Renormdynamics, Discrete Dynamics and Quanputers

Nugzar Makhaldiani

JINR Dubna, mnv@jinr.ru

Talk at the International Workshop Supersymmetries and Quantum Symmetries - SQS'2015
JINR Dubna Russia, August 3 - August 8

We say that we find New Physics when either we find a phenomenon which is forbidden by SM in principal - this is the qualitative level of New physics - or we find significant deviation between precision calculations in SM of an observable quantity and corresponding experimental value.
In 1900, the British physicist Lord Kelvin is said to have pronounced: "There is nothing new to be discovered in physics now. All that remains is more and more precise measurement." Within three decades, quantum mechanics and Einstein's theory of relativity had revolutionized the field. Today, no physicist would dare assert that our physical knowledge of the universe is near completion. To the contrary, each new discovery seems to unlock a Pandora's box of even bigger, even deeper physics questions.

In the Universe, matter has manly two geometric structures, homogeneous, [Weinberg,1972] and hierarchical, [Okun, 1982] .
The homogeneous structures are naturally described by real numbers with an infinite number of digits in the fractional part and usual archimedean metrics. The hierarchical structures are described with p-adic numbers with an infinite number of digits in the integer part and non-archimedean metrics, [Koblitz, 1977].

A discrete, finite, regularized, version of the homogenous structures are homogeneous lattices with constant steps and distance rising as arithmetic progression. The discrete version of the hierarchical structures is hierarchical lattice-tree with scale rising in geometric progression.

There is an opinion that present day theoretical physics needs (almost) all mathematics, and the progress of modern mathematics is stimulated by fundamental problems of theoretical physics.

In QFT existence of a given theory means, that we can control its behavior at some scales (short or large distances) by renormalization theory [Collins, 1984].
If the theory exists, than we want to solve it, which means to determine what happens on other (large or short) scales. This is the problem (and content) of Renormdynamics.
The result of the Renormdynamics, the solution of its discrete or continual motion equations, is the effective QFT on a given scale (different from the initial one).
We can invent scale variable λ and consider QFT on $D+1+1$ dimensional space-time-scale. For the scale variable $\lambda \in (0,1]$ it is natural to consider q-discretization, $0 < q < 1,\ \lambda_n = q^n,\ n = 0, 1, 2,$ ... and p - adic, nonarchimedian metric, with $q^{-1} = p$ - prime integer number.
The field variable $\varphi(x, t, \lambda)$ is complex function of the real, x, t, and p - adic, λ, variables. The solution of the UV renormdynamic problem means, to find evolution from finite to small scales with respect to the scale time $\tau = \ln \lambda/\lambda_0 \in (0, -\infty)$. Solution of the IR renormdynamic problem means to find evolution from finite to the large scales, $\tau = \ln \lambda/\lambda_0 \in (0, \infty)$.

This evolution is determined by Renormdynamic motion equations with respect to the scale-time.
As a concrete model, we take a relativistic scalar field model with lagrangian (see e.g. [Makhaldiani, 1980])

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{g}{n}\varphi^n,\ \mu = 0, 1, ..., D-1 \quad (1)$$

The mass dimension of the coupling constant is

$$[g] = d_g = D - n\frac{D-2}{2} = D + n - \frac{nD}{2}. \quad (2)$$

In the case

$$n = \frac{2D}{D-2} = 2 + \frac{4}{D-2} = 2 + \epsilon(D)$$
$$D = \frac{2n}{n-2} = 2 + \frac{4}{n-2} = 2 + \epsilon(n) \quad (3)$$

the coupling constant g is dimensionless, and the model is renormalizable. We take the euklidean form of the QFT which unifies quantum and statistical physics problems. In the case of the QFT, we can return (in)to minkowsky space by transformation: $p_D = ip_0,\ x_D = -ix_0$.

The main objects of the theory are Green functions - correlation functions - correlators,

$$G_m(x_1, x_2, ..., x_m) = < \varphi(x_1)\varphi(x_2)...\varphi(x_m) >$$
$$= Z_0^{-1} \int d\varphi(x)\varphi(x_1)\varphi(x_2)...\varphi(x_m)e^{-S(\varphi)} \quad (4)$$

where $d\varphi$ is an invariant measure,

$$d(\varphi + a) = d\varphi. \quad (5)$$

For gaussian actions,

$$S = S_2 = \frac{1}{2} \int dxdy\varphi(x)A(x, y)\varphi(y) = \varphi \cdot A \cdot \varphi \quad (6)$$

the QFT is solvable,

$$G_m(x_1, ..., x_m) = \frac{\delta^m}{\delta J(x_1)...J(x_m)} lnZ_J|_{J=0},$$
$$Z_J = \int d\varphi e^{-S_2 + J\cdot\varphi} = \exp(\frac{1}{2} \int dxdyJ(x)A^{-1}(x, y)J(y))$$
$$= \exp(\frac{1}{2} J \cdot A^{-1} \cdot J) \quad (7)$$

This solution is based on the solution of the linear motion equations with sources

$$A(x, y)\varphi(y) = j(x) \quad (8)$$

Nontrivial problem is to calculate correlators for non gaussian QFT.

Effective action

Generating functional for connected correlators is

$$F(J) = \ln Z_J, \; \frac{\delta F(J)}{\delta J(x)} = \frac{1}{Z_J}\frac{\delta Z_J}{\delta J(x)} \equiv < \varphi(x) >_J \equiv \phi(x) - \tag{9}$$

is observable value of the field, generated by source J. We have

$$\frac{\delta}{\delta J}(F(J) - J \cdot \phi)|_{\phi=const} = 0, \tag{10}$$

so

$$\begin{aligned} & J \cdot \phi - F(J) = S_q(\phi) = S(\phi) + R(\phi) \\ & = \sum_{n \geq 1} \frac{1}{n!} \int dx_1 dx_2 ... dx_n \Gamma_n(x_1, x_2, ..., x_n) \phi(x_1)\phi(x_2)...\phi(x_n), \\ & \frac{\delta S_q}{\delta \phi(x)} = J(x); \; \frac{\delta^2 S_q}{\delta \phi(x_1)\delta \phi(x_2)} = \frac{\delta J(x_2)}{\delta \phi(x_1)} = \frac{\delta J(x_1)}{\delta \phi(x_2)} = \Gamma_2(x_1, x_2) \end{aligned} \tag{11}$$

$R(\phi)$ - is quantum corrections to the classical action.
The connected part of the two point correlator - propagator, is

$$\begin{aligned} & < \varphi(x_1)\varphi(x_2) >_c = < \varphi(x_1)\varphi(x_2) > - < \varphi(x_1) >< \varphi(x_2) > \\ & = \frac{1}{Z(J)}\frac{\delta^2 Z(J)}{\delta J(x_1)\delta J(x_2)} - \frac{1}{Z(J)}\frac{\delta Z(J)}{\delta J(x_1)}\frac{1}{Z(J)}\frac{\delta Z(J)}{\delta J(x_2)} = \Gamma_2(x_1, x_2) \end{aligned} \tag{12}$$

Perturbative series have the following qualitative form

$$f(g) = f_0 + f_1 g + ... + f_n g^n + ..., \;\; f_n = n!P(n)$$
$$f(x) = \sum_{n\geq 0} P(n)n!x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x}, \;\; \delta = x\frac{d}{dx} \qquad (13)$$

In usual sense these series are divergent, but with proper nomalization of the expansion parametre g, the coefficients of the series are rational numbers and if experimental dates indicates for some rational value for g, e.g. in QED

$$g = \frac{e^2}{4\pi} = \frac{1}{137.0...} \qquad (14)$$

then we can take corresponding prime number and consider p-adic convergence of the series. In the case of QED, we have

$$f(g) = \sum f_n p^{-n}, \;\; f_n = n!P(n), \;\; p = 137,$$
$$|f|_p \leq \sum |f_n|_p p^n \qquad (15)$$

In the Youkava theory of strong interections (see e.g. [Bogoliubov,1959]), we take $g = 13$,

$$f(g) = \sum f_n p^n, \; f_n = n! P(n), \; p = 13,$$
$$|f|_p \leq \sum |f_n|_p p^{-n} < \frac{1}{1-p^{-1}} \qquad (16)$$

So, the series is convergent. If the limit is rational number, we consider it as an observable value of the corresponding physical quantity. Note also, that the inverse coupling expansions, e.g. in lattice(gauge) theories,

$$f(\beta) = \sum r_n \beta^n, \qquad (17)$$

are also p-adically convergent for $\beta = p^k$. We can take the following scenery. We fix coupling constants and masses, e.g in QED or QCD, in low order perturbative expansions. Than put the models on lattice and calculate observable quantities as inverse coupling expansions, e.g.

$$f(\alpha) = \sum r_n \alpha^{-n},$$
$$\alpha_{QED}(0) = 1/137; \; \alpha_{QCD}(m_Z) = 0.11... = 1/3^2 \qquad (18)$$

In $MSSM$ (see [M.Muehlleitner, CALC 2012], [D.I.Kazakov, 2004]) coupling constants of the SM unifies at $\alpha_u^{-1} = 26.3 \pm 1.9 \pm 1.$
So,

$$23.4 < \alpha_u^{-1} < 29.2 \tag{19}$$

Question: how many primes are in this interval?

$$24,\ 25,\ 26,\ 27,\ 28,\ 29 \tag{20}$$

Only one!
Proposal: take the value $\alpha_u^{-1} = 29.0...$ which will be two orders of magnitude more precise prediction and find the consequences for the SM scale observables.
Remind that for low energy limit of the fine structure constant $\alpha,\ \alpha^{-1} = 137.036...$

Let us consider the formal representation of (13)

$$f(x) = \sum_{n\geq 0} P(n)n!x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x},$$
$$= P(\delta)\int_0^\infty dte^{-t}t^\delta \frac{1}{1-x} = P(\delta)\int_0^\infty dt\frac{e^{-t}}{1+(-x)t},\ \delta = x\frac{d}{dx} \quad (21)$$

This integral is well defined for negative values of x. The Mathematica answer for the corresponding integral is

$$I(x) = \int_0^\infty dt\frac{e^{-t}}{1+xt} = e^{1/x}\Gamma(0,1/x)/x,\ Im(x)\neq 0,\ Re(x)\geq 0,$$
$$I(0) = 1 \quad (22)$$

For $x = 0.001,\ I(x) = 0.999,\ \Gamma(a,z)$ is the incomplete gamma function

$$\Gamma(a,z) = \int_z^\infty dtt^{a-1}e^{-t} \quad (23)$$

The Goldberger-Treiman relation (GTR) [Goldberger and Treiman,1958] plays an important role in theoretical hadronic and nuclear physics. GTR relates the Meson-Nucleon coupling constants to the axial-vector coupling constant in β-decay:

$$m_N g_A(0) = f_\pi g_{\pi N} \tag{24}$$

where m_N is the nucleon mass, $g_A(0)$ is the axial-vector coupling constant in nucleon β-decay at vanishing momentum transfer, f_π is the π decay constant and $g_{\pi N}$ is the $\pi - N$ coupling constant.
If we take

$$\alpha_{\pi N} = \frac{g_{\pi N}^2}{4\pi} = 13, \;\; g_{\pi N} = 12.78 \tag{25}$$

experimental value for f_π from pion decay

$$f_\pi = \frac{130}{\sqrt{2}} = 91.9MeV, \tag{26}$$

Neutron mass,

$$m_N = 940MeV, \tag{27}$$

from (24), we find

$$g_A(0) = \frac{f_\pi g_{\pi N}}{m_N} = \frac{91.9 \times \sqrt{52\pi}}{940} = 1.2496 \simeq 1.25, \tag{28}$$

which coincides with the experimental value from β-decay

$$g_A(0) = 1.25 \tag{29}$$

So, we can say that using GTR we measured the pion-nucleon fine structure constant and find the value

$$\alpha_{\pi N} = \frac{g_{\pi N}^2}{4\pi} = 13 \tag{30}$$

Note that, determination of $g_{\pi N}$ from $NN, N\bar{N}$ and πN data by the Nijmegen group [Rentmeester et al, 1999] gave the following value

$$g_{\pi N} = 13.05 \pm .08,\ \Delta = 1 - \frac{g_A m_N}{g_{\pi N} f_\pi} = .014 \pm .009,$$
$$13.39 < \alpha_{\pi N} < 13.72 \qquad (31)$$

This value is consistent with assumption $g_{\pi N} = 13.$
Due to the smallness of the u and d quark masses, Δ is necessarily very small, and its determination requires a very precise knowledge of the $g_{\pi N}$ coupling (g_A and f_π are already known to enough precision, leaving most of the uncertainty in the determination of Δ to the uncertainty in $g_{\pi N}$).

Note that in an old version of the unified theory [Heisenberg 1966], for the $\alpha_{\pi N}$ the following value were found

$$\alpha_{\pi N} = 4\pi(1 - \frac{m_\pi^2}{3m_p^2}) = 12.5 \qquad (32)$$

Following the pion, the rho is the most prominent meson. Vector mesons play an important role when considering the interaction of hadrons with electromagnetic fields. In the vector meson dominance model the hadrons couple to photons exclusively through intermediate vector mesons. The equality of the ρ meson self-coupling g and the coupling to nucleons $g_{\rho N}$ and pions $g_{\rho\pi}$, the universality of the ρ meson coupling, plays an important role in vector meson dominance [Sakurai, 1969] and is a consequence of the existence of a consistent EFT with ρ mesons, pions, and nucleons. Indeed, one can rewrite the Lagrangian of [Weinberg, 1968] in terms of renormalized fields and couplings, thereby introducing the basic Lagrangian

$$L_R = \bar{N}(i\gamma\partial - M)N - \frac{1}{2}\pi(\partial^2 + m^2)\pi - \frac{1}{4}(\partial_\mu\rho_\nu^a - \partial_\nu\rho_\mu^a)^2 + \frac{1}{2}M_\rho^2\rho^2 + g\bar{N}\gamma^\mu t_a N\rho_\mu^a + g_{\pi\rho}\epsilon^{abc}\pi^a\partial^\mu\pi^b\rho_\mu^c - g(\rho_\mu \times \rho_\nu)\cdot\partial^\mu\rho^\nu - \frac{g^2}{4}(\rho_\mu \times \rho_\nu)^2 \tag{33}$$

Requiring that the results are UV finite introduces relations between the couplings of the theory [Djukanovic et al, 2004], $g_{\pi\rho} = g$. The coupling g is directly related to the width of the ρ meson.

In the previous $\pi\rho N$ model of pion-nucleon interaction [Di Giacomo, Paffuti, Rossi, 1992]

$$L_{\pi N} = g(\bar{N}\gamma^\mu t_a N + \epsilon^{abc}\pi^b \partial^\mu \pi^c)\rho^a_\mu, \quad (34)$$

pion interacts with nucleon through the exchange of the vector meson $\rho(m_\rho = 750\ MeV, T = 1)$, the amplitude of $\rho^0 \to \pi^+\pi^-$ decay is

$$M = g\varepsilon^\mu (k_{\pi^-} - k_{\pi^+})_\mu, \quad (35)$$

the decay width is

$$\Gamma = \frac{1}{2m_\rho}|M|^2 (1 - \frac{4m_\pi^2}{m_\rho^2})^{\frac{1}{2}} \frac{1}{8\pi} = \frac{g^2}{48\pi} m_\rho (1 - \frac{4m_\pi^2}{m_\rho^2})^{\frac{3}{2}} \quad (36)$$

and for fine structure coupling constant we have

$$\alpha_{\pi\rho N} = \frac{g^2}{4\pi} = \frac{\Gamma}{m_\rho} \frac{12}{(1 - \frac{4m_\pi^2}{m_\rho^2})^{\frac{3}{2}}} = \frac{12.}{5(1 - \frac{4\times 14^2}{75^2})^{\frac{3}{2}}} = 3.006 = 3.0.. \quad (37)$$

for $\Gamma = \Gamma_{\rho\pi\pi} = 150 MeV, m_\pi = 140 MeV, m_\rho = 750 MeV$. So, in this strong coupling model the expansion parameter is a prime number, $\alpha_g = 3$.

After integrating out all heavy and trapped particles, we would expect the effective Lagrangian for

$$\pi^0 \to 2\gamma \tag{38}$$

to be given by the unique gauge and Lorentz-invariant term with no more than two derivatives:

$$L_{\pi\gamma\gamma} = g\pi^0 \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{39}$$

where g is an unknown constant with the mass dimension $m^{-1.}$
The rate for $\pi^0 \to 2\gamma$ is

$$\Gamma(\pi^0 \to 2\gamma) = \frac{g^2 m_\pi^3}{\pi} \tag{40}$$

One might naively expect g to be of order

$$g = \frac{a^2}{F_\pi},\ a = \frac{e}{4\pi}, \tag{41}$$

where $F_\pi = 190MeV$ is used as a typical strong interaction mass scale.

In 1949, using the pre-QCD theory of pions and nuclons with interaction lagrangian

$$L_{\pi NN} = iG_{\pi N}\pi^a \bar{N} 2t^a \gamma_5 N, \tag{42}$$

Steinberger calculated the contribution to g from triangle graphs with a single proton loop

$$g = \frac{e^2 G_{\pi N}}{32\pi^2 m_N} = a^2 \frac{G_{\pi N}}{2m_N},\ a = \frac{e}{4\pi}. \tag{43}$$

From Goldberger-Treiman relation we have

$$\frac{G_{\pi N}}{2m_N} = \frac{g_A}{F_\pi}, \tag{44}$$

so,

$$g = \frac{a^2}{F_\pi} g_A,\ g_A = 1.257,\ F_\pi = 184 MeV \tag{45}$$

Using

$$g = \frac{a^2}{F_\pi},\ a = \frac{e}{4\pi}, \tag{46}$$

$$\Gamma(\pi^0 \to 2\gamma) = \frac{g^2 m_\pi^3}{\pi} = \frac{a^4 m_\pi^3}{\pi F_\pi^2} = \frac{\alpha^2 m_\pi^3}{16\pi^3 F_\pi^2} = 1.1 \times 10^{16} s^{-1} \tag{47}$$

The observed rate is

$$\Gamma(\pi^0 \to 2\gamma)_{exp} = (1.19 \pm 0.08) \times 10^{16} s^{-1}, \tag{48}$$

which is in good agreement with the (naive rough) estimation.

Every (good) school boy/girl knows what is

$$\frac{d^n}{dx^n} = \partial^n = (\partial)^n, \tag{49}$$

but what is its following extension

$$\frac{d^\alpha}{dx^\alpha} = \partial^\alpha \ , \quad \alpha \in \Re \ ? \tag{50}$$

Let us consider the integer derivatives of the monomials

$$\begin{aligned}\frac{d^n}{dx^n}x^m &= m(m-1)...(m-(n-1))x^{m-n}, \quad n \leq m, \\ &= \frac{\Gamma(m+1)}{\Gamma(m+1-n)}x^{m-n}. \end{aligned} \quad (51)$$

L.Euler (1707 - 1783) invented the following definition of the fractal derivatives,

$$\frac{d^\alpha}{dx^\alpha}x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}. \quad (52)$$

J.Liouville (1809-1882) takes exponents as a base functions,

$$\frac{d^\alpha}{dx^\alpha}e^{ax} = a^\alpha e^{ax}. \quad (53)$$

The following Cauchy formula

$$I_{0,x}^{n} f = \int_0^x dx_n \int_0^{x_{n-1}} dx_{n-2} ... \int_0^{x_2} dx_1 f(x_1) = \frac{1}{\Gamma(n)} \int_0^x dy (x-y)^{n-1} f(y)($$

permits analytic extension from integer n to complex α,

$$I_{0,x}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x dy (x-y)^{\alpha-1} f(y) \quad (55)$$

J.H. Holmgren invented (in 1863) the following integral transformation,

$$D_{c,x}^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_c^x |x-t|^{\alpha-1} f(t) dt. \quad (56)$$

It is easy to show that

$$D_{c,x}^{-\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)}(x^{m+\alpha} - c^{m+\alpha}),$$
$$D_{c,x}^{-\alpha} e^{ax} = a^{-\alpha}(e^{ax} - e^{ac}), \quad (57)$$

so, $c = 0$, when $m + \alpha \geq 0$, in Holmgren's definition of the fractal calculus, corresponds to the Euler's definition, and $c = -\infty$, when $a > 0$, corresponds to the Liouville's definition.
Holmgren's definition of the fractal calculus reduce to the Euler's definition for finite c, and to the Liouvill's definition for $c = \infty$,

$$D_{c,x}^{-\alpha} f = D_{0,x}^{-\alpha} f - D_{0,c}^{-\alpha} f,$$
$$D_{\infty,x}^{-\alpha} f = D_{-\infty,x}^{-\alpha} f - D_{-\infty,\infty}^{-\alpha} f. \quad (58)$$

We considered the following modification of the $c = 0$ case [Makhaldiani, 2003],

$$D_{0,x}^{-\alpha} f = \frac{|x|^\alpha}{\Gamma(\alpha)} \int\limits_0^1 |1-t|^{\alpha-1} f(xt) dt, = \frac{|x|^\alpha}{\Gamma(\alpha)} B(\alpha,\ \partial x) f(x)$$
$$= |x|^\alpha \frac{\Gamma(\partial x)}{\Gamma(\alpha+\partial x)} f(x),\ f(xt) = t^{x\frac{d}{dx}} f(x). \quad (59)$$

As an example, consider Euler B-function,

$$B(\alpha,\beta) = \int_0^1 dx |1-x|^{\alpha-1}|x|^{\beta-1} = \Gamma(\alpha)\Gamma(\beta) D_{01}^{-\alpha} D_{0x}^{1-\beta} 1 = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (60)$$

We can define also FC as

$$D^\alpha f = (D^{-\alpha})^{-1} f = \frac{\Gamma(\partial x + \alpha)}{\Gamma(\partial x)} (|x|^{-\alpha} f),\ \partial x = \delta + 1,\ \delta = x\partial \quad (61)$$

For the Liouville's case,

$$D_{-\infty,x}^{\alpha} f = (D_{-\infty,x})^\alpha f = (\partial_x)^\alpha f, \quad (62)$$

$$\partial_x^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} e^{-t\partial_x} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} f(x-t)$$
$$= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x dt (x-t)^{\alpha-1} f(t) = D_{-\infty,x}^{-\alpha} f. \quad (63)$$

The integrals can be calculated as

$$D^{-n}f = (D^{-1})^n f, \tag{64}$$

where

$$D^{-1}f = x\frac{\Gamma(\partial x)}{\Gamma(1+\partial x)}f = x\frac{1}{\partial x}f = x(\partial x)^{-1}f = (\partial)^{-1}f = \int_0^x dt f(t). \tag{65}$$

Let us consider Weierstrass C.T.W. (1815 - 1897) fractal function

$$f(t) = \sum_{n\geq 0} a^n e^{i(b^n t + \varphi_n)}, \quad a < 1, \quad ab > 1. \tag{66}$$

For fractals we have no integer derivatives,

$$f^{(1)}(t) = i\sum (ab)^n e^{i(b^n t + \varphi_n)} = \infty, \tag{67}$$

but the fractal derivative,

$$f^{(\alpha)}(t) = \sum (ab^\alpha)^n e^{i(b^n t + \pi\alpha/2 + \varphi_n)}, \tag{68}$$

when $ab^\alpha = a' < 1$, is another fractal (66).
Question: what if $ab = p$ is prime number? Can we define integer derivatives in this case?

p-adic analog of the fractal calculus (56) ,

$$D_x^{-\alpha} f = \frac{1}{\Gamma_p(\alpha)} \int_{Q_p} |x-t|_p^{\alpha-1} f(t) dt, \tag{69}$$

where $f(x)$ is a complex function of the p-adic variable x, with p-adic Γ–function

$$\Gamma_p(\alpha) = \int_{Q_p} dt |t|_p^{\alpha-1} \chi(t) = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}, \tag{70}$$

was considered by V.S. Vladimirov [Vladimirov,1988].
The following modification of p-adic FC is given in [Makhaldiani, 2003]

$$\begin{aligned} D_x^{-\alpha} f &= \frac{|x|_p^{\alpha}}{\Gamma_p(\alpha)} \int_{Q_p} |1-t|_p^{\alpha-1} f(xt) dt \\ &= |x|_p^{\alpha} \frac{\Gamma_p(\partial|x|)}{\Gamma_p(\alpha+\partial|x|)} f(x). \end{aligned} \tag{71}$$

Last expression is applicable for functions of the type $f(x) = f(|x|)$.
For a functions of the form

$$f(x) = \sum a_n |x|_p^n, \tag{72}$$

we have

$$D_x^{-\alpha} f = \sum a_n \frac{\Gamma_p(n+1)}{\Gamma_p(n+1+\alpha)} |x|_p^{n+\alpha}. \tag{73}$$

The basic object of q-calculus [Gasper, Rahman, 1990] is q-derivative

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} = \frac{1 - q^{x\partial}}{(1-q)x} f(x), \tag{74}$$

where either $0 < q < 1$ or $1 < q < \infty$. In the limit $q \to 1, D_q \to \partial_x$.
Now we define the fractal q-calculus,

$$D_q^\alpha f(x) = (D_q)^\alpha f(x)$$
$$= ((1-q)x)^{-\alpha}(f(x) + \sum_{n\geq 1}(-1)^n \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!} f(q^n x)). \tag{75}$$

For the case $\alpha = -1$, we obtain the integral

$$D_q^{-1} f(x) = (1-q)x(1-q^{x\partial})^{-1} f(x) = (1-q)x \sum_{n\geq 0} f(q^n x). \tag{76}$$

In the case of $1 < q < \infty$, we can give a good analytic sense to these expressions for prime numbers $q = p = 2, 3, 5, ..., 29, ..., 137, ...$ This is an *algebra-analytic quantization* of the q-calculus and corresponding physical models. Note also, that p-adic calculus is the natural tool for the physical models defined on the fractal(space)s like Bete lattice (or Brua-Tits trees, in mathematical literature).
Note also a symmetric definition of the calculus

$$D_{qs} f(x) = \frac{f(q^{-1}x) - f(qx)}{(q^{-1}-q)x} f(x). \tag{77}$$

Usual finite difference calculus is based on the following (left) derivative operator

$$D_- f(x) = \frac{f(x) - f(x-h)}{h} = (\frac{1 - e^{-h\partial}}{h}) f(x). \tag{78}$$

We define corresponding fractal calculus as

$$D_-^{\alpha} f(x) = (D_-)^{\alpha} f(x). \tag{79}$$

In the case of $\alpha = -1$, we have usual finite difference sum as regularization of the Riemann integral

$$D_-^{-1} f(x) = h(f(x) + f(x-h) + f(x-2h) + ...). \tag{80}$$

(I believe that) the fractal calculus (and geometry) are the proper language for the quantume (field) theories, and discrete versions of the fractal calculus are proper regularizations of the fractal calculus and field theories.

