tilde{N}, \quad N_f = 2N \]

- The total Virasoro central charge

\[ c_{\text{tot}} = \frac{3}{2} \left( D + \frac{2}{3} c_{\text{wcp}} - 10 \right) \]

Our string is BPS so we have \( \mathcal{N} = (2,2) \) supersymmetry on the world sheet.
For $\mathcal{N} = (2, 2)$ sigma models

$$c = 3 \sum_i (1 - 2q_i)$$

In our theory the $R$ charges of the fields $n^P$ and $\rho^K$ vanish. Thus, we should just count the number of degrees of freedom, namely

$$c_{\text{wcp}} = 3(N + \tilde{N} - 1).$$

The condition of criticality

$$c_{\text{tot}} = \frac{3}{2} [D + 2(2N - 1) - 10] = 0$$

For $D = 4$ the solution is

$$N = \tilde{N} = 2, \quad N_f = 4.$$ 

For these values of $N$ and $\tilde{N}$ the target space of the weighted $CP(N, \tilde{N})$ model is a noncompact Calabi-Yau manifold studied by Witten.
6 Gauge-string duality

To describe physics at weak coupling we use the bulk theory in its original formulation in terms of quarks and gauge bosons. The quarks and gauge bosons have masses of the order of

\[ m \sim g\sqrt{T} \]

At \( g^2 \ll 1 \) they are light, while stringy states with masses of the order of \( \sqrt{T} \) are heavy.

What happen in the strong coupling limit?

Given the necessary conditions are met we conjecture that the thin-string condition is actually satisfied at \( g^2 \sim 1 \).

\[ T \ll m^2 \]

Quarks and gauge bosons are heavy while the stringy states are light. To describe physics in the regime of large \( m \) we should use string theory. We call this gauge-string duality.
Strings in the $U(N)$ theories are stable; they cannot be broken. Thus, we deal with the closed string.

Quarks are condensed in the bulk theory. Therefore, monopoles are confined.

In $U(N)$ gauge theories the confined monopoles are junctions of two non-Abelian vortex strings.

In particular, monopole-antimonopole stringy meson formed in the so called instead-of-confinement phase is a state of the closed string.

- It is stable because it is light
- Have “correct” (adjoint or singlet) quantum numbers with respect to the global flavor group.
- Lie on the linear Regge trajectories even at small spins
7 Conclusions

- In \( \mathcal{N} = 2 \) supersymmetric QCD with gauge group U(2) and \( N_f = 4 \) quark flavors non-Abelian BPS vortex behaves as a critical fundamental superstring.

- Given the conditions of criticality are satisfied we conjecture that at strong coupling non-Abelian string enters a thin string regime so its states lie on linear Regge trajectories even at small spins.