A hypergeometric series, in the most general sense, is a power series in which the ratio of successive coefficients indexed by n is a rational function of n,

$$f(x) = \sum_{n \geq 0} a_n x^n, \ a_{n+1} = R(n) a_n, \ R(n) = \frac{P(\alpha, n)}{Q(\beta, n)} \tag{81}$$

so

$$\begin{aligned} &P(\alpha, \delta) f(x) = Q(\beta, \delta)(f(x) - f(0))/x, \\ &f(x) - f(0) = xR(\delta) f(x), f(x) = (1 - xR(\delta))^{-1} f(0), \delta = x\partial_x \end{aligned} \tag{82}$$

Hypergeometric functions have many particular special functions as special cases, including many elementary functions, the Bessel functions, the incomplete gamma function, the error function, the elliptic integrals and the classical orthogonal polynomials, because the hypergeometric functions are solutions to the hypergeometric differential equation, which is a fairly general second-order ordinary differential equation.

In a generalization given by Eduard Heine (1821 - 1881) in the late nineteenth century, the ratio of successive terms, instead of being a rational function of n, are considered to be a rational function of q^n

$$f(x) = \sum_{n\geq 0} a_n x^n,\ a_{n+1} = R(q^n)a_n,\ R(n) = \frac{P(\alpha, q^n)}{Q(\beta, q^n)},$$
$$P(\alpha, q^\delta)f(x) = Q(\beta, q^\delta)(f(x) - f(0))/x,$$
$$f(x) - f(0) = xR(q^\delta)f(x), f(x) = (1 - xR(q^\delta))^{-1}f(0), \delta = x\partial_x \tag{83}$$

Another generalization, the elliptic hypergeometric series, are those series where the ratio of terms is an elliptic function (a doubly periodic meromorphic function) of n.
There are a number of new definitions of hypergeometric series, by Aomoto, Gelfand and others; and applications for example to the combinatorics of arranging a number of hyperplanes in complex N-space.

Formal solutions for the the hypergeometric functions (82,83), we put in the fieldtheoretic form,

$$f(x) = G(x)f(0),$$
$$G(x) =< \psi(x)\phi(0) >= \frac{\delta^2 \ln Z}{\delta J(x) \delta I(0)} = (1 - xR)^{-1},$$
$$Z = \int d\psi d\phi e^{-S+I\phi+J\psi} = e^{I(1-xR)^{-1}J},$$
$$S = \int \psi(1 - xR)\phi = \int \psi(Q - xP)\varphi, \ \phi = Q\varphi. \tag{84}$$

When we invent interaction terms, we obtain nontrivial HFT. In terms of the fundamental fields, ψ, φ, we have local field model.

Lauricella Hypergeometric functions (LFs)

For LFs (see, e.g. [Miller,1977]), we find the following formulas [Makhaldiani, 2011]

$$
\begin{aligned}
&F_A(a;b_1,...,b_n;c_1,...,c_n;z_1,...,z_n)=\frac{(a)_{\delta_1+...+\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}{(c_1)_{\delta_1}...(c_n)_{\delta_n}}\ e^{z_1+...+z_n}\\
&=\frac{(a)_{\delta_1+...+\delta_n}}{(a_1)_{\delta_1}...(a_n)_{\delta_n}}\ F(a_1,b_1;c_1;z_1)...F(a_n,b_n;c_n;z_n)\\
&=T^{-1}(a)F^n=\Sigma_{m\geq 0}\frac{(a)_{m_1+...+m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c_1)_{m_1}...(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\ |z_1|+...+|z_n|<1;\\
&F_B(a_1,...,a_n;b_1,...,b_n;c;z_1,...,z_n)=\frac{(a_1)_{\delta_1}...(a_n)_{\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}{(c)_{\delta_1+...+\delta_n}}\ e^{z_1+...+z_n}\\
&=\frac{(c_1)_{\delta_1}...(c_n)_{\delta_n}}{(c)_{\delta_1+...+\delta_n}}\ F(a_1,b_1;c_1;z_1)...F(a_n,b_n;c_n;z_n)=T(c)F^n\\
&=\Sigma_{m\geq 0}\frac{(a_1)_{m_1}...(a_n)_{m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c)_{m_1+...+m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\ |z_1|<1,...,|z_n|<1;
\end{aligned}
\tag{85}
$$

$$F_C(a;b;c_1,...,c_n;z_1,...,z_n) = \frac{(a)_{\delta_1+...+\delta_n}(b)_{\delta_1+...+\delta_n}}{(c_1)_{\delta_1}...(c_n)_{\delta_n}}\ e^{z_1+...+z_n}$$
$$= \frac{(a)_{\delta_1+...+\delta_n}(b)_{\delta_1+...+\delta_n}}{(a_1)_{\delta_1}...(a_n)_{\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}\ F(a_1,b_1;c_1;z_1)...F(a_n,b_n;c_n;z_n)$$
$$= T^{-1}(a)T^{-1}(b)F^n = T^{-1}(b)F_A$$
$$= \Sigma_{m\geq 0}\frac{(a)_{m_1+...+m_n}(b)_{m_1+...+m_n}}{(c_1)_{m_1}...(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\ |z_1|^{1/2}+...+|z_n|^{1/2}<1;$$
$$F_D(a;b_1,...,b_n;c;z_1,...,z_n) = \frac{(a)_{\delta_1+...+\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}{(c)_{\delta_1+...\delta_n}}\ e^{z_1+...+z_n}$$
$$= \frac{(a)_{\delta_1+...+\delta_n}(c_1)_{\delta_1}...(c_n)_{\delta_n}}{(a_1)_{\delta_1}...(a_n)_{\delta_n}(c)_{\delta_1+...\delta_n}}\ F(a_1,b_1;c_1;z_1)...F(a_n,b_n;c_n;z_n)$$
$$= T^{-1}(a)T(c)F^n = T(c)F_A = T^{-1}(a)F_B$$
$$= \Sigma_{m\geq 0}\frac{(a)_{m_1+...+m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c_1)_{m_1}...(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\ |z_1|<1,...,|z_n|<1. \qquad (86)$$

Lomidze B_n function (LBn)

In the paper ([Lomidze, 1994]) the following formula were proposed

$$det[x_j^{i-1}\int_{x_{j-1/x_j}}^1 u^{i-1}(1-u)^{r_j-1}\prod_{k=0,k\neq j}^n(\frac{x_ju-x_k}{x_j-x_k})^{r_k-1}du]/det[x_j^{i-1}]$$
$$=\frac{\Gamma(r_0)\Gamma(r_1)...\Gamma(r_n)}{\Gamma(r_o+r_1+...+r_n)},\ 0=x_0<x_1<x_2<...<x_n,\ n\geq 1. \quad (87)$$

Let us put the formula in the following factorized form

$$LB_n(x,r)\equiv det[x_j^{i-1}\int_{x_{j-1/x_j}}^1 duu^{i+r_0-2}(1-u)^{r_j-1}\prod_{k=1,k\neq j}^n(\frac{x_ju-x_k}{x_j-x_k})^{r_k-1}]$$
$$=detV_n(x)B_n(r),\ V_n(x)=[x_j^{i-1}],\ B_n(r)=\frac{\Gamma(r_0)\Gamma(r_1)...\Gamma(r_n)}{\Gamma(r_0+r_1+...+r_n)} \quad (88)$$

Now, it is enough to proof this formula for general values of x_i and particular values of r_i, e.g., $r_i=1$, and for general values of r_i and particular values of x_i, e.g. $x_i=p^i,\ 1\leq i\leq n$. In the case of $r_i=1$, right hand side of the formula is equal to the Vandermonde determinant divided by $n!$ The left hand side is the determinant of the matrix with elements
$A_{ij}=x_j^{i-1}(1-(x_{j-1}/x_j)^i)/i$
When we calculate determinant of this matrix, from the row i, we factorize $1/i, 2\leq i\leq n$ which gives the $1/n!$ the rest matrix we calculate transforming the matrix to the form of the Vandermonde matrix.

This is the half way of the proof. Let us take the concrete values of $x_i = p^i,\ 1 \leq i \leq n$, where p is positive integer and general complex values for $r_i, 0 \leq i \leq n$, and calculate both sides of the equality. For Vandermonde determinant we find for high values of p the following asymptotic

$$detV = p^N,\ N = \sum_{k=2}^{n} k(k-1) = \frac{n(n^2-1)}{3} \tag{89}$$

The matrix elements are

$$\begin{aligned}
B_{ij} &= x_j^{i-1} \int_{x_{j-1}/x_j}^{1} u^{i+r_0-2}(1-u)^{r_j-1} \prod_{k=1,k\neq j}^{n} (\frac{x_j u - x_k}{x_j - x_k})^{r_k-1} du] \\
&= x_j^{i-1}(\prod_{1\leq k<j} (\frac{x_j}{x_j - x_k})^{r_k-1} \prod_{j<k\leq n} (\frac{x_k}{x_k - x_j})^{r_k-1} \int_{x_{j-1}/x_j}^{1} u^{i+r_0-2}(1-u)^{r_j-1} \\
&\cdot \prod_{1\leq k<j} (u - x_k/x_j)^{r_k-1} \prod_{j<k\leq n} (1 - x_j/x_k u)^{r_k-1} du \\
&= p^{(i-1)j} (\int_0^1 u^{i+r_0-2+\sum_{k=1}^{j-1}(r_k-1)} (1-u)^{r_j-1} du \\
&= p^{(i-1)j} B(i + \sum_{k=0}^{j-1}(r_k - 1), r_j)
\end{aligned} \tag{90}$$

For $n=2$ we have

$$B_{11} = \int_0^1 u^{r_0-1}(1-u)^{r_1-1}du = \frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0+r_1)},$$
$$B_{22} = p^2\int_0^1 u^{r_0+r_1-1}(1-u)^{r_2-1}du = \frac{\Gamma(r_0+r_1)\Gamma(r_2)}{\Gamma(r_0+r_1+r_2)},$$
$$LB_2/V_2 = B_{11}B_{22}/p^2 = \frac{\Gamma(r_0)\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_0+r_1+r_2)} \quad (91)$$

For $n=3$,

$$B_{11} = \int_0^1 u^{r_0-1}(1-u)^{r_1-1} = \frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0+r_1)} = B(r_0,r_1),$$
$$B_{22} = p^2\int_0^1 u^{r_0+r_1-1}(1-u)^{r_2-1} = p^2\frac{\Gamma(r_0+r_1)\Gamma(r_2)}{\Gamma(r_0+r_1+r_2)},$$
$$B_{33} = p^6\int_0^1 u^{r_0+r_1+r_2-1}(1-u)^{r_3-1} = p^6\frac{\Gamma(r_0+r_1+r_2)\Gamma(r_3)}{\Gamma(r_0+r_1+r_2+r_3)}$$
$$LB_3/V_3 = B_{11}B_{22}B_{33}/p^8 = \frac{\Gamma(r_0)\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)}{\Gamma(r_0+r_1+r_2+r_3)} \quad (92)$$

Now it is obvious the last step of the proof [Makhaldiani, 2011]

$$LB_n(x,r) = detV_n(x)B(r_0,r_1)...B(r_0+r_1+...+r_{n-1},r_n)$$
$$= detV_n(x)B_n(r)$$
$$V_n(x) = [x_j^{i-1}],\ B_n(r) = \frac{\Gamma(r_0)\Gamma(r_1)..\Gamma(r_n)}{\Gamma(r_0+r_1+...+r_n)} \quad (93)$$

Note that this proof is based on the factorization assumption (88). The proof without this assumption given by I.R.Lomidze is given in [Lomidze, Makhaldiani, 2012].

Let us consider the following action

$$S = \frac{1}{2}\int\limits_{Q_v} dx\Phi(x)D_x^\alpha\Phi, \;\; v = 1, 2, 3, 5, ..., 29, ..., 137, ... \tag{94}$$

Q_1 is real number field, Q_p, p - prime, are p-adic number fields. In the momentum representation

$$S = \frac{1}{2}\int\limits_{Q_v} du\tilde{\Phi}(-u)|u|_v^\alpha\tilde{\Phi}(u), \;\; \Phi(x) = \int\limits_{Q_v} du\chi_v(ux)\tilde{\Phi}(u),$$
$$D^{-\alpha}\chi_v(ux) = |u|_v^{-\alpha}\chi_v(ux). \tag{95}$$

The statistical sum of the corresponding quantum theory is

$$Z_v = \int d\Phi e^{-\frac{1}{2}\int\Phi D^\alpha\Phi} = det^{-1/2}D^\alpha = (\prod_u |u|_v)^{-\alpha/2}. \tag{96}$$

For (symmetrized, 4-tachyon) Veneziano amplitude we have (see, e.g. [Kaku, 2000])

$$B_s(\alpha,\beta) = B(\alpha,\beta) + B(\beta,\gamma) + B(\gamma,\alpha) = \int_{-\infty}^{\infty} dx|1-x|^{\alpha-1}|x|^{\beta-1}, \quad \alpha+\beta+\gamma = 1 \qquad (97)$$

For the p-adic Veneziano amplitude we take

$$B_p(\alpha,\beta) = \int_{Q_p} dx|1-x|_p^{\alpha-1}|x|_p^{\beta-1} = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \qquad (98)$$

Now we obtain the N-tachyon amplitude using fractal calculus. We consider the dynamics of particle given by multicomponent generalization of the action (110), $\Phi \to x^\mu$.

For the closed trajectory of the particle passing through N points, we have

$$A(x_1, x_2, ..., x_N) = \int dt \int dt_1 ... \int dt_N \delta(t - \Sigma t_n)$$
$$v(x_1, t_1; x_2, t_2) v(x_2, t_2; x_3, t_3) ... v(x_N, t_N; x_1, t_1)$$
$$= \int dx(t) \Pi(\int dt_n \delta(x^\mu(t_n) - x^\mu_n)) exp(-S[x(t)])$$
$$= \int \Pi(dk^\mu_n \chi(k_n x_n)) \tilde{A}(k), \quad (99)$$

where

$$\tilde{A}(k) = \int dx V(k_1) V(k_2) ... V(k_N) exp(-S),$$
$$V(k_n) = \int dt \chi(-k_n x(t)) \quad (100)$$

is vertex function.
Motion equation

$$D^\alpha x^\mu - i \Sigma k^\mu_n \delta(t - t_n) = 0, \quad (101)$$

in the momentum representation

$$|u|^\alpha \tilde{x}^\mu(u) - i \Sigma_n k^\mu_n \chi(-u t_n) = 0 \quad (102)$$

have the solution

$$\tilde{x}^\mu(u) = i \Sigma k^\mu_n \frac{\chi(-u t_n)}{|u|^\alpha}, \quad u \neq 0, \quad (103)$$

the constraint

$$\Sigma_n k_n = 0, \tag{104}$$

and the zero mod $\tilde{x}_n^\mu(0)$, which is arbitrary. Integration in (99) with respect to this zero mod gives the constraint (104). On the solution of the equation (101)

$$x^\mu(t) = iD_t^{-\alpha}\Sigma_n k_n^\mu \delta(t-t_n) = \frac{i}{\Gamma(\alpha)}\Sigma_n k_n^\mu |t-t_n|^{\alpha-1}, \tag{105}$$

the action (110) takes value

$$S = -\frac{1}{\Gamma(\alpha)}\Sigma_{n<m} k_n k_m |t_n - t_m|^{\alpha-1},$$
$$\tilde{A}(k) = \int \Pi_{n=1}^N dt_n exp(-S) \tag{106}$$

In the limit, $\alpha \to 1$, for p-adic case we obtain

$$x^\mu(t) = -i\frac{p-1}{p\ lnp}\Sigma_n k_n^\mu ln|t-t_n|,$$
$$S[x(t)] = \frac{p-1}{p\ lnp}\Sigma_{n<m} k_n k_m\ ln|t_n - t_m|,$$
$$\tilde{A}(k) = \int \Pi_{n=1}^N dt_n \Pi_{n<m}|t_n - t_m|^{\frac{p-1}{p\ lnp}k_n k_m}. \tag{107}$$

Now in the limit $p = q^{-1} \to 1$ we obtain the proper expressions of the real case

$$\begin{aligned} x^\mu(t) = -i\Sigma_n k_n^\mu ln|t - t_n|, \\ S[x(t)] = \Sigma_{n<m} k_n k_m \; ln|t_n - t_m|, \\ \tilde{A}(k) = \int \Pi_{n=1}^N dt_n \Pi_{n<m} |t_n - t_m|^{k_n k_m}. \end{aligned} \tag{108}$$

By fractal calculus and vector generalization of the model (110), fundamental string amplitudes were obtained in [Makhaldiani, 1988].

The ring of (rational) adeles can be defined as the restricted product

$$A_Q = R\prod_p{}' Q_p \tag{109}$$

of all the real numbers and the p-adic completions Q_p, or in other words as the restricted product of all completions of the rationals. In this case the restricted product means that for an adele $a = (a_1, a_2, a_3, a_5,)$ all but a finite number of the a_p are p-adic integers.
The group of invertible elements of the adele ring is the idele group. As a locally compact abelian group, the adeles have a nontrivial translation invariant measure. Similarly, the group of ideles has a nontrivial translation invariant measure.
Let us consider the following action

$$S = \frac{1}{2}\int_{Q_v} dx\Phi(x)D_x^\alpha\Phi,\ v = 1, 2, 3, 5, ... \tag{110}$$

In the momentum representation

$$S = \frac{1}{2} \int\limits_{Q_v} du \tilde{\Phi}(-u) |u|_v^\alpha \tilde{\Phi}(u), \tag{111}$$

where

$$\Phi(x) = \int\limits_{Q_v} du \chi_v(ux) \tilde{\Phi}(u),$$
$$D^{-\alpha} \chi_v(ux) = |u|_v^{-\alpha} \chi_v(ux). \tag{112}$$

The statistical sum of the corresponding quantum theory is

$$Z_v = \int d\Phi e^{-\frac{1}{2}\int \Phi D^\alpha \Phi} = det^{-1/2} D^\alpha = (\prod_u |u|_v)^{-\alpha/2}. \tag{113}$$

Adels $a \in A$ are constructed by real $a_1 \in Q_1$ and p-adic $a_p \in Q_p$ numbers (see e.g. [Gelfand et al, 1966])

$$a = (a_1, a_2, a_3, a_5, ..., a_p, ...), \tag{114}$$

with restriction that $a_p \in Z_p = \{x \in Q_p, |x|_p \leq 1\}$ for all but a finite set F of primes p.
A is a ring with respect to the componentwise addition and multiplication.
A prinsipal adel is a sequence $r = (r, r, ..., r, ...)$, $r \in Q$-rational number.
Norm on adels is defined as

$$|a| = \prod_{p \geq 1} |a_p|_p. \tag{115}$$

Note that the norm on principal adels is trivial.
In the adelic generalization of the model (110),

$$\Phi(x) = \prod_{p \geq 1} \Phi_p(x_p), \quad dx = \prod_{p \geq 1} dx_p, \quad D^\alpha_x = \sum_{p \geq 1} D^\alpha_{x_p}, \tag{116}$$

where by $D^\alpha_{x_1}$ we denote fractal derivative (257), x_1 is real and $|\ |_1$ is real norm.

If

$$\int dx_p |\Phi(x_p)|^2 = 1, \tag{117}$$

then

$$\int dx |\Phi(x)|^2 = 1, \quad S = \sum_{p \geq 1} S_p, \tag{118}$$

so

$$Z = \prod_{p \geq 1} Z_p = \prod_{p \geq 1} (\prod_u |u|_p)^{-\alpha/2} = (\prod_u \prod_{p \geq 1} |u|_p)^{-\alpha/2} = 1,$$
$$\lambda \sim lnZ = 0, \tag{119}$$

if $u \in Q$.

QCD is the theory of the strong interactions with, as only inputs, one mass parameter for each quark species and the value of the QCD coupling constant at some energy or momentum scale in some renormalization scheme. This last free parameter of the theory can be fixed by Λ_{QCD}, the energy scale used as the typical boundary condition for the integration of the Renormdynamic equation for the strong coupling constant. This is the parameter which expresses the scale of strong interactions, the only parameter in the limit of massless quarks. While the evolution of the coupling with the momentum scale is determined by the quantum corrections induced by the renormalization of the bare coupling and can be computed in perturbation theory, the strength itself of the interaction, given at any scale by the value of the renormalized coupling at this scale, or equivalently by Λ_{QCD}, is one of the above mentioned parameters of the theory and has to be taken from experiment.

The RD equations play an important role in our understanding of Quantum Chromodynamics and the strong interactions. The beta function and the quarks mass anomalous dimension are among the most prominent objects for QCD RD equations. The calculation of the one-loop β-function in QCD has lead to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions [Gross,Wilczek,1973, Politzer,1973, 't Hooft, 1972].
The MS-scheme ['t Hooft, 1972 2] belongs to the class of massless schemes where the β-function does not depend on masses of the theory and the first two coefficients of the β-function are scheme-independent.

The Lagrangian of QCD with massive quarks in the covariant gauge

$$\begin{aligned}
L &= -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{q}_n(i\gamma D - m_n)q_n \\
&-\frac{1}{2\xi}(\partial A)^2 + \partial^\mu \bar{c}^a(\partial_\mu c^a + gf^{abc}A^b_\mu c^c) \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \\
(D_\mu)_{kl} &= \delta_{kl}\partial_\mu - igt^a_{kl}A^a_\mu,
\end{aligned} \tag{120}$$

$A^a_\mu, a = 1, ..., N_c^2 - 1$ are gluon; $q_n, n = 1, ..., n_f$ are quark; c^a are ghost fields; ξ is gauge parameter; t^a are generators of fundamental representation and f^{abc} are structure constants of the Lie algebra

$$[t^a, t^b] = if^{abc}t^c, \tag{121}$$

we will consider an arbitrary compact semi-simple Lie group G. For QCD, $G = SU(N_c), N_c = 3$.

The RD equation for the coupling constant is

$$\dot{a} = \beta(a) = \beta_2 a^2 + \beta_3 a^3 + \beta_4 a^4 + \beta_5 a^5 + O(a^6),$$
$$a = \frac{\alpha_s}{4\pi} = (\frac{g}{4\pi})^2,$$
$$\int_{a_0}^{a} \frac{da}{\beta(a)} = t - t_0 = \ln \frac{\mu}{\mu_0}, \qquad (122)$$

μ is the 't Hooft unit of mass, the renormalization point in the MS-scheme. To calculate the β-function we need to calculate the renormalization constant Z of the coupling constant, $a_b = Za$, where a_b is the bare (unrenormalized) charge.

The expression of the β-function can be obtained in the following way

$$0 = d(a_b\mu^{2\varepsilon})/dt = \mu^{2\varepsilon}(\varepsilon Za + \frac{\partial(Za)}{\partial a}\frac{da}{dt})$$
$$\Rightarrow \frac{da}{dt} = \beta(a,\varepsilon) = \frac{-\varepsilon Za}{\frac{\partial(Za)}{\partial a}} = -\varepsilon a + \beta(a),$$
$$\beta(a) = a\frac{d}{da}(aZ_1) \quad (123)$$

where

$$\beta(a,\varepsilon) = \frac{D-4}{2}a + \beta(a),\ D = 4-2\varepsilon \quad (124)$$

is $D-$dimensional $\beta-$function and Z_1 is the residue of the first pole in ε expansion

$$Z(a,\varepsilon) = 1 + Z_1\varepsilon^{-1} + ... + Z_n\varepsilon^{-n} + ... \quad (125)$$

Since Z does not depend explicitly on μ, the β-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ.

For quark anomalous dimension, RD equation is

$$\dot{b} = \gamma(a) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \gamma_4 a^4 + O(a^5),$$
$$b = \ln m_q,$$
$$b(t) = b_0 + \int_{t_0}^{t} dt\gamma(a(t)) = b_0 + \int_{a_0}^{a} da\gamma(a)/\beta(a). \qquad (126)$$

To calculate the quark mass anomalous dimension $\gamma(g)$ we need to calculate the renormalization constant Z_m of the quark mass $m_b = Z_m m,\ m_b$ is the bare (unrenormalized) quark mass. Than we find the function $\gamma(g)$ in the following way

$$0 = \dot{m}_b = \dot{Z}_m m + Z_m \dot{m} = Z_m m((\ln Z_m)^{\cdot} + (\ln m)^{\cdot})$$
$$\Rightarrow \gamma(a) = -\frac{d\ln Z_m}{dt}$$
$$= -\frac{d\ln Z_m}{da}\frac{da}{dt} = -\frac{d\ln Z_m}{da}(-\varepsilon a + \beta(a)) = a\frac{dZ_{m1}}{da}, \qquad (127)$$

where RD equation in $D-$dimension is

$$\dot{a} = -\varepsilon a + \beta(a) = \beta_1 a + \beta_2 a^2 + ... \qquad (128)$$

and Z_{m1} is the coefficient of the first pole in the $\varepsilon-$expantion of the Z_m in MS-scheme

$$Z_m(\varepsilon, g) = 1 + Z_{m1}(g)\varepsilon^{-1} + Z_{m2}(g)\varepsilon^{-2} + ... \qquad (129)$$

Since Z_m does not depend explicitly on μ and m, the γ_m-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ.

RD equation,

$$\dot{a} = \beta_1 a + \beta_2 a^2 + ... \tag{130}$$

can be reparametrized,

$$a(t) = f(A(t)) = A + f_2 A^2 + ... + f_n A^n + ... = \sum_{n \geq 1} f_n A^n,$$

$$\dot{A} = b_1 A + b_2 A^2 + ... = \sum_{n \geq 1} b_n A^n, \tag{131}$$

$$\begin{aligned}
&\dot{a} = \dot{A} f'(A) = (b_1 A + b_2 A^2 + ...)(1 + 2 f_2 A + ... + n f_n A^{n-1} + ...) \\
&= \beta_1 (A + f_2 A^2 + ... + f_n A^n + ...) + \beta_2 (A^2 + 2 f_2 A^3 + ...) + ... \\
&+ \beta_n (A^n + n f_2 A^{n+1} + ...) + ... \\
&= \beta_1 A + (\beta_2 + \beta_1 f_2) A^2 + (\beta_3 + 2\beta_2 f_2 + \beta_1 f_3) A^3 + \\
&... + (\beta_n + (n-1)\beta_{n-1} f_2 + ... + \beta_1 f_n) A^n + ... \\
&= \sum_{n, n_1, n_2 \geq 1} A^n b_{n_1} n_2 f_{n_2} \delta_{n, n_1 + n_2 - 1} \\
&= \sum_{n, m \geq 1; m_1, ..., m_k \geq 0} A^n \beta_m f_1^{m_1} ... f_k^{m_k} f(n, m, m_1, ..., m_k),
\end{aligned}$$

$$f(n, m, m_1, ..., m_k) = \frac{m!}{m_1! ... m_k!} \delta_{n, m_1 + 2m_2 + ... + k m_k} \delta_{m, m_1 + m_2 + ... + m_k}, \tag{132}$$

$$
\begin{aligned}
&b_1 = \beta_1,\ b_2 = \beta_2 + f_2\beta_1 - 2f_2 b_1 = \beta_2 - f_2\beta_1, \\
&b_3 = \beta_3 + 2f_2\beta_2 + f_3\beta_1 - 2f_2 b_2 - 3f_3 b_1 = \beta_3 + 2(f_2^2 - f_3)\beta_1, \\
&b_4 = \beta_4 + 3f_2\beta_3 + f_2^2\beta_2 + 2f_3\beta_2 - 3f_4 b_1 - 3f_3 b_2 - 2f_2 b_3, ... \\
&b_n = \beta_n + ... + \beta_1 f_n - 2f_2 b_{n-1} - ... - n f_n b_1, ...
\end{aligned} \tag{133}
$$

so, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but β_1.
We can fix any higher coefficient with zero value, if we take

$$
f_2 = \frac{\beta_2}{\beta_1},\ f_3 = \frac{\beta_3}{2\beta_1} + f_2^2,\ ...\ , f_n = \frac{\beta_n + ...}{(n-1)\beta_1}, ... \tag{134}
$$

In this case we have exact classical dynamics in the (external) space-time and simple scale dynamics,

$$
\begin{aligned}
&g = (\mu/\mu_0)^{-2\varepsilon} g_0 = e^{-2\varepsilon\tau} g_0; \\
&\varphi(\tau, t, x) = e^{-(D-2)/2\tau} \varphi_0(t, x), \\
&\psi(\tau, t, x) = e^{-(D-1)/2\tau} \psi_0(t, x)
\end{aligned} \tag{135}
$$

We will consider in applications the case when only one of higher coefficient is nonzero.
In the critical dimension of space-time, $\beta_1 = 0$, and we can change by reparametrization any coefficient but β_2 and β_3.

From the relations (133), in the critical dimenshion ($\beta_1 = 0$), we find that, we can define the minimal form of the RD equation

$$\dot{A} = \beta_2 A^2 + \beta_3 A^3, \tag{136}$$

e.g. $b_4 = 0$ when

$$f_3 = \frac{\beta_4}{\beta_2} + \frac{\beta_3}{\beta_2} f_2 + f_2^2, \tag{137}$$

f_2 remains arbitrary and we can make choice $f_2 = 0$. We can solve (136) as implicit function,

$$u^{\beta_3/\beta_2} e^{-u} = ce^{\beta_2 t},\ u = \frac{1}{A} + \frac{\beta_3}{\beta_2} \tag{138}$$

than, as in the noncritical case, explicit solution will be given by reparametrization representation. If we know somehow the coefficients β_n, e.g. for first several exact and for others asymptotic values (see e.g. [Kazakov,Shirkov,1980]) than we can construct reparametrization function (131) and find the dynamics of the running coupling constant. This is similar to the action-angular canonical transformation of the analytic mechanics (see e.g. [Faddeev, Takhtajan, 1990]). Statement: The series for a is p-adically convergent, when β_n and A are rational numbers.

Let us take the the anomalous dimension of some quantity

$$\gamma(a) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + ... \tag{139}$$

and make reparametrization

$$a = f(A) = A + f_2 A^2 + f_3 A^3 + ... \tag{140}$$

$$\gamma(a) = \gamma_1(A + f_2 A^2 + f_3 A^3 + ...) + \gamma_2(A^2 + 2f_2 A^3 + ...) + \gamma_3(A^3 + ...)$$
$$= \Gamma_1 A + \Gamma_2 A^2 + \Gamma_3 A^3 + ...$$
$$\Gamma_1 = \gamma_1,\ \Gamma_2 = \gamma_2 + \gamma_1 f_2,\ \Gamma_3 = \gamma_3 + 2\gamma_2 f_2 + \gamma_1 f_3, ...$$

When $\gamma_1 \neq 0$, we can take $\Gamma_n = 0,\ n \geq 2$, if we define f_n as

$$f_2 = -\frac{\gamma_2}{\gamma_1},\ f_3 = -\frac{\gamma_3 + 2\gamma_2 f_2}{\gamma_1} = -\frac{\gamma_3 - 2\gamma_2^2/\gamma_1}{\gamma_1}, ... \tag{142}$$

So, we get the exact value for the anomalous dimension

$$\gamma(A) = \gamma_1 A = \gamma_1 f^{-1}(a) = \gamma_1(a + \gamma_2/\gamma_1 a^2 + \gamma_3/\gamma_1 a^3 + ...) \tag{143}$$

We will call RDF functions $g_n = f_n(t)$, which are solutions of the RD motion equations

$$\dot{g}_n = \beta_n(g), 1 \leq n \leq N. \tag{144}$$

In the simplest case of one coupling constant, the function $g = f(t)$, is constant $g = g_c$ when $\beta(g_c) = 0$, or is invertible (monotone). Indeed,

$$\dot{g} = f'(t) = f'(f^{-1}(g)) = \beta(g). \tag{145}$$

Each monotone interval ends by UV and IR fixed points and describes corresponding phase of the system.
Note that, the simplest case of the classical dynamics, the hamiltonian system with one degree of freedom, is already two dimensional, so we have not an analog of one charge renormdynamics. Than the regular hamiltonian systems of the classical mechanics are defined on the even dimensional phase space, so there is not an analog of the three dimensional renormdynamics for the coupling constants of the SM.

Based on real experiments and computer simulations, quantum gauge theory in four dimensions is believed to have a mass gap. This is one of the most fundamental facts that makes the Universe the way it is.
In the lattice (gauge) theory approach to the renormdynamics (see, e.g. [Makhaldiani, 1986]), recently running coupling constant dynamics were calculated for $SU(3)$ Yang-Mills model [Bogolubsky et al,2009]. The result is in agreement with perturbative calculations at small scales; at an intermediate scale the coupling constant reaches its maximum($\simeq 1.$); than decrease. So, at the maximum, we may have nontrivial zero of the $\beta-$function, which corresponds to the conformal invariance of the gluodynamics at this point. Beyond this point we have another phase, strong coupling phase with decreasing coupling constant similar (identical?!) to the abelian (monopole?) theory.

Note that, in the case of the two coupling constants,

$$\dot{g}_1 = \beta_1(g_1, g_2),$$
$$\dot{g}_2 = \beta_2(g_1, g_2), \quad (146)$$

we can reformulate RD as

$$g_1 \equiv g; g_2 = f_2(t) \equiv \tau,$$
$$\frac{dg_1}{dg_2} = \frac{dg}{d\tau} \equiv \dot{g} = \beta(g, \tau) = \frac{\beta_1(g, \tau)}{\beta_2(g, \tau)} \quad (147)$$

and RDF must fulfil corresponding restrictions. E.g. if

$$g_1 = f_1(t) = g = f(\tau) = f(f_2(t)), g_2 = f_2(t) = \tau \quad (148)$$

So, if we approximate the form of the curve near maximum as

$$a(t) = a_c - b|t - t_c|^n, \quad (149)$$

for the $\beta-$function we obtain

$$\dot{a} = \beta(a, t) = sign(t_c - t)bn(\frac{a_c - a}{b})^{\frac{n-1}{n}}. \quad (150)$$

Of course this is not usual $\beta-$function, function of a only. It depends also on t. For $t > t_c$ we have perturbative phase. For $n > 1, \ \beta(a_c, t) = 0$. Explicit dependence on time variable in one coupling case indicates on implicit two coupling case.

We have seen that the quantitative values and qualitative content of the given field theory depends on the scale (parameter, e.g. $\mu-$renormalization point, $g = g(\mu),\ A = A(\mu)$). In QCD e.g. the effective action have the following form

$$S(\mu) = \frac{1}{g^2(\mu)} \int d^D x \mathcal{L}(A(\mu)), \tag{151}$$

so variation with respect the change of scale gives

$$\delta S = -2\frac{\beta(g)}{g^3}\delta g S + \frac{1}{g^2}\int d^D x \frac{\delta \mathcal{L}}{\delta A}\delta A \tag{152}$$

and the following two statements are equivalent,

$$\delta S = 0,\ \beta(g) = 0 \Leftrightarrow \delta S = 0,\ \frac{\delta \mathcal{L}}{\delta A} = 0 \tag{153}$$

So, from renorminvariance of the effective action, $\delta S = 0$, follows that at the conformal symmetric points, fixed points of RD, $(\beta(g) = 0)$, the motion equations for fields are satisfied. Generalization for the several coupling constants and other models is obvious. The solutions of the motion equations are selfsimilar, their are generally fractals. In string theory, the connection between conformal invariance of the effective theory on the parametric world sheet and the motion equations of the fields on the embedding space is well known [Green, Schwarz, Witten,1987]. More recent topic in this direction is AdS/CFT Duality [Maldacena, 1998].

Based on real experiments and computer simulations, quantum gauge theory in four dimensions is believed to have a mass gap. This is one of the most fundamental facts that makes the Universe the way it is.
The AdS/CFT duality provides a gravity description in a (d + 1)-dimensional AdS space-time in terms of a flat d-dimensional conformally-invariant quantum field theory defined at the AdS asymptotic boundary [Maldacena, 1998],
[Gubser,Klebanov,Polyakov, 1998],[Witten, 1998]. Thus, in principle, one can compute physical observables in a strongly coupled gauge theory in terms of a classical gravity theory. The β-function for the nonperturbative effective coupling obtained from the LF holographic mapping in a positive dilaton modified AdS background is [Brodsky, de Tèramond, Deur, 2010]

$$\beta(\alpha_{AdS}) = \frac{d\alpha_{AdS}}{\ln Q^2} = -\frac{Q^2}{4k^2}\alpha_{AdS}(Q^2)$$
$$= \alpha_{AdS}(Q^2)\ln\frac{\alpha_{AdS}(Q^2)}{\alpha(0)} \leq 0 \quad (154)$$

where the physical QCD running coupling in its nonperturbative domain is

$$\alpha_{AdS}(Q^2) = \alpha(0)e^{-Q^2/4k^2} \quad (155)$$

For the QCD running coupling [Diakonov, 2003]

$$\alpha(q^2) = \frac{4\pi}{9\ln(\frac{q^2+m_g^2}{\Lambda^2})} \tag{156}$$

where $m_g = 0.88GeV,\ \Lambda = 0.28GeV$, the $\beta-$function of renormdynamics is

$$\beta(\alpha) = -\frac{\alpha^2}{k}(1 - c\exp(-\frac{k}{\alpha})) = -\frac{\alpha^2}{k} + \frac{c\alpha^2}{k}\exp(-\frac{k}{\alpha}),$$
$$k = \frac{4\pi}{9} = 1.40,\ c = \frac{m_g^2}{\Lambda^2} = (3.143)^2 = 9.88 \tag{157}$$

for nontrivial (IR) fixed point we have

$$\alpha_{IR} = \frac{k}{\ln c} = 0.61 \tag{158}$$

For $\alpha(0) = 2$, we predict the gluon mass as

$$m_g = \Lambda e^{\frac{k}{2\alpha(0)}} = 1.42\Lambda = m_N/3,\ \Lambda = 220MeV. \tag{159}$$

The ghost-gluon interaction in Landau gauge has been determined either from DSEs [Zwanziger, 2002],[Lerche,von Smekal, 2002], or the Exact Renormalization Group Equations (ERGEs)
[Pawlowski et al, 2004],[Fischer,Gies, 2004] and yield an IR fixed point

$$\alpha(0) = \frac{2\pi}{3N_c}\frac{\Gamma(3-2k)\Gamma(3+k)\Gamma(1+k)}{\Gamma(2-k)^2\Gamma(2k)} = \frac{8.9115}{N_c} = 2.970,$$
$$N_c = 3,\ k = (93-\sqrt{1201})/98 = 0.5954 \qquad (160)$$

Note that, from this formula for $k = 0.6036$ we have $\alpha(0) = 3$ and for $k = 0.36$ we have $\alpha(0) = 2$.

While it has been well established in the perturbative regime at high energies, QCD still lacks a comprehensive solution at low and intermediate energies, even 40 years after its invention. In order to deal with the wealth of non-perturbative phenomena, various approaches are followed with limited validity and applicability. This is especially also true for lattice QCD, various functional methods, or chiral perturbation theory, to name only a few. In neither one of these approaches the full dynamical content of QCD can yet be included. Basically, the difficulties are associated with a relativistically covariant treatment of confinement and the spontaneous breaking of chiral symmetry, the latter being a well-established property of QCD at low and intermediate energies. As a result, most hadron reactions, like resonance excitations, strong and electroweak decays etc., are nowadays only amenable to models of QCD. Most famous is the constituent-quark model (CQM), which essentially relies on a limited number of effective degrees of freedom with the aim of encoding the essential features of low- and intermediate-energy QCD.
The CQM has a long history, and it has made important contributions to the understanding of many hadron properties, think only of the fact that the systematization of hadrons in the standard particle-data base follows the valence-quark picture.

It was noted [Voloshin, Ter-Martyrosian,1984]that parton densities given by the following solution

$$M_2(Q^2) = \frac{3}{25} + \frac{2}{3}\omega^{-32/81} + \frac{16}{75}\omega^{-50/81},$$
$$\bar{M}_2(Q^2) = M_2^s(Q^2) = \frac{3}{25} - \frac{1}{3}\omega^{-32/81} + \frac{16}{75}\omega^{-50/81},$$
$$M_2^G(Q^2) = \frac{16}{25}(1 - \omega^{-50/81}),$$
$$\omega = \frac{\alpha_s(m^2)}{\alpha_s(Q^2)},\ Q^2 \in (5, 20)GeV^2,\ b = 9,\ \alpha_s(Q^2) \simeq 0.2 \quad (161)$$

of the Altarelli-Parisi equation

$$\dot{M} = AM, \ \dot{M} = Q^2 \frac{dM}{dQ^2}, \ a = (\frac{g}{4\pi})^2,$$
$$M^T = (M_2, \bar{M}_2, M_2^s, M_2^G),$$
$$M_2 = \int_0^1 dxx(u(x) + d(x)), \ \bar{M}_2 = \int_0^1 dxx(\bar{u}(x) + \bar{d}(x)),$$
$$M_2^s = \int_0^1 dxx(s(x) + \bar{s}(x)), \ M_2^G = \int_0^1 dxxG(x),$$
$$A = -a(Q^2) \begin{pmatrix} 32/9 & 0 & 0 & -2/3 \\ 0 & 32/9 & 0 & -2/3 \\ 0 & 0 & 32/9 & -2/3 \\ -32/9 & -32/9 & -32/9 & 2 \end{pmatrix}, \quad (162)$$

with the following "valence quark" initial condition at a scale m

$$M_2(m^2) = 1, \ \bar{M}_2(m^2) = M_2^s(m^2) = M_2^G(m^2) = 0, \ \alpha_s(m^2) = 2, \quad (163)$$

gives the experimental values

$$M_2 = 0.44, \ \bar{M}_2 = M_2^s = 0.04, \ M_2^G = 0.48 \quad (164)$$

So, for valence quark VQCD, $\alpha_s(m^2) = 2$. We have seen, that for $\pi\rho N$ model $\alpha_{\pi\rho N} = 3$, and for πN model $\alpha_{\pi N} = 13$. It is nice that $\alpha_s^2 + \alpha_{\pi\rho N}^2 = \alpha_{\pi N}$. This relation can be seen, e.g., by considering pion propagator in the low energy πN model and in superposition of higher energy VQCD and $\pi\rho N$ models.
Note that $g^2 = 25,\ g = 5$, corresponds to the

$$\alpha_g = \frac{g^2}{4\pi} = 1.989 \simeq 2 \qquad (165)$$

To $\alpha_s = 2$ corresponds

$$g = \sqrt{4\pi\alpha_s} = \sqrt{8\pi} = 5.013 = 5+ \qquad (166)$$

In the relativistic string-gauge field duality [Maldacena, 1998] (see review [Aharony et al, 2000]), the string coupling constant g_s and the gauge field fine structure constant α_s are related: $g_s = \alpha_s$. The statement that the later is (prime) integer means (prime) integer quantization of the string coupling constant.

There are many motivations to think that the SM is not the ultimate and complete theory of Nature, among which the naturalness argument plays a predominant role. The instability of the Higgs mass with respect to radiative corrections requires in fact an incredible high level of fine tuning in the precision of their cancellation in the SM in order to have an Higgs mass at the EW scale. Beside the supersymmetric solution to this problem, another possibility is to postulate the Higgs boson as a composite state arising as a bound state from a strongly interacting sector at the TeV scale [Kaplan, Georgi 1984]. Being composite the Higgs will be insensitive to radiative corrections above the composite scale.
With the discovery of the Higgs particle with mass $125\ GeV$, a nice number $m_W/m_H \simeq 2/3$ appear, which, at least for me, indicates for composed nature of W and H, with a same mass of about $40\ GeV$ two and three valence constituents correspondingly. The fermion constituents ψ_n^a of W and scalar constituents φ_n^a of H compose scalar super multiplet (φ_n^a, ψ_n^a) with a flavor index n and color index a.

Phenomenological approach to the nonrelativistic potential-model study of Υ and ψ spectra leads to a static Coulombic Power-law potential of the form

$$V(r) = a(r)r^{2-d(r)} \sim \begin{array}{c} 1/r,\ r \sim 0.1fm \\ r,\ r \sim 1.fm \end{array} \qquad (167)$$

E.g. in the case of the Υ and small r

$$V(r) = \frac{4}{3}\frac{\alpha_s}{r},\ \alpha_s = \frac{2\pi}{b\ln r\Lambda},\ b = 9. \qquad (168)$$

This behavior corresponds not only to the running fine structure constant but also to the running space dimension. Confinement-the point-like hadrons on the scales higher than hadronic, corresponds to the zero dimensional space for hadron constituents.
RD equations of QCD beyond the critical dimention has explicit dependence on the space dimension. When the dimension becomes running we should consider two dimensional renormdinamics

$$\begin{array}{l} \dot{a_1} = \beta_1(a_1, a_2),\ a_1 = a, \\ \dot{a_2} = \beta_2(a_1, a_2),\ a_2 = d \end{array} \qquad (169)$$

If we have a solution $x_n = x_{0n}$ (a state) of the following system of motion equations (of the corresponding dynamical system)

$$\dot{x}_n = f_n(x),\ 1 \leq n \leq N, \tag{170}$$

we can consider the question of stability of the solution, the existence of the solutions of the type $x_n = x_{0n} + g_n$, for small values of g_n. If there are solutions with rising g_n, of the corresponding motion equations

$$\begin{aligned} &\dot{g}_n = \beta_n(g), \\ &\beta_n(g) = f_n(x_0 + g) - f_n(x_0) = \beta_{1nm} g_m + \beta_{2nmk} g_m g_k + ..., \\ &\beta_{kn...m} = f^{(n...m)}(x_0) \end{aligned} \tag{171}$$

we say that the solution x_{0n} is not stable.

The linear approximation, we transform into diagonal form,

$$\dot{g}_n = \beta_{1nm} g_m,\ h_n = A_{nm} g_m,$$
$$\dot{h}_n = \lambda_n h_n,\ \lambda_n \delta_{nm} = (A\beta_1 A^{-1})_{nm}, \tag{172}$$

if all of the λ_n are purely imaginary $\lambda_n = i\omega_n$, we have stable solution (in the linear approximation): small deviations remain small. If real parts of all λ_n are negative, we have asymptotic stability: deviations decrease. If some λ_n are zero, we have undefined case. In regular case, when the matrix β_1 has inverse, by reparametrization trick we can construct the formal solution of the nonlinear equation for g_n, and try to investigate its convergence properties.

In the case of several integrals of motion, $H_n,\ 1 \leq n \leq N$, we can formulate Renormdynamics as Nambu - Poisson dynamics (see e.g. [Makhaldiani, 2007])

$$\dot{\varphi}(x) = [\varphi(x), H_1, H_2, ..., H_N], \tag{173}$$

where φ is an observable as a function of the coupling constants $x_m,\ 1 \leq m \leq M$.
In the case of Standard model [Weinberg,1995], we have three coupling constants, $M = 3$.

The renormdynamic motion equations

$$\dot{g}_n = \beta_n(g),\ 1 \le n \le N \tag{174}$$

can be presented as nonlinear part of a hamiltonian system with linear part

$$\dot{\Psi}_n = -\frac{\partial \beta_m}{\partial g_n}\Psi_m, \tag{175}$$

hamiltonian and canonical Poisson bracket as

$$H = \sum_{n=1}^{N} \beta(g)_n \Psi_n,\ \{g_n, \Psi_m\} = \delta_{nm} \tag{176}$$

In this extended version, we can define optimal control theory approach [Pontryagin, 1983] to the unified field theories. We can start from the unified value of the coupling constant, e.g. $\alpha^{-1}(M) = 29.0...$ at the scale of unification M, put the aim to reach the SM scale with values of the coupling constants measured in experiments, and find optimal threshold corrections to the RD coefficients [Makhaldiani, 2010].

For connected vertex functions Γ_n, (11)

$$\Gamma_n(x_1, x_2, ..., x_n; g, m, \mu) = Z^{n/2}(\mu)\Gamma_{0n}(x_1, x_2, ..., x_n; g_0, m_0),$$
$$(D - \frac{n}{2}\gamma)\Gamma_n(x; g, m, \mu) = 0; \quad (177)$$

For effective action S_q,

$$(D - \frac{1}{2}\gamma \int dx \phi(x) \frac{\delta}{\delta\phi(x)})S_q(\phi) = 0,$$
$$(D - \frac{1}{2}\gamma\phi\frac{\partial}{\partial\phi})V(\phi) = 0, \ V(\phi) = S_q(\phi(x))|_{\phi(x)=\phi=const}, \quad (178)$$

where $V(\phi)$ is effective potential.
For the effective potential in the RD (conformal) fixed point, $\gamma(g) = \gamma(g_c) \equiv \gamma_c$ we have the following wave equation and corresponding (auto model) solution

$$(\partial_t - \frac{\gamma_c}{2}\partial_z)V = 0,$$
$$V(\phi, \mu) = f(z + vt) = F(\frac{\phi}{\mu^v}), \ t = \ln\frac{\mu}{\mu_0}, z = \ln\frac{\phi}{\phi_0}, v = \frac{\gamma_c}{2} \quad (179)$$

The fundamental quark and gluon degrees of freedom are the relevant ones at high temperatures and/or densities. Since these degrees of freedom are confined in the low temperature and density regime there must be a quark and/or gluon (de)confinement phase transition.
It is difficult to describe the phase transition because there is not known a local parameter which can be linked to confinement. We consider the fractal dimension of the hadronic/quark-gluon space as order parameter of (de)confinement phase transition. It has value less than 3 in the abelian, hadronic, phase, and more than 3, in nonabelian, quark-gluon, phase.

Perturbation theory results for QCD (QED) give negative (positive) $\beta-$function, in one loop approximation

$$\dot{a} = \beta_2 a^2,$$
$$QCD : \beta_2 = (\frac{n_f}{6} - \frac{11}{4}),$$
$$QED : \beta_2 = \frac{1}{3} \quad (180)$$

So, running coupling constant vanishes at higher (low) energy. For QCD this property named as asymptotic freedom gives the scaling behavior of observable quantities in good agreement with experimental data. Small value of the coupling constant may describe small deviation from the scaling. Infrared zero value of the QED coupling constant contradicts with experiments. Small value of the coupling constant equal to the observable value of the fine structure constant $\alpha^{-1} = 137.036$, in the infrared (low energy) limit, will be good solution of the zero-charge problem. For this, we will consider the QCD (QED) beyond the critical dimension of the space-time.

Corresponding $\beta-$function

$$\beta(a,\varepsilon) = -\varepsilon a + \beta(a), \tag{181}$$

has stable ultraviolet (infrared) fixed point for negative (positive) value of ε,

$$\varepsilon = \beta(a)/a. \tag{182}$$

According to the LEP and Tevatron data, the standard model coupling constants at the Z-boson mass scale take the values (see, e.g. [D.I.Kazakov, 2004])

$$\begin{array}{l}\alpha_1(m_Z) = 0.017,\ \alpha_1(m_Z)^{-1} = 58.8 \\ \alpha_2(m_Z) = 0.034,\ \alpha_2(m_Z)^{-1} = 29.4 \\ \alpha_3(m_Z) = 0.118,\ \alpha_3(m_Z)^{-1} = 8.47\end{array} \tag{183}$$

Our aim is to consider RD equation in critical dimension for weak interaction part of the SM ($\varepsilon_2 = 0$); RD equations for the electromagnetic and strong interaction parts beyond critical dimension ($\varepsilon_1, \varepsilon_3 \neq 0$); reach unification (equality) of the three couplings at the TeV scale in the point $\alpha_u^{-1} = 31.0$

The solution of the one loop RD equation beyond critical dimension

$$\dot{a} = -\varepsilon a + ka^2,$$
$$a = \frac{\alpha}{4\pi} = (\frac{g}{4\pi})^2,\ t = \ln\frac{Q^2}{m_Z^2}, \tag{184}$$

is

$$a_n(t)^{-1} = \frac{k_n}{\varepsilon} + c_n e^{\varepsilon_n t},\ n = 1, 3$$
$$c_n = a_n(m_Z)^{-1} - \frac{k_n}{\varepsilon_n},$$
$$k_n = (\frac{41}{10}, -7). \tag{185}$$

The solution of the RD equation in critical dimension

$$\dot{a}_2 = k_2 a_2^2,\ \ k_2 = -\frac{19}{6} \tag{186}$$

is

$$a_2^{-1}(t) = a_2^{-1}(m_Z) + k_2 t \tag{187}$$

From the last expression, having unification value, $\alpha_2^{-1}(t_u) = \alpha_u^{-1} = 31.0$ we define the unification scale

$$\begin{aligned} & t_u = (a_2^{-1}(t_u) - a_2^{-1}(m_Z))/k_2 \\ & = 4\pi \times 1.6 \times \frac{6}{19} = 6.35, \\ & Q_u = 23.9 m_Z = 2182 GeV, \\ & m_Z = 91.2 GeV \end{aligned} \tag{188}$$

Solution of the RD equation beyond the critical dimension for electrodynamic constant,

$$\dot{a} = -\varepsilon a + ba^2,\ b = \frac{41}{10}, \tag{189}$$

is

$$a^{-1}(t) = \frac{b}{\varepsilon} + (a^{-1}(m_Z) - \frac{b}{\varepsilon})e^{\varepsilon t} \tag{190}$$

The condition of the unification

$$(b\varepsilon^{-1} - a^{-1}(t_u)) = (b\varepsilon^{-1} - a^{-1}(m_Z))e^{\varepsilon t_u} \tag{191}$$

defines the value $\varepsilon_1 = -0.093$ Unification takes place in dimension $d = 4 - 2\varepsilon_1 = 4.186$

For the strong coupling constant beyond the critical dimension,

$$\dot{a} = -\varepsilon a - ba^2, \ b = 7, \tag{192}$$

the solution is

$$a^{-1}(t) = -\frac{b}{\varepsilon} + (\frac{b}{\varepsilon} + a^{-1}(m_Z))e^{t\varepsilon}, \tag{193}$$

the unification condition

$$(b\varepsilon^{-1} + a^{-1}(t_u)) = (b\varepsilon^{-1} + a^{-1}(m_Z))e^{\varepsilon t_u} \tag{194}$$

defines $\varepsilon = 0.168$ Unification takes place in the dimension
$d = 4 - 2\varepsilon = 3.66$

Let us consider unification at the point $\alpha^{-1}(t_u) = 29.0$, the low energy unification,

$$
\begin{aligned}
t_{ul} &= (\alpha_2^{-1}(t_{ul}) - a_2^{-1}(m_Z))/k_2 \\
&= -4\pi \times 0.4 \times \frac{6}{19} = -1.59, \\
Q_{ul} &= 0.45 m_Z = 41.2 GeV
\end{aligned}
\tag{195}
$$

For electrodynamic case unification condition

$$
\frac{41}{10} - 4\pi 29\varepsilon = (\frac{41}{10} - 4\pi 58.8\varepsilon)e^{-1.59\varepsilon}, \tag{196}
$$

gives the values $\varepsilon_1 = 0.453,\ d_{el} = 3.09 = 2.09 + 1$ dimensional space-time. For strong coupling constant unification condition

$$
7 + 4\pi\varepsilon \times 29 = (7 + 4\pi\varepsilon \times 8.47)e^{-1.59\varepsilon} \tag{197}
$$

gives $\varepsilon_3 = -0.8121,\ d_{sl} = 5.624$

The low energy value of the QED $\alpha^{-1} = 137.036...$
Let us find the scale at which $\alpha^{-1} = 137$ if

$$\alpha^{-1}(m_Z) = \frac{5}{3\cos^2\theta_W}\alpha_1^{-1}(m_Z) = 128.978 \pm 0.027 \simeq 129,$$
$$\sin^2\theta_W = 0.23146 \pm 0.00017 \simeq 0.2315,$$
$$\alpha_1^{-1}(m_Z) = 58.8,$$
$$\alpha^{-1}(m_Z) = \frac{5}{3 \times 0.7685} \times 58.8 = 127.52 \simeq 128 \tag{198}$$

Now take one loop RD evolution to the 137,

$$t_l = (a_1^{-1}(t_l) - a_1^{-1}(m_Z))/k_1$$
$$= -4\pi \times 8. \times \frac{10}{41} = -24.5,$$
$$Q_l \simeq 5 \times 10^{-6} m_Z \simeq 5 \times 10^{-4} m_p \simeq m_e \tag{199}$$

Theoretical equations describing the physical world deal with dimensionless quantities and their solutions depend on dimensionless fundamental parameters, like $\alpha^{-1} \simeq 137$. But experiments, from which these theories are extracted and by which they could be tested, involve measurements, i.e. comparisons with standard dimensionful scales. Without standard dimensionful units and hence without certain conventions physics is unthinkable.
According to the high school physics, there are three basic quantities in Nature: Length, Mass and Time. All other quantities, such as electric charge or temperature, occupied a lesser status since they could all be re-expressed in terms of these basic three. As a result, there are three basic units: centimeter (cm), gram (g) and second (s), reflected in the three-letter name "CGS" system (or perhaps meter, kilogram and second in the alternative, but still three-letter, "MKS" system).

In quantum mechanics, there is a minimum quantum of action given by Planck's constant $\hbar$; in special relativity there is a maximum velocity given by the velocity of light c; in classical gravity the strength of the force between two objects is determined by Newton's constant of gravitation G. In terms of length, time and mass their dimensions are

$$
\begin{aligned}
[c] &= LT^{-1}, \\
[\hbar] &= L^2T^{-1}M \\
[G] &= L^3T^{-2}M^{-1}
\end{aligned}
\tag{200}
$$

Max Planck identified a century ago three basic units, the Planck length l_p, the Planck time t_p and Planck mass m_p:

$$l_p = \sqrt{\frac{G\hbar}{c^3}} = 1.616 \times 10^{-35} m$$
$$t_p = \sqrt{\frac{G\hbar}{c^5}} = 5.390 \times 10^{-44} s$$
$$m_p = \sqrt{\frac{c\hbar}{G}} = 2.177 \times 10^{-8} kg \quad (201)$$

Note that, unlike $\hbar$ and c, the dimension of G depends on dimension of space-time D:

$$F = G\frac{mM}{r^{D-2}} = ma, \Downarrow$$
$$[G_D] = L^{D-1}T^{-2}M^{-1} \tag{202}$$

so,

$$\begin{aligned} &\hbar G_D = l_{pD}^{D+1} t_{pD}^{-3}, \\ &c = l_{pD} t_{pD}^{-1}, \Downarrow \\ &l_{pD}^{D-2} = \frac{\hbar G_D}{c^3}, \\ &t_{pD}^{D-2} = \frac{\hbar G_D}{c^{D+1}}, \\ &m_{pD}^{D-2} = \frac{c^{5-D} \hbar^{D-3}}{G_D} \end{aligned} \tag{203}$$

After compactification to four dimensions,

$$G_D = v G_4 \tag{204}$$

where v - the volume of the compactifying manifold has the four-dimensional interpretation as the vacuum expectation value of scalar modulus fields coming from the internal components of the metric tensor, it depends on the choice of vacuum but does not introduce any more fundamental constants into the lagrangian.

Note that in the gravity coupling constant and corresponding unites (203), the dimention D can takes also non integer-fractal values.
In the 1870's G.J. Stoney [Stoney, 1881], the physicist who coined the term "electron" and measured the value of elementary charge e, introduced as universal units of Nature for L, T, M :

$$l_S = \frac{e}{c^2}\sqrt{G},$$
$$t_S = \frac{e}{c^3}\sqrt{G}$$
$$= \frac{l_S}{c},$$
$$m_S = \frac{e}{\sqrt{G}},$$
$$l_S m_S = \frac{e^2}{c^2} \tag{205}$$

The expression for m_S has been derived by equating the Coulomb and Newton forces,

$$e^2 = Gm^2 \Rightarrow m_S = \frac{e}{\sqrt{G}} \tag{206}$$

The expressions for l_S and t_S has been derived from m_S, c and e on dimensional grounds,

$$[\frac{e^2}{r^2}] = [ma] = MLT^{-2} \Rightarrow e^2 = m_S L^3 T^{-2} = m_S l_S c^2$$
$$\Rightarrow l_S = \frac{e^2}{c^2 m_S} = \frac{e\sqrt{G}}{c^2} \tag{207}$$

Note that, we can define the units of Nature from fundamental length-l, charge-e and speed of light-c

$$t = l/c,\ m = (\frac{e}{c})^2/l,\ G = (\frac{lc^2}{e})^2 \tag{208}$$

When M. Planck discovered in 1899 h he introduced [Planck, 1899] as universal units of Nature for L, T, M:

$$m_P = \sqrt{\frac{hc}{G}} = \frac{m_S}{\sqrt{\alpha}},$$
$$l_P = \frac{h}{cm_P} = \frac{l_S}{\sqrt{\alpha}} = 11.7 l_S,$$
$$t_P = \frac{l_P}{c} = \frac{t_S}{\sqrt{\alpha}}, \tag{209}$$

Max Planck invented the system of fundamental unites c, h, G and k. G. Gamov, D. Ivanenko and L. Landau [Gamov, Ivanenko, Landau, 1928] considered the system without the parameter k, as fundamental one. Bronshtein [Bronshtein, 1933] and Zelmanov [Zelmanov, 1967], developed the idea of the cube of theories. The cube is located along three orthogonal axes marked by c (actually by $1/c$), $\hbar, G$. The vertex (000) corresponds to nonrelativistic mechanics, (c00) - to special relativity, ($0\hbar0$) - to non-relativistic quantum mechanics, ($c\hbar0$) - to quantum field theory, ($c0G$) - to general relativity, ($c\hbar$G) - to futuristic quantum gravity and the Theory of Everything, TOE, modern version of which is M-theory. There is a hope that in the framework of TOE the values of dimensionless fundamental parameters will be ultimately calculated. Note that 3-dimensional TS-$c\hbar$G where invented for 3-dimensional space models, d-dimensional theory may need d-dimensional TS, but, as we have seen, when extra dimensions are compactified the TS remain 3-dimensional; Stoney's fundamental constants are more fundamental just because they are less than Planck's constants :)

The meter was defined in 1791 as a 1/40 000 000 part of Paris meridian.The gram is the mass of one cubic cm of water. The cm and sec are connected with the size and rotation of the earth. An important step forward was made in the middle of XX century, when the standards of cm and sec were defined in terms of of wave-length and frequency of a certain atomic line.
Enormously more universal and fundamental are c and $\hbar$ given to us by Nature herself as units of velocity $[v] = [L/T]$ and angular momentum $[J] = [MvL] = [ML^2/T]$ or action $[S] = [ET] = [Mv^2T] = [ML^2/T]$.

It is important that c is not only the speed of light in vacuum. What is much more significant is the fact that it is the maximal velocity of any object in Nature, the photon being only one of such objects. The fundamental character of c would not be diminished in a world without photons. The fact that c is the maximal v leads to new phenomena, unknown in newtonian physics and described by relativity. Therefore Nature herself suggests c as fundamental unit of velocity.
c is more fundamental than α because it is the basis of relativity theory which unifies space and time, as well as energy, momentum and mass.
The quantity $\hbar$ is also fundamental: it is the quantum of the angular momentum J and a natural unit of the action S. When J or S are close to $\hbar$, the whole realm of quantum mechanical phenomena appears. Particles with integer J (bosons) tend to be in the same state (i.e. photons in a laser, or Rubidium atoms in a drop of Bose-Einstein condensate). Particles with half-integer J (fermions) obey the Pauli exclusion principle which is so basic for the structure of atoms, atomic nuclei and neutron stars.
Symmetry between fermions and bosons, dubbed supersymmetry or SUSY, is badly broken at low energies, but many theorists believe that it is restored near the Planck mass in particular in superstrings and M-theories.
It is natural when dealing with quantum mechanical problems to use $\hbar$ as the unit of J and S.

The status of G and its derivatives, m , l , t , is at present different from that of c and $\hbar$, because the quantum theory of gravity is still under construction. The majority of experts connect their hopes with extra spatial dimensions and superstrings. The characteristic length of a superstring $l_s(M^2_{GUT}) = l_P/\sqrt{\alpha(M^2_{GUT})}$. Possible modifications of Newton's potential at sub-millimetre distances demonstrates that the position of G is not as firm as that of c and $\hbar$. If the theory of gravity reduce to more fundamental structures, like old theory of weak interections with its coupling constant G reduce to SM, than gravitation coupling constant become calculable in terms of the fundamental theory.
The Newtonian potential around the sun is for non-vanishing Λ modified to [Axenides, Floratos, Perivolaropoulos, 2000], [Gibbons, Hawking, 1977]

$$V(r) = \frac{GM}{r} + \frac{\Lambda c^2}{6} r^2 \tag{210}$$

where M is the mass of the sun and r the distance from the sun.

Mathematically temperature T is defined as a derivative of internal energy E of a system over its entropy S:

$$Z(\beta) = \sum_n e^{-\beta E_n} = \sum_{E_n} N(E_n) e^{-\beta E_n} = \sum_{E_n} e^{-\beta F_n} = e^{-\beta F},$$
$$F = E - TS = E - tS_B,\ T = \beta^{-1} = kt,\ S_B = kS,$$
$$(\frac{\partial F}{\partial S})_T = 0 \Rightarrow T = \frac{dE}{dS},$$
$$k = 8.69 \times 10^{-5} eV/K = 1.38 \times 10^{-23} J/K. \quad (211)$$

As temperature is an average energy of an ensemble of particles, it is natural to measure it in units of energy. So, the Boltzmann's constant k connects microscopic phenomena to macroscopic one but it is not necessary to have different unit for measuring temperature and corresponding dimensional coefficient $k, T = kt$. We can put $k = 1$ and measuring the temperature in energy unites. In this sense, the Boltzmann's Constant k has not the fundamental meaning.

There are different opinions about the number of fundamental constants [Duff, Okun, Veneziano, 2001]. According to Okun there are three fundamental dimensionful constants in Nature: Planck's constant, $\hbar$; the velocity of light, c; and Newton's constant, G. According to Veneziano, there are only two: the string length l_s and c. According to Duff, there are not fundamental constants at all.

5-dimensional Einstein-Hilbert action

$$S = (12\pi^2 G_5)^{-1} \int d^5x \sqrt{-g_5} R_5 \tag{212}$$

Decomposing 5-dimensional metric as

$$g_5 = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu / M^2 & \phi A_\mu / M \\ \phi A_\nu / M & \phi^2 \end{pmatrix}, \tag{213}$$

we obtain

$$S = (16\pi G_4)^{-1} \int d^4x \sqrt{-g_4} \phi (R_4 - \frac{\phi^2}{4M^2} F^2) \tag{214}$$

where the 4-dimensional gravitational constant G_4 is

$$G_4 = G_5 \frac{3\pi}{4} / \int dx_5 \tag{215}$$

The scalar field couples explicitly to the kinetic term of the vector field and cannot be eliminated by a redefinition of the metric. Such dependencies of the masses and couplings are generic for higher-dimensional theories and in particular string theory. It is actually one of the definitive predictions for string theory that there exists a dilaton, that couples directly to matter [Taylor, Veneziano, 1988] and whose vacuum expectation value determines the string coupling constants [Witten, 1984,2]. In the Nambu-Goto string model

$$\frac{S}{\hbar} = \frac{1}{s} \int d(Area),\ \ s = l_s^2 \tag{216}$$

where l_s is the characteristic size of strings. The characteristic length of a superstring

$$l_s(M_{GUT}^2) = \frac{l_p}{\sqrt{\alpha(M_{GUT}^2)}} \tag{217}$$

We have seen, that α_{GUT}^{-1} in MSSM is equal to 29, so, in String Minimal SM (SMSM)

$$s = \frac{l_p^2}{\alpha_{GUT}} = \frac{l_s^2}{\alpha_{GUT}\alpha(m_e)} = 29 \times 137 e^2 G/c^4 \tag{218}$$

where l_s is Stony's fundamental length,

$$l_s = \frac{l_p}{\sqrt{\alpha(m_e)}}, \ \alpha(m_e)^{-1} = 137. \tag{219}$$

the parameter s is the one which replace the gravitational constant in old triumvirate of fundamental units $G, c, \hbar \Rightarrow s, c, \hbar$. Important consequence of this statement is that a string theory phenomenon we observe in everyday live as gravitation force.
String theory only needs two fundamental dimensionful constants c and s, i.e. one fundamental unit of speed and one of area. The role of Planck constant plays s.
There is, in relativity, a fundamental unit of speed c; there is, in QM, a fundamental unit of action $\hbar$; there is, in string theory, a fundamental unit of action - area, s.

In string theory we would like to freeze the moduli at values that provide the correct values of the coupling constant and unification scale of grand unified theories (GUTs). For instance, the dilaton and compactification volume V_6 should be frozen at values such that

$$\alpha_{GUT} \sim e^{\phi} \sim \frac{m_s^2}{m_P^2} \sim g_s,\ m_{GUT}^2 \sim \alpha_{GUT}^{4/3} g_s^{-1/3} m_P^2,\ g_s = V_6 M^6 e^{\phi} \quad (220)$$

where m_{GUT}, m_s, m_P are GUT, string and the Planck scales, g_s is the string coupling.
The tree-level low-energy effective action of string theory reads:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} e^{-\phi} (\lambda_s^{-2} (\mathcal{R} + \partial_\mu \phi \partial^\mu \phi + H_{\mu\nu\rho} H^{\mu\nu\rho}) + F_{\mu\nu} F^{\mu\nu}) \quad (221)$$

where $H_{\mu\nu\rho}$ is Kalb-Ramond antisymmetric tensor field strength.
Couplings are VEVs which, hopefully, become dynamically determined. In particular, a scalar field, the so-called dilaton ϕ, controls all sorts of couplings, gravitational and gauge alike,

$$\alpha_{gauge} \sim e^{\phi} \sim \frac{l_p^2}{l_s^2} = G_N T,\ T = \frac{\hbar}{l_s^2} \quad (222)$$

where l_s is string length, T is string tension.

In mathematics we have two kind of structures, discrete and continuous one. If a physical quantity has discrete values, it might not have dimension. If the values are continuous - the quantity might have dimension, unit of measure. These structures may depend on scale, e.g. on macroscopic scale condensed state of matter (and time) is well described as continuous medium, so we use dimensional units of length (and time). On the scale of atoms, the matter has discrete structure, so we may count lattice sites and may not use unit of length. If at small (e.g. at Plank) scale space (and/or time) is discrete then we not need an unit of length (time) for measuring, there is the fundamental length and we can just count.

Let us consider $l-$particle semi-inclusive distribution

$$F_l(n,q) = \frac{d^l\sigma_n}{\bar{d}q_1...\bar{d}q_l} = \frac{1}{n!}\int\prod_{i=1}^{n}\bar{d}q_i'\delta(p_1+p_2-\Sigma_{i=1}^{l}q_i-\Sigma_{i=1}^{n}q_i')$$
$$\cdot|M_{n+l+2}(p_1,p_2,q_1,...,q_l,q_1',...,q_n';g(\mu),m(\mu)),\mu)|^2,$$
$$\bar{d}p\equiv\frac{d^3p}{E(p)},\ E(p)=\sqrt{p^2+m^2}. \quad (223)$$

From the renormdynamic equation

$$DM_{n+l+2} = \frac{\gamma}{2}(n+l+2)M_{n+l+2}, \tag{224}$$

we obtain

$$\begin{aligned}
&DF_l(n,q) = \gamma(n+l+2)F_l(n,q),\\
&DF_l(q) = \gamma(<n> +l+2)F_l(q),\\
&D<n^k(q)>= \gamma(<n^{k+1}(q)> - <n^k(q)><n(q)>),\\
&DC_k = \gamma<n(q)>(C_{k+1} - C_k(1+k(C_2-1)))\\
&F_l(q) \equiv \frac{d^l\sigma}{\bar{d}q_1 ... \bar{d}q_l} = \sum_n \frac{d^l\sigma_n}{\bar{d}q_1 ... \bar{d}q_l}, \ <n^k(q)>= \frac{\sum_n n^k d^l\sigma_n/\bar{d}q^l}{\sum_n d^l\sigma_n/\bar{d}q^l}\\
&C_k = \frac{<n^k(q)>}{<n(q)>^k}
\end{aligned} \tag{225}$$

From dimensional considerations, the following combination of cross sections [Koba et al, 1972] must be universal function

$$< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \tag{226}$$

Corresponding relation for the inclusive cross sections is [Matveev et al, 1976].

$$< n(p) > \frac{d\sigma_n}{d\bar{p}} / \frac{d\sigma}{d\bar{p}} = \Psi(\frac{n}{< n(p) >}). \tag{227}$$

Indeed, let us define $n-$dimension of observables [Makhaldiani, 1980]

$$[n] = 1, \ [\sigma_n] = -1, \sigma = \Sigma_n \sigma_n, \ [\sigma] = 0, \ [< n >] = 1. \tag{228}$$

The following expression does not depend on any dimensional quantities and must have a corresponding universal form

$$P_n =< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \tag{229}$$

Let us find an explicit form of the universal functions using renormdynamic equations.

From the definition of the moments we have

$$C_k = \int_0^\infty dx x^k \Psi(x), \tag{230}$$

so they are universal parameters,

$$\begin{aligned} &DC_k = 0 \Rightarrow C_{k+1} = (1 + k(C_2 - 1))C_k \Rightarrow \\ &C_k = (1 + (k-1)(C_2 - 1))...(1 + 2(C_2 - 1))C_2. \end{aligned} \tag{231}$$

Now we can invert momentum transform and find (see [Makhaldiani, 1980] and appendix) universal functions [Ernst, Schmit, 1976], [Darbaidze et al, 1978].

$$\begin{aligned} &\Psi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn z^{-n-1} C_n = \frac{c^c}{\Gamma(c)} z^{c-1} e^{-cz}, \\ &C_2 = 1 + \frac{1}{c} \end{aligned} \tag{232}$$

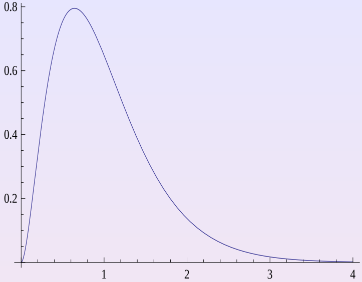

Figure: KNO distribution (232), $\Psi(z)$, with $c = 2.8$

The value of the parameter c can be measured from the dispersion low,

$$D = \sqrt{< n^2 > - < n >^2} = \sqrt{C_2 - 1} < n >= A < n >,$$
$$A = \frac{1}{\sqrt{c}} \simeq 0.6, \ c = 2.8;$$
$$(c = 3, \ A = 0.58) \qquad (233)$$

which is in accordance with $n-$dimension counting.

We can calculate also $1/<n>$ correction to the scaling function (see appendix)

$$<n>\frac{\sigma_n}{\sigma}=\Psi=\Psi_0(\frac{n}{<n>})+\frac{1}{<n>}\Psi_1(\frac{n}{<n>}),$$
$$C_k=C_k^0+\frac{1}{<n>}C_k^1,$$
$$C_k^0=\int_0^\infty dx x^k\Psi_0(x),\ C_k^1=\int_0^\infty dx x^k\Psi_1(x),$$
$$\Psi_1(z)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}dn z^{-n-1}C_n^1=\frac{C_2^1c^2}{2}(z-2+\frac{c-1}{cz})\Psi_0 \quad (234)$$

The characteristic function we define as

$$\Phi(t) = \int_0^\infty dx e^{tx} \Psi(x) = (1 - t/c)^{-c}, \; Re(t) < c \tag{235}$$

For the moments of the distribution, we have

$$\Phi^{(k)}(0) = C_k = (-c)(-c-1)...(-c-k+1)(-1/c)^k = \frac{\Gamma(c+k)}{\Gamma(c)c^k} \tag{236}$$

Note that it is an infinitely divisible characteristic function, i.e.

$$\Phi(t) = (\Phi_n(t))^n, \; \Phi_n(t) = (1 - t/c)^{-c/n} \tag{237}$$

If we calculate observable(mean) value of x, we find

$$\begin{aligned} &< x >= \Phi'(0) = n\Phi(0)_n{}' = n < x >_n, \\ &< x >_n = \frac{< x >}{n} \end{aligned} \tag{238}$$

For the second moment and dispersion, we have

$$
\begin{aligned}
&< x^2 >= \Phi^{(2)}(0) = n < x^2 >_n + n(n-1) < x >_n^2, \\
&\mathrm{D}^2 = < x^2 > - < x >^2 = n(< x^2 >_n - < x >_n^2) = n\mathrm{D}_n^2 \\
&\mathrm{D}_n^2 = \frac{\mathrm{D}^2}{n} = \frac{\mathrm{D}^2}{< x >} < x >_n
\end{aligned}
\tag{239}
$$

In a sense, any Hamiltonian quantum (and classical) system can be described by infinitely divisible distributions, because in the functional integral formulation, we use the following step

$$U(t) = e^{-itH} = (e^{-i\frac{t}{N}H})^N \tag{240}$$

In the case of our scalar field theory (1),

$$\begin{aligned} L(\varphi) &= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{g}{n}\varphi^n \\ &= g^{\frac{2}{2-n}}(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{1}{n}\phi^n) \end{aligned} \tag{241}$$

so, to the constituent field ϕ_N corresponds higher value of the coupling constant,

$$g_N = gN^{\frac{n-2}{2}} \tag{242}$$

For weak nonlinearity, $n = 2 + 2\varepsilon, d = 2/\varepsilon + 2, g_N = g(1 + \varepsilon \ln N + O(\varepsilon^2))$

Closed equation of renormdynamics for the generating function of the observables

Let us consider a generating function of the topological crossections

$$F(h,g,m,\mu)=\Sigma_{n\geq 2}h^n\sigma_n,$$
$$\sigma_n=\frac{1}{n!}\frac{d^n}{dh^n}F|_{h=0},$$
$$\sigma=F|_{h=1},\quad <n>=\frac{d}{dh}\ln F|_{h=1},... \tag{243}$$

It is natural that for the generating function we have closed renormdynamic equation [Makhaldiani, 1980]

$$(D-\gamma(\frac{h\partial}{\partial h}+2))F=0,$$
$$F(h(\mu),g(\mu),m(\mu),\mu)=F(h(\bar{\mu}),g(\bar{\mu}),m(\bar{\mu}),\bar{\mu})\exp(2\int_{\bar{\mu}}^{\mu}\frac{d\rho}{\rho}\gamma(g(\rho))),$$
$$\bar{h}=\bar{h}(\bar{\mu})=h(\mu)\exp(\int_{\mu}^{\bar{\mu}}\frac{d\rho}{\rho}\gamma(g(\rho))),$$
$$\bar{m}=\bar{m}(\bar{\mu})=m(\mu)\exp(\int_{\mu}^{\bar{\mu}}\frac{d\rho}{\rho}\eta(g(\rho))),\ \int_g^{\bar{g}}\frac{dg}{\beta(g)}=\ln\frac{\bar{\mu}}{\mu} \tag{244}$$

Let us find generating function in the case of KNO scaling. From the definition of Generating function and using topological cross section from KNO, we find

$$F(h) = \sum_n h^n \frac{\sigma}{<n>}\Psi(\frac{n}{<n>}) = \frac{\sigma}{<n>}\sum \Psi(\frac{n}{<n>})h^n$$
$$= \frac{\sigma}{<n>}\Psi(\frac{\delta}{<n>})\frac{h^2}{1-h},\ \delta \equiv h\frac{d}{dh},\ q^\delta f(h) = f(qh), \qquad (245)$$

Now we can find more concrete form of the generating function, with the explicit form of KNO function,

$$(\frac{\delta}{<n>})^{c-1}\exp(-c\frac{\delta}{<n>})\frac{h^2}{1-h} = (\frac{\delta}{<n>})^{c-1}\frac{q^2h^2}{1-qh}$$
$$= \frac{1}{<n>^{c-1}}\frac{1}{\Gamma(1-c)}\int_0^\infty \frac{dt}{t^c}\frac{q^2h^2e^{-2t}}{1-qhe^{-t}}, \qquad (246)$$

so

$$F(h)_{KNO} = \frac{c^c}{\Gamma(c)}\frac{\sigma}{<n>^c}\frac{1}{\Gamma(1-c)}\int_0^\infty \frac{dt}{t^c}\frac{q^2h^2e^{-2t}}{1-qhe^{-t}},$$
$$q = \exp(-\frac{c}{<n>}) \qquad (247)$$

Indeed, if we expend and than integrate under this formula, we hind

$$F(h) = \frac{c^c}{\Gamma(c)} \frac{\sigma}{< n >^c} \sum_{n \geq 2} h^n n^{c-1} \exp(-\frac{c}{< n >} n) \tag{248}$$

which corresponds to the considered explicit form of the KNO function.

Negative binomial distribution (NBD) is defined as

$$P(n) = \frac{\Gamma(n+r)}{n!\Gamma(r)} p^n (1-p)^r, \ \sum_{n \geq 0} P(n) = 1, \tag{249}$$

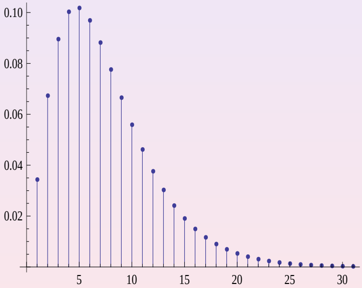

Figure: $P(n), (249), r = 2.8, p = 0.3, < n >= 6$

NBD provides a very good parametrization for multiplicity distributions in e^+e^- annihilation; in deep inelastic lepton scattering; in proton-proton collisions; in proton-nucleus scattering.

Hadronic collisions at high energies (LHC) lead to charged multiplicity distributions whose shapes are well fitted by a single NBD in fixed intervals of central (pseudo)rapidity η [ALICE,2010].

It is interesting to understand how NBD fits such a different reactions?

Let us consider NBD for normed topological cross sections

$$\frac{\sigma_n}{\sigma} = P(n) = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(1+\frac{k}{<n>})^{-n}(1+\frac{<n>}{k})^{-k}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{<n>}{<n>+k})^{n}(\frac{k}{k+<n>})^{k},$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}\frac{(\frac{k}{<n>})^k}{(1+\frac{k}{<n>})^{k+n}},$$
$$r = k > 0,\ p = \frac{<n>}{<n>+k}. \quad (250)$$

The generating function for NBD is

$$F(h) = (1+\frac{<n>}{k}(1-h))^{-k} = (1+\frac{<n>}{k})^{-k}(1-ah))^{-k},$$
$$a = p = \frac{<n>}{<n>+k}. \quad (251)$$

Indeed,

$$
\begin{aligned}
(1-ah))^{-k} &= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1}e^{-t(1-ah)}\\
&= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1}e^{-t}\sum_0^\infty \frac{(tah)^n}{n!}\\
&= \sum_0^\infty \frac{\Gamma(n+k)a^n}{\Gamma(k)n!}h^n,\\
P(n) &= (1+\frac{<n>}{k})^{-k}\frac{\Gamma(n+k)}{\Gamma(k)n!}(\frac{<n>}{<n>+k})^n\\
&= \frac{k^k\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)}(<n>+k)^{-(n+k)}<n>^n\\
&= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)} \qquad (252)
\end{aligned}
$$

Note that KNO characteristic function (235) coincides with the NBD generating function (251) when $t =< n > (h-1),\ c = k.$
The Bose-Einstein distribution is a special case of NBD with $k = 1$.

If k is negative, the NBD becomes a positive binomial distribution, narrower than Poisson (corresponding to negative correlations).
For negative (integer) values of $k = -N$, we have Binomial GF

$$F_{bd} = (1 + \frac{< n >}{N}(h-1))^N = (a + bh)^N,\ a = 1 - \frac{< n >}{N}, b = \frac{< n >}{N},$$
$$P_{bd}(n) = C_N^n(\frac{< n >}{N})^n(1 - \frac{< n >}{N})^{N-n} \quad (253)$$

(In a sense) we have a (quantum) spectrum for the parameter k, which contains any (positive) real values and (with finite number of states) the negative integer values, $(0 \leq n \leq N)$

From the generating function we have

$$< n^2 >= (\frac{hd}{dh})^2 F(h)|_{h=1} = \frac{k+1}{k} < n >^2 + < n >, \qquad (254)$$

for dispersion we obtain

$$D = \sqrt{< n^2 > - < n >^2} = \frac{1}{\sqrt{k}} < n > (1 + \frac{k}{< n >})^{1/2}$$
$$= \frac{1}{\sqrt{k}} < n > + \frac{\sqrt{k}}{2} + O(1/ < n >), \qquad (255)$$

so the dispersion low for KNO and NBD distributions are the same, with $c = k$, for high values of the mean multiplicity.
The factorial moments of NBD,

$$F_m = (\frac{d}{dh})^m F(h)|_{h=1} = \frac{< n(n-1)...(n-m+1) >}{< n >^m} = \frac{\Gamma(m+k)}{\Gamma(m)k^m}, \qquad (256)$$

and usual normalized moments of KNO (236) coincides.

Using fractal calculus (see e.g. [Makhaldiani, 2003]),

$$D_{0,x}^{-\alpha} f = \frac{|x|^\alpha}{\Gamma(\alpha)} \int_0^1 |1-t|^{\alpha-1} f(xt)dt, = \frac{|x|^\alpha}{\Gamma(\alpha)} B(\alpha,\ \partial x) f(x)$$
$$= |x|^\alpha \frac{\Gamma(\partial x)}{\Gamma(\alpha+\partial x)} f(x),\ f(xt) = t^{x\frac{d}{dx}} f(x). \quad (257)$$

we can define factorial and cumulant moments for not only negative integer values of q, but for any complex indexes,

$$F_{-q} =< n >^q D_{0,x}^{-q} G_{NBD}(x)|_{x=0} = \frac{k^q \Gamma(k-q)}{\Gamma(k)},$$
$$K_{-q} =< n >^q D_{0,x}^{-q} \ln G_{NBD}(x)|_{x=0} = k^{q+1}\Gamma(-q),$$
$$H_{-q} = \frac{\Gamma(k+1)\Gamma(-q)}{\Gamma(k-q)} \quad (258)$$

Let us show that NBD is a discrete distribution corresponding to the KNO scaling,

$$\lim_{<n>\to\infty} < n > P_n|_{\frac{n}{<n>}=z} = \Psi(z) \qquad (259)$$

Indeed, using the following asymptotic formula

$$\Gamma(x+1) = x^x e^{-x}\sqrt{2\pi x}(1 + \frac{1}{12x} + O(x^{-2})), \qquad (260)$$

we find

$$< n > P_n =< n > \frac{(n+k-1)^{n+k-1}e^{-(n+k-1)}}{\Gamma(k)n^n e^{-n}}\frac{k^k}{n^k} < n > z^k e^{-k\frac{n+k}{<n>}}$$
$$= \frac{k^k}{\Gamma(k)}z^{k-1}e^{-kz} + O(1/ < n >) \qquad (261)$$

We can calculate also $1/ < n >$ correction term to the KNO from the NBD. The answer is

$$\Psi = \frac{k^k}{\Gamma(k)}z^{k-1}e^{-kz}(1 + \frac{k^2}{2}(z - 2 + \frac{k-1}{kz})\frac{1}{< n >}) \qquad (262)$$

This form coincides with the corrected KNO (234) for $c = k$ and $C_2^1 = 1$.

We have seen that KNO characteristic function (235) and NBD GF (251) have almost same form. This relation become in coincidence if

$$c = k,\ t = (h-1)\frac{< n >}{k} \tag{263}$$

Now the definition of the characteristic function (235) can be read as

$$\int_0^\infty e^{-<n>z(1-h)}\Psi(z)dz = (1 + \frac{< n >}{k}(1-h))^{-k} \tag{264}$$

which means that Poisson GF weighted by KNO distribution gives NBD GF. Because of this, the NBD is the gamma-Poisson (mixture) distribution.

Fore high values of $x_2 = k$ the NBD distribution reduces to the Poisson distribution

$$F(x_1, x_2, h) = (1 + \frac{x_1}{x_2}(1-h))^{-x_2} \Rightarrow e^{-x_1(1-h)} = e^{-<n>}e^{h<n>}$$
$$= \sum P(n)h^n,$$
$$P(n) = e^{-<n>}\frac{<n>^n}{n!} \qquad (265)$$

For the Poisson distribution

$$\frac{d^2F(h)}{dh^2}|_{h=1} =< n(n-1) >=< n >^2,$$
$$D^2 =< n^2 > - < n >^2 =< n > . \qquad (266)$$

In the case of NBD, we had the following dispersion low

$$D^2 = \frac{1}{k} < n >^2 + < n >, \qquad (267)$$

which coincides withe previous expression for high values of k.
Poisson GF belongs to the class of the infinitely divisible distributions,

$$F(h, < n >) = (F(h, < n > /k))^k \qquad (268)$$

For high values of $< n >$, the Poisson distribution reduces to the Gauss distribution

$$P(n) = e^{-<n>} \frac{< n >^n}{n!} \Rightarrow \frac{1}{\sqrt{2\pi < n >}} \exp(-\frac{(n- < n >)^2}{2 < n >}) \qquad (269)$$

For high values of k in the integral relation (264), in the KNO function dominates the value $z_c = 1$ and both sides of the relation reduce to the Poisson GF.

An useful property of the negative binomial distribution with parameters

$$< n >, k$$

is that it is (also) the distribution of a sum of k independent random variables drawn from a Bose-Einstein distribution[1] with mean $< n > /k$,

$$P_n = \frac{1}{< n > +1}(\frac{< n >}{< n > +1})^n$$
$$= (e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2})e^{-\beta\hbar\omega(n+1/2)},\ T = \frac{\hbar\omega}{\ln\frac{<n>+1}{<n>}}$$
$$\sum_{n\geq 0} P_n = 1,\ \sum nP_n =< n >= \frac{1}{e^{\beta\hbar\omega-1}},\ T \simeq \hbar\omega < n >,\ < n >\gg 1,$$
$$P(x) = \sum_n x^n P_n = (1+ < n > (1-x))^{-1}. \tag{270}$$

[1] A Bose-Einstein, or geometrical, distribution is a thermal distribution for single state systems.

This is easily seen from the generating function in (251), remembering that the generating function of a sum of independent random variables is the product of their generating functions.
Indeed, for

$$n = n_1 + n_2 + ... + n_k, \qquad (271)$$

with n_i independent of each other, the probability distribution of n is

$$P_n = \sum_{n_1,...,n_k} \delta(n - \sum n_i) p_{n_1} ... p_{n_k},$$
$$P(x) = \sum_n x^n P_n = p(x)^k \qquad (272)$$

This has a consequence that an incoherent superposition of N emitters that have a negative binomial distribution with parameters $k, < n >$ produces a negative binomial distribution with parameters $Nk, N < n >$.

So, for the GF of NBD we have (N=2)

$$F(k,<n>)F(k,<n>) = F(2k, 2<n>) \tag{273}$$

And more general formula (N=m) is

$$F(k,<n>)^m = F(mk, m<n>) \tag{274}$$

We can put this equation in the closed nonlocal form

$$Q_q F = F^q, \tag{275}$$

where

$$Q_q = q^D, \quad D = \frac{kd}{dk} + \frac{<n>d}{d<n>} = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2} \tag{276}$$

Note that temperature defined in (270) gives an estimation of the Glukvar temperature when it radiates hadrons. If we take $\hbar\omega = 100MeV$, to $T \simeq T_c \simeq 200MeV$ corresponds $<n> \simeq 1.5$ If we take $\hbar\omega = 10MeV$, to $T \simeq T_c \simeq 200MeV$ corresponds $<n> \simeq 20$.
We see that universality of NBD in hadron-production is similar to the universality of black body radiation.

p-adic string amplitudes can be obtained as tree amplitudes of the field theory with the following lagrangian and motion equation (see e.g. [Brekke, Freund, 1993])

$$L = \frac{1}{2}\Phi Q_p \Phi - \frac{1}{p+1}\Phi^{p+1},$$

$$Q_p\Phi = \Phi^p,\ Q_p = p^D \quad (277)$$

$$D = -\frac{1}{2}\triangle,\ \triangle = -\partial^2_{x_0} + \partial^2_{x_1} + ... + \partial^2_{x_{n-1}}, \quad (278)$$

Φ - is real scalar field on D-dimensional space-time with coordinates $x = (x_0, x_1, ..., x_{D-1})$. We have trivial, $\Phi = 0$ and $\Phi = 1$, and following nontrivial solutions of the equation (277)

$$\Phi(x_0, x_1, ..., x_{D-1}) = p^{\frac{D}{2(p-1)}} e^{\frac{1-p^{-1}}{2\ln p}(x_0^2 - x_1^2 - x_2^2 - ... - x_{D-1}^2)} \quad (279)$$

The equation (277) permits factorization of its solutions $\Phi(x) = \Phi(x_0)\Phi(x_1)...\Phi(x_{D-1})$, every factor of which fulfils one dimensional equation

$$p^{\varepsilon \partial_x^2} \Phi(x) = \Phi(x)^p, \ \varepsilon = \pm \frac{1}{2} \tag{280}$$

The trivial solution of the equations are $\Phi = 0$ and $\Phi = 1$. For nontrivial solution of (280), we have

$$\begin{aligned} p^{\varepsilon \partial_x^2} \Phi(x) &= e^{a\partial^2} \Phi(x) = \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{\infty} dy e^{-\frac{1}{4a} y^2 + y\partial} \Phi(x) \\ &= \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{\infty} dy e^{-\frac{1}{4a} y^2} \Phi(x+y) = \Phi(x)^p, \ a = \varepsilon \ln p \end{aligned} \tag{281}$$

If we (de quantize) put, $p = q$, and take (classical) limit, $q \to 1$, the motion equation reduce to

$$\varepsilon \partial_x^2 \Phi = \Phi \ln \Phi, \tag{282}$$

with solution

$$\Phi(x) = e^{\frac{1}{2}} e^{\frac{x^2}{4\varepsilon}}. \tag{283}$$

It is obvious that the anzac

$$\Phi = Ae^{bx^2}, \tag{284}$$

can pass the equation (281). Indeed, the solution is

$$\Phi(x) = p^{\frac{1}{2(p-1)}} e^{\frac{1-p^{-1}}{4\varepsilon \ln p} x^2},$$
$$\Phi(x_0, x_1, ..., x_{D-1}) = p^{\frac{D}{2(p-1)}} e^{\frac{1-p^{-1}}{2\ln p}(x_0^2 - x_1^2 - x_2^2 - ... - x_{D-1}^2)} \tag{285}$$

Now, we can define the following class of motion equations

$$Q_q F = F^q, \tag{286}$$

where

$$Q_q = q^D, \quad D = D_1(x_1) + ... + D_l(x_l), \tag{287}$$

$D_k(x)$ is some (differential) operator depending on x. In the case of the NBD GF,

$$D_k(x) = \frac{xd}{dx}. \tag{288}$$

For this (Qlike) class of equations, we have factorization property

$$F = F(x_1, ..., x_l) = F_1(x_1)...F_l(x_l),$$
$$q^{D_k(x)} F_k(x) = c_k F_k(x)^q, 1 \leq k \leq l, \; c_1 c_2 ... c_l = 1. \tag{289}$$

For NBD distribution we have corresponding multiplication(convolution)formulas

$$(P \star P)_n \equiv \sum_{m=0}^{n} P_m(k, <n>)P_{n-m}(k, <n>)$$
$$= P_n(2k, 2<n>) = Q_2 P_n(k, <n>), ... \quad (290)$$

So, we can say, that star-product on the distributions of NBD corresponds ordinary product for GF.
It will be nice to have similar things for string field theory(SFT) [Kaku, 2000].
SFT motion equation is

$$Q\Phi = \Phi \star \Phi \quad (291)$$

For stringfield GF F we may have

$$QF = F^2. \quad (292)$$

By construction we know the solution of the nice equation (275) as GF of NBD, F. We obtain corresponding differential equations, if we consider $q = 1 + \varepsilon$, for small ε,

$$(D(D-1)...(D-m+1) - (lnF)^m)\Psi = 0,$$
$$(\frac{\Gamma(D+1)}{\Gamma(D+1-m)} - (\ln F)^m)\Psi = 0,$$
$$(D_m - \Phi^m)\Psi = 0, m = 1, 2, 3, ...$$
$$D_m = \frac{\Gamma(D+1)}{\Gamma(D+1-m)}, \Phi = \ln F, \tag{293}$$

with the solution $\Psi = F = \exp(\Phi)$. In the case of the NBD and p-adic string, we have correspondingly

$$D = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2};$$
$$D = -\frac{1}{2}\triangle, \ \triangle = -\partial^2_{x_0} + \partial^2_{x_1} + ... + \partial^2_{x_{n-1}}. \tag{294}$$

These equations have meaning not only for integer m.

For high mean multiplicities we have corresponding equations for KNO

$$Q_2\Psi(z) = \Psi\star\Psi \equiv \int_0^z \Psi(t)\Psi(z-t)dt$$
$$= z\int_0^1 dt t^{\delta_1}(1-t)^{\delta_2}\Psi(z_1)\Psi(z_2)|_{z_1=z_2=z}$$
$$= z\frac{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}{\Gamma(\delta_1+\delta_2+2)}\Psi(z_1)\Psi(z_2)|_{z_1=z_2=z} \quad (295)$$

Due to the explicit form of the operator D, these equations and corresponding solutions have the symmetry under the change of the variables

$$k \to ak, \ <n> \to b<n>. \quad (296)$$

When

$$a = \frac{<n>}{k}, \ b = \frac{k}{<n>}, \quad (297)$$

we obtain the symmetry with respect to the transformations $k \leftrightarrow <n>, x_1 \leftrightarrow x_2$.

The Riemann zeta function $\zeta(s)$ is defined for complex $s = \sigma + it$ and $\sigma > 1$ by the expansion

$$\begin{aligned}\zeta(s) &= \sum_{n\geq 1} n^{-s},\ Re\ s > 1,\\ &= \delta_x^{-s}\frac{x}{1-x}|_{x\to 1} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\delta_x t}\frac{x}{1-x}|_{x\to 1}\\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{t\partial_\tau}\frac{1}{e^\tau - 1}|_{\tau\to 0}\\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}dt}{e^t - 1},\ x = e^{-\tau}.\end{aligned} \tag{298}$$

All complex zeros, $s = \alpha + i\beta$, of $\zeta(\sigma + it)$ function lie in the critical stripe $0 < \sigma < 1$, symmetrically with respect to the real axe and critical line $\sigma = 1/2$. So it is enough to investigate zeros with $\alpha \leq 1/2$ and $\beta > 0$. These zeros are of three type, with small, intermediate and big ordinates.

The Riemann hypothesis [Titchmarsh,1986] states that the (non-trivial) complex zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$.
At the beginning of the XX century Polya and Hilbert made a conjecture that the imaginary part of the Riemann zeros could be the oscillation frequencies of a physical system (ζ - (mem)brane).
After the advent of Quantum Mechanics, the Polya-Hilbert conjecture was formulated as the existence of a self-adjoint operator whose spectrum contains the imaginary part of the Riemann zeros.
The Riemann hypothesis (RH) is a central problem in Pure Mathematics due to its connection with Number theory and other branches of Mathematics and Physics.

The functional equation is (see e.g. [Titchmarsh,1986])

$$\zeta(1-s) = \frac{2\Gamma(s)}{(2\pi)^s}\cos(\frac{\pi s}{2})\zeta(s) \tag{299}$$

From this equation we see the real (trivial) zeros of zeta function:

$$\zeta(-2n) = 0, \ n = 1, 2, ... \tag{300}$$

Also, at s=1, zeta has pole with reside 1.
From Field theory and statistical physics point of view, the functional equation (299) is duality relation, with self dual (or critical) line in the complex plane, at $s = 1/2 + i\beta$,

$$\zeta(\frac{1}{2} - i\beta) = \frac{2\Gamma(s)}{(2\pi)^s}\cos(\frac{\pi s}{2})\zeta(\frac{1}{2} + i\beta), \tag{301}$$

we see that complex zeros lie symmetrically with respect to the real axe.
On the critical line, (nontrivial) zeros of zeta corresponds to the infinite value of the free energy,

$$F = -T\ln\zeta. \tag{302}$$

At the point with $\beta = 14.134725...$ is located the first zero. In the interval $10 < \beta < 100$, zeta has 29 zeros. The first few million zeros have been computed and all lie on the critical line. It has been proved that uncountably many zeros lie on critical line.
The first relation of zeta function with prime numbers is given by the following formula,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \ Res > 1. \tag{303}$$

Another formula, which can be used on critical line, is

$$\zeta(s) = (1-2^{1-s})^{-1}\sum_{n\geq 1}(-1)^{n+1}n^{-s},\ Re\ s > 0$$
$$= \frac{e^{i\pi(\delta_x+1)}}{(1-2^{1-s})\delta_x^s}\frac{x}{1-x}|_{x\to 1}$$
$$= \frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}e^{i\pi}e^{(i\pi-t)\delta_x}\frac{1}{x^{-1}-1}|_{x\to 1}$$
$$= \frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}e^{(t-i\pi)\partial_\tau}\frac{e^{i\pi}}{e^\tau-1}|_{\tau\to 0}$$
$$= \frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}dt}{e^t+1},$$
$$\int_0^\infty \frac{t^{s-1}dt}{e^t+1} = \int_0^\infty dt t^{s-1}e^{-t}\sum_{n\geq 0}(-1)^n e^{-nt}$$
$$= \Gamma(s)\sum_{n\geq 1}(-1)^{n+1}n^{-s} \tag{304}$$

Let us consider the values $q = n, n = 1, 2, 3, ...$ and take sum of the corresponding equations (286), we find

$$\zeta(-D)F = \frac{F}{1-F} \tag{305}$$

In the case of the NBD we know the solutions of this equation.
Now we invent a Hamiltonian H with spectrum corresponding to the set of nontrivial zeros of the zeta function, in correspondence with Riemann hypothesis,

$$\begin{aligned}
&-D_n = \frac{n}{2} + iH_n,\ H_n = i(\frac{n}{2} + D_n),\\
&D_n = x_1\partial_1 + x_2\partial_2 + ... + x_n\partial_n,\ H_n^+ = H_n = \sum_{m=1}^{n} H_1(x_m),\\
&H_1 = i(\frac{1}{2} + x\partial_x) = -\frac{1}{2}(x\hat{p} + \hat{p}x),\ \hat{p} = -i\partial_x
\end{aligned} \tag{306}$$

The Hamiltonian $H = H_n$ is hermitian, its spectrum is real. The case $n = 1$ corresponds to the Riemann hypothesis.

The case $n = 2$, corresponds to NBD,

$$\zeta(1+iH_2)F = \frac{F}{1-F},\ \zeta(1+iH_2)|_F = \frac{1}{1-F},$$
$$F(x_1,x_2;h) = (1+\frac{x_1}{x_2}(1-h))^{-x_2} \quad (307)$$

Let us scale $x_2 \to \lambda x_2$ and take $\lambda \to \infty$ in (307), we obtain

$$\zeta(\frac{1}{2}+iH_1(x))e^{-(1-h)x} = \frac{1}{e^{(1-h)x}-1},$$
$$\frac{1}{\zeta(\frac{1}{2}+iH(x))}\frac{1}{e^{\varepsilon x}-1} = e^{-\varepsilon x},$$
$$H(x) = i(\frac{1}{2}+x\partial_x) = -\frac{1}{2}(x\hat{p}+\hat{p}x),\ H^+ = H, \varepsilon = 1-h. \quad (308)$$

Now we scale $x \to xy$, multiply the equation by y^{s-1} and integrate

$$\frac{1}{\zeta(\frac{1}{2}+iH(x))}\int_0^\infty dy\frac{y^{s-1}}{e^{\varepsilon xy}-1} = \int_0^\infty dy e^{-\varepsilon xy}y^{s-1} = \frac{1}{(\varepsilon x)^s}\Gamma(s),$$
$$\frac{1}{\zeta(\frac{1}{2}+iH(x))}\int_0^\infty dy\frac{y^{s-1}}{e^{\varepsilon xy}-1}$$
$$= \frac{1}{\zeta(\frac{1}{2}+iH(x))}x^{-s}\varepsilon^{-s}\Gamma(s)\zeta(s), \quad (309)$$

so

$$\zeta(\frac{1}{2}+iH(x))x^{-s} = \zeta(s)x^{-s} \Rightarrow H(x)\psi_E = E\psi_E,$$
$$\psi_E = cx^{-s},\ s = \frac{1}{2}+iE, \quad (310)$$

we have correct way and can return to the previous step (308) and take the following transformation

$$\frac{1}{e^{\varepsilon xy}-1}=\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} dE x^{-iE-1/2}\varphi(E,\varepsilon y),$$
$$\varphi(E,\varepsilon y)=\int_0^\infty dx\frac{x^{iE-\frac{1}{2}}}{e^{\varepsilon xy}-1}=\frac{\Gamma(\frac{1}{2}+iE)}{(\varepsilon y)^{iE+1/2}}\zeta(\frac{1}{2}+iE),$$
$$\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} dE x^{-iE-1/2}\varphi(E,\varepsilon y)\frac{1}{\zeta(1/2+iE)}=e^{-\varepsilon xy} \quad (311)$$

If we take the following formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}dt}{e^t - 1}, \tag{312}$$

which says that ζ function is the Mellin transformation, we can find

$$\Gamma(1 + iH_2)\frac{F}{1-F} = \int_0^\infty \frac{dt/t}{e^t - 1} F^{1/t}, \tag{313}$$

or

$$\Gamma(1 + iH_2)\Phi = \int_0^\infty \frac{dt/t}{e^t - 1} (\frac{\Phi}{1+\Phi})^{1/t},$$
$$\Phi = \frac{F}{1-F} = \frac{1}{(1 + \frac{x_1}{x_2}(1-h))^{x_2} - 1} \tag{314}$$

We can obtain also the following equation with argument of ζ_N on critical axis

$$\zeta_N(\frac{1}{2} + iH_1(x_2))F(x_1, x_2, h) = \sum_{n=1}^{N} \frac{1}{(1 + \frac{x_1}{nx_2}(1-h))^{nx_2}}$$
$$= \sum_{n=1}^{N} F(x_1, nx_2, h),$$
$$\zeta_N(\frac{1}{2} + iH_1(x_2))F(\lambda x_1, x_2, h) = \sum_{n=1}^{N} \frac{1}{(1 + \frac{\lambda x_1}{nx_2}(1-h))^{nx_2}}$$
$$= \sum_{n=1}^{N} F(\lambda x_1, nx_2, h) \simeq N e^{-\lambda(1-h)x_1}, N \gg 1. \quad (315)$$

Let us calculate next therm in the $1/\lambda$ expansion in the (307)

$$F(x_1,\lambda x_2,h) = (1+\frac{\varepsilon x_1}{\lambda x_2})^{-\lambda x_2} = e^{-\lambda x_2 \ln(1+\varepsilon\frac{x_1}{\lambda x_2})}$$
$$= e^{-\varepsilon x_1} e^{\frac{(\varepsilon x_1)^2}{2\lambda x_2}+O(\lambda^{-2})} = e^{-\varepsilon x_1}(1+\frac{(\varepsilon x_1)^2}{2\lambda x_2}+O(\lambda^{-2})),$$
$$(F^{-1}-1)^{-1} = (e^{\lambda x_2 \ln(1+\varepsilon\frac{x_1}{\lambda x_2})}-1)^{-1}$$
$$= \frac{1}{e^{\varepsilon x_1}-1}(1+\frac{e^{\varepsilon x_1}}{e^{\varepsilon x_1}-1}\frac{(\varepsilon x_1)^2}{2\lambda x_2}+O(\lambda^{-2})) \quad (316)$$

The zero order term, λ^0 we already considered. The next, λ^{-1} order term gives the following relations

$$\zeta(-\delta_1-\delta_2)\frac{x_1^2}{x_2}e^{-\varepsilon x_1} = \frac{1}{x_2}\zeta(1-\delta_1)x_1^2 e^{-\varepsilon x_1} = \frac{x_1^2 e^{\varepsilon x_1}}{x_2(e^{\varepsilon x_1}-1)^2},$$
$$\zeta(1-\delta)x^2 e^{-\varepsilon x} = \frac{x^2 e^{\varepsilon x}}{(e^{\varepsilon x}-1)^2} = x^2 e^{-\varepsilon x} + O(e^{-2\varepsilon x})$$
$$\zeta(1-\delta)\Psi = E\Psi + O(e^{-2\varepsilon x}), \Psi = x^2 e^{-\varepsilon x}, E = 1. \quad (317)$$

There have been a number of approaches to understanding the Riemann hypothesis based on physics (for a comprehensive list see [Watkins]) According to the idea of Berry and Keating, [Berry,Keating,1997] the real solutions E_n of

$$\zeta(\frac{1}{2} + iE_n) = 0, \tag{318}$$

are energy levels, eigenvalues of a quantum Hermitian operator (the Riemann operator) associated with the one-dimensional classical hyperbolic Hamiltonian

$$H_c = xp, \tag{319}$$

where x and p are the conjugate coordinate and momentum.

They suggest a quantization condition generating Riemann zeros. This Hamiltonian breaks time-reversal invariance since $(x, p) \to (x, -p) \Rightarrow H \to -H$. The classical Hamiltonian $H = xp$ of linear dilation, i.e. multiplication in x and contraction in p, gives the Hamiltonian equations:

$$\begin{aligned} \dot{x} &= x, \\ \dot{p} &= -p, \end{aligned} \tag{320}$$

with the solution

$$\begin{aligned} x(t) &= x_0 e^{t}, \\ p(t) &= p_0 e^{-t} \end{aligned} \tag{321}$$

for any nonzero $E = x_0 p_0 = x(t)p(t)$ is hyperbola in phase space.

The system is quantized by considering the dilation operator in the x space

$$H = \frac{1}{2}(xp + px) = -i\hbar(\frac{1}{2} + x\partial_x), \tag{322}$$

which is the simplest formally Hermitian operator corresponding to the classical Hamiltonian. The eigenvalue equation

$$H\psi_E = E\psi_E, \tag{323}$$

is satisfied by the eigenfunctions

$$\psi_E(x) = cx^{-\frac{1}{2}+\frac{i}{\hbar}E}, \tag{324}$$

where the complex constant c is arbitrary, since the solutions are not square-integrable. To the normalization

$$\int_0^\infty dx\psi_E(x)^*\psi_{E'}(x) = \delta(E - E'), \tag{325}$$

corresponds $c = 1/\sqrt{2\pi}$.

We have seen that

$$\zeta(\frac{1}{2}+iH)e^{-\varepsilon x}=\frac{1}{e^{\varepsilon x}-1},$$
$$H=-i(\frac{1}{2}+x\partial_x)=x^{1/2}px^{1/2},p=-i\partial_x, \tag{326}$$

than

$$e^{-\varepsilon x}=\int dEx^{-1/2+iE}\varphi(E,\varepsilon),\varphi(E,\varepsilon)=\frac{1}{2\pi}\int_0^\infty dxx^{-1/2-iE}e^{-\varepsilon x}$$
$$=\frac{\varepsilon^{-1/2+iE}}{2\pi}\Gamma(1/2+iE);$$
$$\zeta(\frac{1}{2}+iE)\varphi(E,\varepsilon)=\frac{1}{2\pi}\int_0^\infty dx\frac{x^{-1/2-iE}}{e^{\varepsilon x}-1}$$
$$=\frac{\varepsilon^{-1/2+iE}}{2\pi}\Gamma(1/2+iE)\zeta(\frac{1}{2}+iE). \tag{327}$$

From the equation (308) we have

$$\zeta(\frac{1}{2}+iH_1(x))e^{-\varepsilon x}=\frac{1}{e^{\varepsilon x}-1},\ H_1=i(\frac{1}{2}+x\partial_x),$$
$$\zeta(-x\partial_x)(1-\varepsilon x+\frac{(\varepsilon x)^2}{2}+...)=\frac{1}{\varepsilon x}(1-(\frac{\varepsilon x}{2}+\frac{(\varepsilon x)^2}{6}+...)+$$
$$+(\frac{\varepsilon x}{2}+...)^2+...), \tag{328}$$

so

$$\zeta(0)=-\frac{1}{2},\ \zeta(-1)=-\frac{1}{12},... \tag{329}$$

Note that, a little calculation shows that, the $(\varepsilon x)^2$ terms cancels on the r.h.s, in accordance with $\zeta(-2)=0$.

More curious question concerns with the term $1/\varepsilon x$ on the r.h.s. To it corresponds the term with actual infinitesimal coefficient on the l.h.s.

$$\frac{1}{\zeta(1)}\frac{1}{\varepsilon x}, \tag{330}$$

in the spirit of the nonstandard analysis (see, e.g. [Davis,1977]), we can imagine that such a terms always present but on the r.h.s we may not note them.
For other values of zeta function we will use the following expansion

$$\frac{1}{e^x-1} = \frac{1}{x+\frac{x^2}{2}+\frac{x^3}{3!}+...} = \frac{1}{x}-\frac{1}{2}+\sum_{k\geq 1}\frac{B_{2k}x^{2k-1}}{(2k)!},$$
$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},... \tag{331}$$

and obtain

$$\zeta(1-2n) = -\frac{B_{2n}}{2n},\ n\geq 1. \tag{332}$$

Let us imagine space-time development of the the multiparticle process and try to describe it by some (phenomenological) dynamical equation. We start to find the equation for the Poisson distribution and than naturally extend them for the NBD case.
Let us define an integer valued variable $n(t)$ as a number of events (produced particles) at the time $t,\ n(0) = 0$. The probability of event $n(t),\ P(t, n)$, is defined from the following motion equation

$$P_t \equiv \frac{\partial P(t,n)}{\partial t} = r(P(t,n-1) - P(t,n)),\ n \geq 1$$
$$P_t(t,0)) = -rP(t,0),$$
$$P(t,n) = 0,\ n < 0, \quad (333)$$

so

$$P(t,0) \equiv P_0(t) = e^{-rt},$$
$$P(t,n) = Q(t,n)P_0(t),$$
$$Q_t(t,n) = rQ(t,n-1),\ Q(t,0) = 1. \quad (334)$$

To solve the equation for Q, we invent its generating function

$$F(t,h) = \sum_{n \geq 0} h^n Q(t,n), \tag{335}$$

and solve corresponding equation

$$F_t = rhF,\ F(t,h) = e^{rth} = \sum h^n \frac{(rt)^n}{n!},\ Q(t,n) = \frac{(rt)^n}{n!}, \tag{336}$$

so

$$P(t,n) = e^{-rt} \frac{(rt)^n}{n!} \tag{337}$$

is the Poisson distribution.
If we compare this distribution with (269), we identify $< n >= rt$, as if we have a free particle motion with velocity r and the distance is the mean multiplicity. This way we have a connection between n-dimension of the multiplicity and the usual dimension of trajectory.

As the equation gives right solution, its generalization may give more general distribution, so we will generalize the equation (333). For this, we put the equation in the closed form

$$P_t(t,n) = r(e^{-\partial_n} - 1)P(t,n)$$
$$= \sum_{k\geq 1} D_k \partial^k P(t,n), \ D_k = (-1)^k \frac{r}{k!}, \tag{338}$$

where the $D_k, \ k \geq 1$, are generalized diffusion coefficients.
For other values of the coefficients, we will have other distributions.

For mean square deviation of the trajectory we have

$$< (x-\bar{x})^2 >=< x^2 > - < x >^2 \equiv D(x)^2 \sim t^{2/d_f}, \qquad (339)$$

where d_f is fractal dimension. For smooth classical trajectory of particles we have $d_f = 1$; for free stochastic, Brownian, trajectory, all diffusion coefficients are zero but D_2, we have $d_f = 2$. In the case of Poisson process we have,

$$D(n)^2 =< n^2 > - < n >^2 \sim t, \ d_f = 2. \qquad (340)$$

In the case of the NBD and KNO distributions

$$D(n)^2 \sim t^2, \ d_f = 1. \qquad (341)$$

As we have seen, rasing k, KNO reduce to the Poisson, so we have a dimensional (phase) transition from the phase with dimension 1 to the phase with dimension 2. It is interesting, if somehow this phase transition is connected to the other phase transitions in strong interaction processes. For the Poisson distribution GF is solution of the following equation,

$$\dot{F} = -r(1-h)F, \qquad (342)$$

For the NBD corresponding equation is

$$\dot{F} = \frac{-r(1-h)}{1+\frac{rt}{k}(1-h)}F = -R(t)F, \ \ R(t) = \frac{r(1-h)}{1+\frac{rt}{k}(1-h)}. \tag{343}$$

If we change the time variable as $t = T^{d_f}$, we reduce the dispersion low from general fractal to the NBD like case. Corresponding transformation for the evolution equation is

$$F_T = -d_f T^{d_f-1} R(T^{d_F})F, \tag{344}$$

we ask that this equation coincides with NBD motion equation, and define rate function $R(T)$

$$d_f T^{d_f-1} R(T^{d_F}) = \frac{r(1-h)}{1+\frac{rT}{k}(1-h)}, \tag{345}$$

now the following equation defines a production processes with fractal dimension d_F

$$F_t = -R(t)F, \ \ R(t) = \frac{r(1-h)}{d_F t^{\frac{d_F-1}{d_F}}(1+\frac{rt^{1/d_F}}{k}(1-h))} \tag{346}$$

Now we would like to consider a model of multiparticle production based on the d-dimensional sphere, and (try to) motivate the values of the NBD parameter k. The volum of the d-dimensional sphere with radius r, in units of hadron size r_h is

$$v(d,r) = \frac{\pi^{d/2}}{\Gamma(d/2+1)}(\frac{r}{r_h})^d \tag{347}$$

Note that,

$$v(0,r) = 1,\ v(1,r) = 2\frac{r}{r_h},$$
$$v(-1,r) = \frac{1}{\pi}\frac{r_h}{r} \tag{348}$$

If we identify this dimensionless quantity with corresponding coulomb energy formula,

$$\frac{1}{\pi} = \frac{e^2}{4\pi}, \tag{349}$$

we find $e = \pm 2$.

For less then -1 even integer values of d, and $r \neq 0,\ v = 0$. For negative odd integer $d = -2n + 1$

$$v(-2n+1, r) = \frac{\pi^{-n+1/2}}{\Gamma(-n+3/2)}(\frac{r_h}{r})^{2n-1},\ n \geq 1, \tag{350}$$

$$v(-3, r) = -\frac{1}{2\pi^2}(\frac{r_h}{r})^3,\ v(-5, r) = \frac{3}{4\pi^3}(\frac{r_h}{r})^5 \tag{351}$$

Note that,

$$v(2, r)v(3, r)v(-5, r) = \frac{1}{\pi},\ v(1, r)v(2, r)v(-3, r) = -\frac{1}{\pi} \tag{352}$$

We postulate that after collision,it appear intermediate state with almost spherical form and constant energy density. Than the radius of the sphere rise, dimension decrease, volume remains constant. At the last moment of the expansion, when the crossection of the one dimensional sphere - string become of order of hadron size, hadronic string divide in k independent sectors which start to radiate hadrons with geometric (Boze-Einstein) distribution, so all of the string final state radiate according to the NBD distribution.

So, from the volume of the hadronic string,

$$v = \pi(\frac{r}{r_h})^2 \frac{l}{r_h} = \pi k, \tag{353}$$

we find the NBD parameter k,

$$k = \frac{\pi^{d/2-1}}{\Gamma(d/2+1)} (\frac{r}{r_h})^d \tag{354}$$

Knowing, from experimental date, the parameter k, we can restrict the region of the values of the parameters d and r of the primordial sphere (PS),

$$r(d) = (\frac{\Gamma(d/2+1)}{\pi^{d/2-1}} k)^{1/d} r_h,$$
$$r(3) = (\frac{3}{4}k)^{1/3} r_h, \ r(2) = k^{1/2} r_h, \ r(1) = \frac{\pi}{2} k r_h \tag{355}$$

If the value of $r(d)$ will be a few r_h, the matter in the PS will be in the hadronic phase. If the value of r will be of order $10r_h$, we can speak about deconfined, quark-gluon, Glukvar, phase. From the formula (355), we see, that to have for the r, the value of order $10r_h$, in $d = 3$ dimension, we need the value for k of order 1000, which is not realistic.

So in our model, we need to consider the lower than one, fractal, dimensions. It is consistent with the following intuitive picture. Confined matter have point-like geometry, withe dimension zero. Primordial sphere of Glukvar have nonzero fractal dimension, which is less than one,

$$k = 3,\ r(0.7395)/r_h = 10.00,$$
$$k = 4,\ r(0.8384)/r_h = 10.00 \quad (356)$$

From the experimental data we find the parameter k of the NBD as a function of energy, $k = k(s)$. Then, by our spherical model, we construct fractal dimension of the Glukvar as a function of $k(s)$.
If we suppose that radius of the primordial sphere r is of order (or less) of r_h. Than we will have higher dimensional PS, e.g.

d	r/r_h	k
3	1.3104	3.0002
4	1.1756	3.0003
6	1.1053	2.9994
8	1.1517	3.9990

With extra dimensions gravitation interactions may become strong at the LHC energies,

$$V(r) = \frac{m_1 m_2}{m^{2+d}} \frac{1}{r^{1+d}} \tag{357}$$

If the extra dimensions are compactified with(in) size R, at $r >> R$,

$$V(r) \simeq \frac{m_1 m_2}{m^2 (mR)^d} \frac{1}{r} = \frac{m_1 m_2}{M_{Pl}^2} \frac{1}{r}, \tag{358}$$

where (4-dimensional) Planck mass is given by

$$M_{Pl}^2 = m^{2+d} R^d, \tag{359}$$

so the scale of extra dimensions is given as

$$R = \frac{1}{m} (\frac{M_{Pl}}{m})^{\frac{2}{d}} \tag{360}$$

If we take $m = 1TeV, (GeV^{-1} = 0.2fm)$

$$R(d) = 2 \cdot 10^{-17} \cdot (\frac{M_{Pl}}{1TeV})^{\frac{2}{d}} \cdot cm,$$
$$R(1) = 2 \cdot 10^{15} cm,$$
$$R(2) = 0.2 \ cm \ !$$
$$R(3) = 10^{-7} cm \ !$$
$$R(4) = 2 \cdot 10^{-9} cm,$$
$$R(6) \sim 10^{-11} cm \qquad (361)$$

Note that lab measurements of $G_N (= 1/M_{Pl}^2, M_{Pl} = 1.2 \cdot 10^{19} GeV)$ have been made only on scales of about 1 cm to 1 m; 1 astronomical unit(AU) (mean distance between Sun and Earth) is $1.5 \cdot 10^{13} cm$; the scale of the periodic structure of the Universe, $L = 128 Mps \simeq 4 \cdot 10^{26} cm$. It is curious which (small) value of the extra dimension corresponds to L?

$$\begin{aligned} d &= 2\frac{\ln \frac{M_{Pl}}{m}}{\ln(mL)} = 0.74, \ m = 1TeV, \\ &= 0.81, \ m = 100GeV, \\ &= 0.07, \ m = 10^{17} GeV. \end{aligned} \qquad (362)$$

Motion equations of physics (applied mathematics in general) connect different observable quantities and reduce the number of independently measurable quantities. More fundamental equation contains less number of independent quantities. When (before) we solve the equations, we invent dimensionless invariant variables, than one solution can describe all of the class of phenomena.
In the z - Scaling (zS) approach to the inclusive multiparticle distributions (MPD) (see, e.g. [Tokarev, Zborovsky, 2007a]), different inclusive distributions depending on the variables $x_1, ...x_n$, are described by universal function $\Psi(z)$ of fractal variable z,

$$z = x_1^{-\alpha_1}...x_n^{-\alpha_n}. \tag{363}$$

It is interesting to find a dynamical system which generates this distributions and describes corresponding MPD.

We can find a good function if we know its derivative. Let us consider the following RD like equation

$$z\frac{d}{dz}\Psi = V(\Psi),$$
$$\int_{\Psi(z_0)}^{\Psi(z)} \frac{dx}{V(x)} = \ln\frac{z}{z_0} \quad (364)$$

In $x-$representation,

$$\ln z = -\sum_{k=1}^{n}\alpha_k \ln x_k,\ \delta_z = z\frac{d}{dz} = -\sum_k \frac{\delta_k}{n_h\alpha_k},$$
$$\sum_{k=1}^{n}\frac{x_k}{n_h\alpha_k}\frac{\partial}{\partial x_k}\Psi(x_1, ..., x_n) + V(\Psi) = 0, \quad (365)$$

e.g.

$$z = \delta_z z = -\sum_{k=1}^{n}\frac{x_k}{n_h\alpha_k}\frac{\partial}{\partial x_k}x_1^{-\alpha_1}...x_n^{-\alpha_n} = z,\ n_h = n. \quad (366)$$

In the case of NBD GF (275), we have

$$n = 2,\ x_1 = k,\ x_2 =< n >,\ \alpha_1 = \alpha_2 = 1,\ n_h = 1,$$
$$\Psi = F,\ V(\Psi) = -\Psi \ln \Psi. \qquad (367)$$

In the case of the $z-$scaling, [Tokarev, Zborovsky, 2007a],

$$n = 4,\ x_3 = y_a,\ x_4 = y_b,$$
$$\alpha_1 = \delta_1,\ \alpha_2 = \delta_2,\ \alpha_3 = \varepsilon_a,\ \alpha_4 = \varepsilon_b, n_h = 4, \qquad (368)$$

for infinite resolution, $\alpha_n = 1,\ n = 1, 2, 3, 4$. In z variable the equation for Ψ has universal form. In the case of $n = 2, \alpha_1 = \alpha_2 = 1,\ n_h = 1$, we find that $V(\Psi) = -\Psi \ln \Psi$,

$$z\frac{d}{dz}\Psi(z) = -\Psi \ln \Psi,$$
$$\Psi(z) = e^{c/z} = (\Psi(z_0)^{z_0})^{\frac{1}{z}} = \Psi(z_0)^{\frac{z_0}{z}},$$
$$c = z_0 \ln \Psi(z_0) < 0,\ z \in (0, \infty),\ \Psi(z) \in (0, 1). \qquad (369)$$

Note that the fundamental equation is invariant with respect to the scale transformation $z \to \lambda z$, but the solution is not, the scale transformation transforms one solution into another solution. This is an example of the spontaneous breaking of the (scale) symmetry by the states of the system.

As a dimensionless physical quantity $\Psi(z)$ may depend only on the running coupling constant $g(\tau),\ \tau = \ln z/z_0$

$$z\frac{d}{dz}\Psi = \dot{\Psi} = \frac{d\Psi}{dg}\beta(g) = U(g) = U(f^{-1}(\Psi)) = V(\Psi),$$
$$\Psi(\tau) = f(g(\tau)),\ g = f^{-1}(\Psi(\tau)) \tag{370}$$

According to the paper [Tokarev, Zborovsky, 2007a], for high values of z, $\Psi(z) \sim z^{-\beta}$; for small z, $\Psi(z) \sim$ const.
So, for high z,

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = -\beta\Psi(z); \tag{371}$$

for smaller values of z, $\Psi(z)$ rise and we expect nonlinear terms in $V(\Psi)$,

$$V(\Psi) = -\beta\Psi + \gamma\Psi^2. \tag{372}$$

With this function, we can solve the equation for Ψ(see appendix) and find

$$\Psi(z) = \frac{1}{\frac{\gamma}{\beta} + cz^{\beta}}. \tag{373}$$

RD equation of the z-Scaling,

$$z\frac{d}{dz}\Psi(z) = V(\Psi),\ V(\Psi) = V_1\Psi + V_2\Psi^2 + ... + V_n\Psi^n + ... \qquad (374)$$

can be reparametrized,

$$\Psi(z) = f(\psi(z)) = \psi(z) + f_2\psi^2 + ... + f_n\psi^n + ...$$
$$z\frac{d}{dz}\psi(z) = v(z) = v_1\psi(z) + v_2\psi^2 + ... + v_n\psi^n + ...$$
$$(v_1\psi(z) + v_2\psi^2 + ... + v_n\psi^n + ...)(1 + 2f_2\psi + ... + nf_n\psi^{n-1} + ...)$$
$$= V_1(\psi + f_2\psi^2 + ... + f_n\psi^n + ...)$$
$$+V_2(\psi^2 + 2f_2\psi^3 + ...) + ... + V_n(\psi^n + nf_2\psi^{n+1} + ...) + ...$$
$$= V_1\psi + (V_2 + V_1f_2)\psi^2 + (V_3 + 2V_2f_2 + V_1f_3)\psi^3 +$$
$$... + (V_n + (n-1)V_{n-1}f_2 + ... + V_1f_n)\psi^n + ...$$
$$v_1 = V_1,$$
$$v_2 = V_2 - f_2V_1,$$
$$v_3 = V_3 + 2V_2f_2 + V_1f_3 - 2f_2v_2 - 3f_3v_1 = V_3 + 2(f_2^2 - f_3)V_1, ...$$
$$v_n = V_n + (n-1)V_{n-1}f_2 + ... + V_1f_n - 2f_2v_{n-1} - ... - nf_nv_1, \qquad (375)$$

so, by reparametrization, we can change any coefficient of potential V but V_1.

We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{V_2}{V_1},\ f_3 = \frac{V_3}{2V_1} + f_2^2 = \frac{V_3}{2V_1} + (\frac{V_2}{V_1}^2),\ ...$$
$$f_n = \frac{V_n + (n-1)V_{n-1}f_2 + ... + 2V_2 f_{n-1}}{(n-1)V_1}, ... \tag{376}$$

We will consider the case when only one of higher coefficient is nonzero and give explicit form of the solution Ψ.

Let us consider more general potential V

$$z\frac{d}{dz}\Psi = V(\Psi) = -\beta\Psi(z) + \gamma\Psi(z)^{1+n} \tag{377}$$

Corresponding solution for Ψ is

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz^{n\beta})^{\frac{1}{n}}} \tag{378}$$

More general solution contains three parameters and may better describe the data of inclusive distributions.

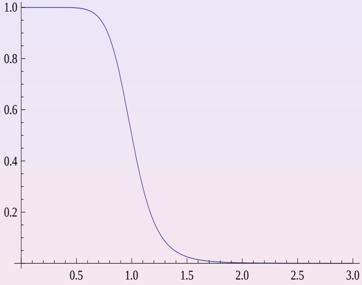

Figure: z-scaling distribution (378), $\Psi(z, 9, 9, 1, 1)$

In the case of $n = 1$ we reproduce the previous solution.

Another "natural" case is $n = 1/\beta$,

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz)^{\beta}} \tag{379}$$

In this case, we can absorb (interpret) the combined parameter by shift and scaling

$$z \to \frac{1}{c}(z - \frac{\gamma}{\beta}) \tag{380}$$

Another interesting point of view is to predict the value of β

$$\beta = \frac{1}{n} = 0.5; 0.33; 0.25; 0.2; ..., \;\; n = 2, 3, 4, 5, ... \tag{381}$$

For experimentally suggested value $\beta \simeq 9, n = 0.11$

In the case of $n=-\varepsilon,\ \beta=\gamma=1/\varepsilon,\ c=\varepsilon k$, we will have

$$V(\Psi)=-\Psi\ln\Psi,\ \Psi(z)=e^{\frac{k}{z}} \tag{382}$$

This form of $\Psi-$function interpolates between asymptotic values of Ψ and predicts its behavior in the intermediate region.
The three parameter function is restricted by the normalization condition

$$\int_0^\infty \Psi(z)dz=1,$$
$$B(\frac{\beta-1}{\beta n},\frac{1}{\beta n})=(\frac{\beta}{\gamma})^{\frac{\beta-1}{\beta n}}\frac{\beta n}{c^{\beta n}}, \tag{383}$$

so remains only two free parameter. When $\beta n=1$, we have

$$c=(\beta-1)(\frac{\beta}{\gamma})^{\beta-1} \tag{384}$$

If $\beta n=1$ and $\beta=\gamma$, than $c=\beta-1$.
In general

$$c^{\beta n}=(\frac{\beta}{\gamma})^{\frac{\beta-1}{\beta n}}\frac{\beta n}{B(\frac{\beta-1}{\beta n},\frac{1}{\beta n})} \tag{385}$$

RD equation of the z-scaling (377), after substitution,

$$\Psi(z) = (\varphi(z))^{\frac{1}{n}}, \tag{386}$$

reduce to the $n = 1$ case with scaled parameters,

$$\dot{\varphi} = -\beta n\varphi + \gamma n\varphi^2, \tag{387}$$

this substitution could be motivated also by the structure of the solution (378),

$$\Psi(z, \beta, \gamma, n, c) = \Psi(z, \beta n, \gamma n, 1, c)^{\frac{1}{n}} = \Psi(z, \beta, \gamma, \beta n, c)^{\beta}. \tag{388}$$

General RD equation takes form

$$\dot{\varphi} = nv_1\varphi + nv_2\varphi^{1+\frac{1}{n}} + nv_3\varphi^{1+\frac{2}{n}} + ... + nv_n\varphi^2 + nv_{n+1}\varphi^{2+\frac{1}{n}} + ... \tag{389}$$

The dimension of the space(-time) is the model dependent concept. E.g. for the fundamental bosonic string model (in flat space-time) the dimension is 26; for superstring model the dimension is 10 [Kaku, 2000].
Let us imagine that we have some action-functional formulation with the fundamental motion equation

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = V(\Psi) = -\beta\Psi + \gamma\Psi^{1+n}. \tag{390}$$

Than, the corresponding Lagrangian contains the following mass and interaction parts

$$-\frac{\beta}{2}\Psi^2 + \frac{\gamma}{2+n}\Psi^{2+n} \tag{391}$$

The action gives renormalizable (effective quantum field theory) model when

$$d+2 = \frac{2N}{N-2} = \frac{2(2+n)}{n} = 2 + \frac{4}{n} = 2 + 4\beta, \tag{392}$$

so, measuring the parameter β inside hadronic and nuclear matters, we find corresponding (fractal) dimension.

From fundamental equation we obtain

$$(z\frac{d}{dz})^2\Psi \equiv \ddot{\Psi} = V'(\Psi)V(\Psi) = \frac{1}{2}(V^2)'$$
$$= \beta^2\Psi - \beta\gamma(n+2)\Psi^{n+1} + \gamma^2(n+1)\Psi^{2n+1} \qquad (393)$$

Corresponding action Lagrangian is

$$L = \frac{1}{2}\dot{\Psi}^2 + U(\Psi),\ U = \frac{1}{2}V^2 = \frac{1}{2}\Psi^2(\beta - \gamma\Psi^n)^2$$
$$= \frac{\beta^2}{2}\Psi^2 - \beta\gamma\Psi^{2+n} + \frac{\gamma^2}{2}\Psi^{2+2n} \qquad (394)$$

This potential, $-U$, has two maximums, when $V = 0$, and minimum, when $V' = 0$, at $\Psi = 0$ and $\Psi = (\beta/\gamma)^{1/n}$, and $\Psi = (\beta/(n+1)\gamma)^{1/n}$, correspondingly.

We define time-space-scale field $\Psi(t, x, \eta)$, where $\eta = \ln z-$ is scale coordinate variable, with corresponding action functional

$$A = \int dtd^d x d\eta (\frac{1}{2} g^{ab} \partial_a \Psi \partial_b \Psi + U(\Psi)) \tag{395}$$

The renormalization constraint for this action is

$$N = 2 + 2n = \frac{2(2+d)}{2+d-2} = 2 + \frac{4}{d},\ dn = 2,\ d = 2/n = 2\beta. \tag{396}$$

So we have two models for spase-time dimension, (392) and (396),

$$d_1 = 4\beta;\ d_2 = 2\beta \tag{397}$$

The coordinate η characterise (multiparticle production) physical process at the (external) space-time point (x,t). The dimension of the space-time inside hadrons and nuclei, where multiparticle production takes place is

$$d + 1 = 1 + 2\beta \tag{398}$$

Note that this formula reminds the dimension of the spin s state, $d_s = 2s + 1$. If we take $\beta (= s) = 0; 1/2; 1; 3/2; 2; ...$ We will have $d + 1 = 1; 2; 3; 4; 5; ...$

Note that as we invent Ψ as a real field, we ought to take another normalization

$$\int d^d x |\Psi|^2 = 1 \qquad (399)$$

for the solutions of the motion equation. This case extra values of the parameter β is possible, $\beta > d/2$.

We can take a renormdynamic scheme were $\Psi(g)$ is running coupling constant. The variable z is a formation length and has dimension -1, RD equation for Ψ in φ^3_D model is

$$z\frac{d}{dz}\Psi = \frac{6-D}{2}\Psi + \gamma\Psi^2 \tag{400}$$

$$\beta = \frac{D-6}{2} \tag{401}$$

For high values of z, $\beta = 9$, so $D = 24$. This value of D corresponds to the physical (transverse) degrees of freedom of the relativistic string, to the dimension of the external space in which this relativistic string lives. This is also the number of the quark - lepton matter degrees of freedom, $3 \cdot 6 + 6$. So, in these high energy reactions we measured the dimension of the space-time and matter and find the values predicted by relativistic string and SM. For lower energies, in this model, D monotonically decrees until $D = 6$, than the model (may) change form on the $\varphi^4_D, \beta = D - 4$. So we have two scenarios of behavior. In one of them the dimension of the space-time inside hadrons has value 6 and higher. In another the dimension is 4 and higher.

Perturbative QCD indicates that we have a fixed point of RD in dimension slightly higher than 4, and ordinary to hadron phase transition corresponds to the dimensional phase transition from slightly lower than 4, in QED, to slightly higher than 4 dimension in QCD. In general scalar field model φ_D^n,

$$\beta = -d_g = \frac{nD}{2} - n - D. \qquad (402)$$

For φ^3 model, $\beta = 9$ corresponds to $D = 24$. In tha case of the $O(N)-$sigma model

$$\beta = D - 2, \qquad (403)$$

For the experimental value of $\beta = 9$, we have the dimension of the $M-$theory, $D = 11!$

One of the characteristic features in every high energy collision experiment is the production of large numbers of secondaries (mostly pions). From the very beginning of the history of the multiparticle production processes, it was realized that a possible way to treat them was to employ some sort of statistical approach [Heisenberg, 1949],[Fermi, 1936],[Pomeranchuk, 1951]. In the statistical bootstrap model proposed by Hagedorn [Hagedorn, 1965], the exponential growth of the number of hadronic resonances with mass is one of the most fundamental issues

$$\frac{d^3\sigma}{dp^3} = N \int dm\rho(m)e^{-\beta\sqrt{p_l^2+p_t^2+m^2}}, \tag{404}$$

where $\rho(m)$ denotes the density of resonances given by

$$\rho(m) = \frac{e^{\beta_H m}}{(m^2+m_0^2)^{\frac{5}{4}}},\ \beta_H = \frac{1}{k_B T_H}, \tag{405}$$

T_H, the Hagedorns temperature, is a parameter to be deduced from data on resonance production. The other parameter is $\beta = 1/(k_B T)$, with T explicitly governing the observed energy distribution and therefore identified with the temperature of the hadronizing system. In the followings we put $k_B = 1$.

One of the aims in the study of multiparticle production processes is therefore the best possible estimation of this quantity. To this end we would like to investigate the measured transverse momentum (p_t) distributions integrated over longitudinal degrees of freedom,

$$\frac{d\sigma}{2\pi p_t dp_t} = N \int dm\rho(m) m_t K_1(\beta m_t), \tag{406}$$

where

$$\int_0^\infty dx e^{-\sqrt{x^2+a^2}} = aK_1(a) \tag{407}$$

and for modified Bessel functions,

$$K_a(x) = \int_0^\infty dt \cosh(at) e^{-x\cosh t} \tag{408}$$

This simple formula can explain the RHIC data only in the limited range of transverse momenta, namely for $p_t < 6\ GeV/c$, [Biyajima et al, 2005]. For larger values of p_t data exhibit a power-like tail.

The Main thermodynamic relation (MTR) and the von Neumann-Shannon entropy

We call MTR the following relation

$$F = E - TS. \tag{409}$$

Let us obtain MTR. From statistical sum we have

$$Z = \sum_n e^{-\beta E_n} = e^{-\beta F},\ \beta = \frac{1}{T}$$
$$E = \frac{\sum_n E_n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} = -\frac{\partial \ln Z}{\partial \beta} = \frac{\partial (\beta F)}{\partial \beta} = F - T\frac{\partial F}{\partial T},$$
$$F = E + T\frac{\partial F}{\partial T}. \tag{410}$$

The von Neumann-Shannon entropy is defined as

$$S = -\sum_n p_n \ln p_n,$$
$$\sum_n p_n = 1,\ 0 \le p_n \le 1. \tag{411}$$

For Gibbs weights-probabilities,

$$p_n = \frac{e^{-\beta E_n}}{\sum_m e^{-\beta E_m}} = e^{-\beta(E_n - F)},\ \beta = \frac{1}{T}$$
$$S = -\sum_n p_n \beta(F - E_n) = \beta(E - F),$$
$$F = E - TS,\ E = \sum_n E_n p_n, \qquad (412)$$

so, we obtain MTR (409) and using (410), we have

$$S = -\frac{\partial F}{\partial T}. \qquad (413)$$

The von Neumann-Shannon entropy has the following additive property

$$S(A+B) = S(A) + S(B), \qquad (414)$$

when the subsystems A and B of the system $A+B$ are independent, i.e. $p(A+B) = p(A)p(B) = p_1 p_2$.

Indeed,

$$S(A+B) = -\sum_{n,m} p_{1n}p_{2m}(\ln p_{1n} + \ln p_{2m}) = S(A) + S(B),$$
$$\sum_n p_{1n} = \sum_n p_{2n} = 1. \quad (415)$$

Let us find minimum and maximum values of the entropy and corresponding distributions. The entropy is nonnegative. For the finite number of the levels, $E_n,\ n = 1, 2, ..., N$, to the minimum values corresponds all of the values $p_n = \epsilon \to 0$, but one, which is $p_1 = 1 - (N-1)\epsilon \to 1$, by constraint. For that values, $S = 0$.
To the maximum of the entropy $S = \ln N$ corresponds equal partition $p_n = \frac{1}{N}$. Indeed, let us find maximum of the following function

$$f = -\sum_n p_n \ln p_n + \lambda\varphi(p_n),\ \varphi(p_n) = \sum_n p_n - 1,$$
$$\frac{\partial f}{\partial p_n} = -\ln p_n - 1 + \lambda = 0 \Rightarrow p_n = e^{\lambda - 1} = p,$$
$$\frac{\partial f}{\partial \lambda} = \sum_n p_n - 1 = 0 \Rightarrow p_n = p = \frac{1}{N},\ \lambda = 1 - \ln N,$$
$$S = \ln N. \quad (416)$$

For the simplest composed system, $N = 2$ and maximum $S = \ln 2$. Now we can define p_n as a monotone function of energy $p_n = p_n(\beta E_n),\ p_n(0) = \frac{1}{N};\ p_1(\infty) = 1,\ p_n(\infty) = 0,\ n = 2, 3, ..., N$. The Gibbs weights-probabilities fulfils these conditions. If a system A with energy E_A reduce to two independent subsystems (B, E_B) and $(C, E_C) : p(E_A) = p(E_B)p(E_C),\ E_A = E_B + E_C$, than definitely $p(E) = \beta e^{-\beta E}-$ the Gibbs distribution.

The Rényi entropies are defined for an arbitrary real parameter q as [Rényi, 1970]

$$S_q^r = \frac{\ln \sum_n p_n^q}{1-q},$$
$$\sum_n p_n = 1,\ 0 \le p_n \le 1,$$
$$S_1^r = \lim_{q\to 1} \frac{\ln \sum_n p_n^q}{1-q} = -\sum_n p_n \ln p_n = S \tag{417}$$

The Rényi entropies are additive. Indeed,

$$S_q^r(A) + S_q^r(B) = \frac{\ln \sum_n p_{1n}^q + \ln \sum_m p_{2m}^q}{1-q} = \frac{\ln \sum_{nm} (p_{1n} p_{2m})^q}{1-q}$$
$$= S_q^r(A+B) \tag{418}$$

The Tsallis entropy [Tsallis,1988-2004] is the following one parameter deformation

$$S_q^t = \frac{1 - \sum_n p_n^q}{1 - q},$$
$$\sum_n p_n = 1,\ 0 \le p_n \le 1, \tag{419}$$

of the the von Neumann-Shannon entropy

$$S_1^t = \lim_{q \to 1} \frac{1 - \sum_n p_n^q}{1 - q} = -\sum_n p_n \ln p_n = S \tag{420}$$

The Tsallis entropy is not additive. We have

$$S_q^t(A + B) = S_q^t(A) + S_q^t(B) - (1 - q) S_q^t(A) S_q^t(B) \tag{421}$$

The Tsallis distribution $p(a)$ of some variable a is defined as

$$p_q(a) = (2 - q)(1 + (q - 1)a)^{\frac{1}{1-q}},\ \int_0^\infty da p_q(a) = 1. \tag{422}$$

In the limit $q \to 1$ and $a = \beta E$, the Tsallis distribution becomes the usual exponential (Boltzmann-Gibbs) distribution,

$$p(E) = \beta e^{-\beta E}, \int_0^\infty dEp(E) = 1. \tag{423}$$

Note that, when $q - 1 = 1/k$ and $a =< n > (1 - h)$ the Tsallis distribution reduce to the generating function of the NBD

$$p_q(a) = (1 - \frac{1}{k})(1 + \frac{< n >}{k}(1 - h))^{-k}, \ q = 1 + 1/k, \ a =< n > (1 - h), \ k$$

In our interpretation of the parameter k as the number of the independent radiating sources, it is positive integer equal to the number of sources. In a recent description of the multiparticle production spectrum at LHC, [Wong, Wilk, 2012], the value $q = 1.172$ were identified. It corresponds to the value $k = 5.814$

We assume that $k = 6$ and propose to find q from the fit to the data. Corresponding value from the Tsallis distribution is $q = 1.1667$.

The obvious question is: to what physics corresponds the value $k = 6$. And again, obvious answer is: the value is the number of constituent valence quarks of the two protons in the initial state of the multiparticle production processes.

Let us calculate the Tsallis entropy of the following distribution

$$p_n = N(1+\frac{E_n}{kT})^{-k},\ k=\frac{1}{1-q},\ \beta=\frac{1}{T},\ \sum_n p_n = 1, \tag{425}$$

$$S_q^t = \frac{1-\sum_n p_n^q}{1-q} = \frac{1-\sum p_n N^{q-1}(1+(1-q)\beta E_n)}{1-q}$$
$$= \frac{1-N^{q-1}+N^{q-1}(1-q)\beta E}{1-q} = \beta_N(E-F),$$
$$F = E - T_N S,\ \beta_N = \beta N^{q-1},\ F = \frac{N^{q-1}-1}{(q-1)\beta_N},$$
$$N^{-1} = Z = \sum_n (1+\frac{\beta E_n}{k})^{-k} = \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1} e^{-t} \sum_n e^{-t\beta E_n/k}$$
$$= \int_0^\infty db f(b) \sum_n e^{-bE_n} = (1+(1-q)\beta_N F)^{\frac{1}{q-1}},$$
$$f(b) = \frac{(kT)^k}{\Gamma(k)} b^{k-1} e^{-kTb} \tag{426}$$

So, we have a state with a mixture of the systems with different temperatures with Gamma distribution named as superstatistics [Beck, Cohen, 2003].
For the Rényis entropies of the same distribution,

$$S_q^r = \frac{\ln \sum_n p_n N^{q-1}(1+(1-q)\beta E_n)}{1-q} = \frac{\ln[N^{q-1}(1+(1-q)\beta E)]}{1-q}$$
$$= -\ln N + \frac{\ln(1+(1-q)\beta E)}{1-q} = \beta(E_r - F),$$
$$F = E_r - TS_q^r,\ N = e^{\beta F},\ E_r = \frac{\ln(1+(1-q)\beta E)}{(1-q)\beta},\ T = \beta^{-1} \quad (427)$$

From the definition of β_N and F,

$$\beta_N = N^{q-1}\beta = \frac{\beta}{1+(1-q)\beta_N F},\ N = (1+(1-q)\beta_N F)^{-\frac{1}{q-1}} \quad (428)$$

we find

$$F = \frac{\beta - \beta_N}{\beta_N^2(1-q)},\ \beta_N = \frac{-1 \pm \sqrt{1+4(1-q)\beta F}}{2(1-q)F} \quad (429)$$

To the positive values of β_N, β and F corresponds two states with

$$\beta_N = \frac{1 \pm \sqrt{1-4(q-1)\beta F}}{2(q-1)F},\ 1 < q = 1+\frac{1}{k} < 1+\frac{1}{4\beta F},\ k > 4\beta F \quad (430)$$

and one state

$$\beta_N = \frac{\sqrt{1+4(1-q)\beta F}-1}{2(1-q)F},\ q < 1 \quad (431)$$

The higher temperature phase for $q > 1$, in the classical limit $q \to 1$, reduce to the classical temperature,

$$\beta_N = \beta(1+(q-1)\beta F + \ldots) \quad (432)$$

For the low-temperature phase,

$$T_N = \frac{2(q-1)F}{2-2(q-1)\beta F + ...} = (q-1)F(1+(q-1)\beta F + ...)$$
$$= (q-1)F + (q-1)^2\beta F^2 + ... \quad (433)$$

For $pp-$multiparticle productions, we have seen that $q-1 = 1/k, \ k = 6$, so in that processes

$$T_N = \frac{F}{6}(1 + \frac{F}{6}\beta + ...) \quad (434)$$

Let us calculate the von Neumann-Shannon entropy

$$S = -\sum_n p_n \ln p_n,$$
$$\sum_n p_n = 1,\ 0 \le p_n \le 1, \tag{435}$$

for fermi and bose oscillators.
The energy spectrum of the bose-oscillator is

$$E_n = \hbar\omega(n + \frac{1}{2}),\ n = 0, 1, 2, ... \tag{436}$$

Corresponding statistical sum is

$$Z_B = \sum_{n\ge 0} e^{-a(n+\frac{1}{2})} = \frac{e^{-\frac{a}{2}}}{1 - e^{-a}} = \frac{1}{2\sinh\frac{a}{2}},\ a = \frac{\hbar\omega}{kT} \tag{437}$$

For fermi-oscillator we have

$$E_n = \hbar\omega(n - \frac{1}{2}),\ n = 0, 1$$
$$Z_F = \sum_{n=0}^{1} e^{-a(n-\frac{1}{2})} = e^{\frac{a}{2}} + e^{-\frac{a}{2}} = 2\cosh\frac{a}{2} \quad (438)$$

For super-oscillator system composed from one fermi- and one bose-oscillators,

$$Z = Z_B Z_F = e^{-\beta F} = \coth\frac{a}{2} = 1 + 2e^{-\frac{a}{2}} + ...,\ a = \frac{\hbar\omega}{kT} \gg 1 \quad (439)$$

For fermi oscillator

$$p_0 = \frac{e^{\frac{a}{2}}}{e^{\frac{a}{2}} + e^{-\frac{a}{2}}} = \frac{1}{1 + e^{-a}},\ p_1 = \frac{e^{-\frac{a}{2}}}{e^{\frac{a}{2}} + e^{-\frac{a}{2}}} = \frac{1}{e^a + 1},\ p_0 + p_1 = 1$$
$$S_F(a) = \frac{\ln(1 + e^{-a})}{1 + e^{-a}} + \frac{\ln(1 + e^a)}{1 + e^a} = \frac{\ln(1 + q)}{1 + q} + \frac{\ln(1 + q^{-1})}{1 + q^{-1}},$$
$$0 \leq S_F \leq \ln 2 \quad (440)$$

S_F is symmetric under the dual transformation:
$a \leftrightarrow -a,\ q \leftrightarrow q^{-1};\ S_F(0) = \ln 2,\ S_F(\infty) = 0$

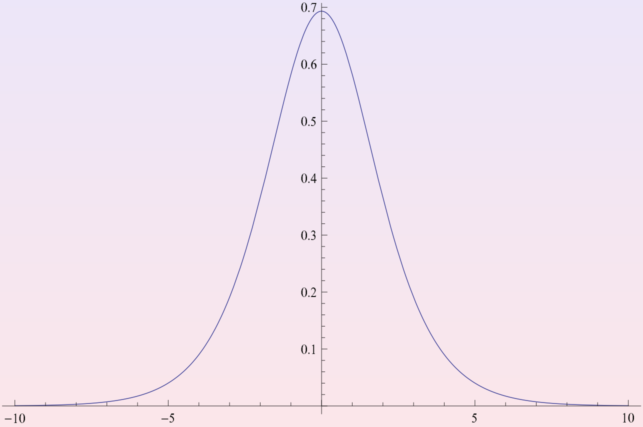

Figure: $S_F(a)$-entropy distribution, $S_F(0) = \ln 2 = 0.693147$

Fermi and bose statistics

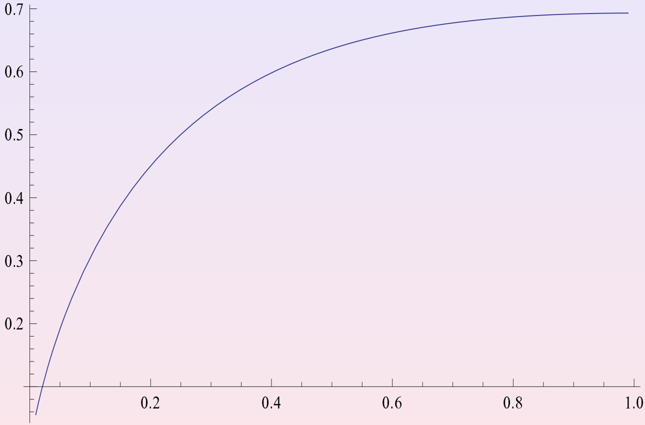

Figure: $S_F(q)$-entropy distribution, $S_F(1) = \ln 2 = 0.693147$

Now, let us calculate the entropy for bose oscillator,

$$
\begin{aligned}
& p_n = (1-q)q^n, \; q = e^{-a} \\
& S_B = -\sum_{n=0}^{\infty} p_n \ln p_n = \ln \frac{1}{1-q} + \ln \frac{1}{q} < n >, \\
& < n >= \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} (1-q) q \frac{d}{dq} q^n \\
& = q(1-q) \frac{d}{dq} \frac{1}{1-q} = \frac{q}{1-q} = \frac{1}{e^a - 1}, \\
& S_B(q) = \ln \frac{1}{1-q} + \frac{q}{1-q} \ln \frac{1}{q}, \; 0 \le q \le 1, \\
& S_B(0) = 0, \; S_B(1-\epsilon) = \ln \frac{1}{\epsilon} + 1 + O(\epsilon)
\end{aligned}
\tag{441}
$$

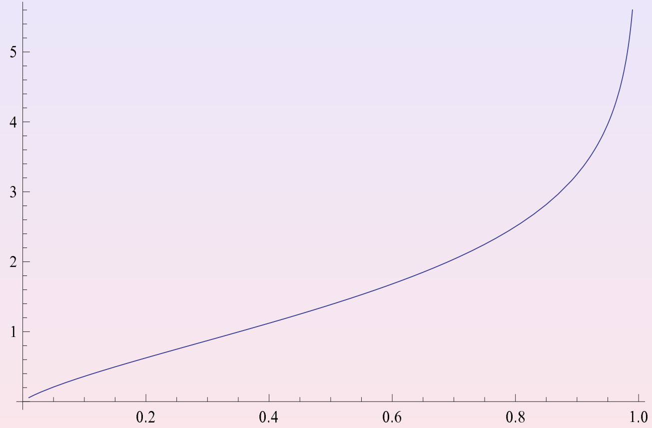

Figure: $S_B(q)$-entropy distribution,

For super-oscillator system composed from one fermi-oscillator and one bose-oscillator,

$$p_{nm} = p_n p_m = \tan\frac{a}{2} e^{-a(n+m)}, \; n = 0,1; \; m = 0,1,2,...$$
$$S_{FB} = S_F + S_B = \frac{\ln(1+q)}{1+q} + \frac{\ln(1+q^{-1})}{1+q^{-1}}$$
$$+ \ln\frac{1}{1-q} + \frac{q}{1-q}\ln\frac{1}{q} \tag{442}$$

The figure of the supersymmetric oscillator entropy is similar with the figure of the bose oscilator entropy.

Let us consider the following distribution

$$\Psi(z) = N(1+az)^{-k},\ k>1,\ \int_0^\infty \Psi(z)dz = 1 \Rightarrow N = (k-1)a. \quad (443)$$

The RD equation which define $\Psi(z)$ as a solution is

$$z\frac{d\Psi}{dz} = -k\Psi + r\Psi^q,\ r = \frac{k}{N^{\frac{1}{k}}},\ q = 1 + \frac{1}{k} \quad (444)$$

In the case of the Tsallis distribution we will have

$$\begin{aligned} &\Psi(E) = p(E) = N(1+aE)^{-k}, \\ &N = (2-q)\beta,\ a = (q-1)\beta,\ k = \frac{1}{q-1}. \end{aligned} \quad (445)$$

Having right equation, we see that the parameter a is an integration constant. If we want to have a transition from power-like to exponential form, we need to correlate the constant a and the parameter k as in the case of Tsallis distribution.

For negative values of $k = -m$ and $a = -b$ we have binomial distribution

$$\Psi(z) = N(1 - bz)^m, \int_0^{z_b} \Psi(z)dz = 1 \Rightarrow N = (m+1)b, \ z_b = \frac{1}{b}, \quad (446)$$

$$z\frac{d\Psi}{dz} = m\Psi - r\Psi^q, \ r = mN^{\frac{1}{m}}, \ 0 < q = 1 - \frac{1}{m} < 1 \quad (447)$$

To the classical exponential distribution corresponds

$$b = (1-q)\beta, \ m = \frac{1}{1-q}, \ \lim_{q \to 1} \Psi(z) = e^{-\beta z} \quad (448)$$

We have seen, that the generating function of NBD is

$$F(h) = (1 + (1-h)\frac{<n>}{k})^{-k} \quad (449)$$

Where the parameter k has clear physical sense, it is the number of identical independent sources radiating as black body with mean-multiplicity $<n>/k$.
Interpolating distribution for inclusive crossection is

$$\frac{d\sigma}{dp} = F(p) = N(1 + (1-q)\beta p)^{-k} = N(1+\beta p)^{-k}(1-a)^{-k},$$
$$a = \frac{q\beta p}{1+\beta p} \quad (450)$$

The semiinclusive crossection we define expending the inclusive crossection as generating function

$$F(p) = \sum_n F_n(p) = N(1+\beta p)^{-k}(1 + ka + \frac{k(k+1)}{1\cdot 2}a^2 + ...),$$
$$\frac{d\sigma_n}{dp} = F_n(p) = N(1+\beta p)^{-k}\frac{\Gamma(k+n)a^n}{\Gamma(k)n!},$$
$$< n(p) >= \frac{\sum_n nF_n}{\sum_n F_n} = a\frac{d}{da}\ln F = \frac{ka}{1-a},$$
$$a = \frac{< n(p) >}{< n(p) > +k} \quad (451)$$

So, for semiinclusive crossection we have NBD

$$p_n = \frac{d\sigma_n/dp}{d\sigma/dp} = \frac{\Gamma(k+n)}{\Gamma(k)n!}\frac{(k/ < n(p) >)^k}{(1 + k/ < n(p) >)^{n+k}} \quad (452)$$

and for inclusive-KNO distribution [Matveev et al, 1976] we obtain

$$< n(p) > \frac{d\sigma_n/dp}{d\sigma/dp} = \frac{\Gamma(k+n)}{\Gamma(k)n!}\frac{k(k/<n(p)>)^{k-1}}{(1+k/<n(p)>)^{n+k}}$$
$$= \frac{k^k}{\Gamma(k)} z^{k-1} e^{-kz} (1 + \frac{k^2}{2}(z - 2 + \frac{k-1}{kz})\frac{1}{<n(p)>} + O(\frac{1}{<n>^2})),$$
$$z = z(p) = \frac{n}{<n(p)>} \qquad (453)$$

Nowadays there are several big collaborations in science, e.g. LHC. Scientific value of LHC depends on three components, the highest quality of accelerator, highest quality of detectors and distributed data processing. The first two components need good mathematical and physical modeling. Third component and the collaboration as a social structure are not under (anther) the control by scientific methods and corresponding modeling. By definition, scientific collaborations (SC) have a main scientific aim: to obtain answer on the important scientific question(s) and maybe gain extra scientific bonus: new important questions and discoveries.
SC is more open information system than e.g. finance or military systems. So, it is possible to describe and optimize SC by scientific methods. Profit from scientific modeling of SC maybe also for other information systems and social structures.

Let us consider a general dynamical system described by the following system of the ordinary differential equations [Arnold, 1978]

$$\dot{x}_n = v_n(x),\ 1 \leq n \leq N, \tag{454}$$

$\dot{x}_n$ stands for the total derivative with respect to the parameter t. When the number of the degrees of freedom is even, and

$$v_n(x) = \varepsilon_{nm} \frac{\partial H_0}{\partial x_m},\ 1 \leq n, m \leq 2M, \tag{455}$$

the system (549) is Hamiltonian one and can be put in the form

$$\dot{x}_n = \{x_n, H_0\}_0, \tag{456}$$

where the Poisson bracket is defined as

$$\{A, B\}_0 = \varepsilon_{nm} \frac{\partial A}{\partial x_n} \frac{\partial B}{\partial x_m} = A \frac{\overleftarrow{\partial}}{\partial x_n} \varepsilon_{nm} \frac{\overrightarrow{\partial}}{\partial x_m} B, \tag{457}$$

and summation rule under repeated indices has been used.

Let us consider the following Lagrangian

$$L = (\dot{x}_n - v_n(x))\psi_n \tag{458}$$

and the corresponding equations of motion

$$\dot{x}_n = v_n(x), \dot{\psi}_n = -\frac{\partial v_m}{\partial x_n}\psi_m. \tag{459}$$

The system (551) extends the general system (549) by linear equation for the variables ψ. The extended system can be put in the Hamiltonian form [Makhaldiani, Voskresenskaya, 1997]

$$\dot{x}_n = \{x_n, H_1\}_1, \dot{\psi}_n = \{\psi_n, H_1\}_1, \tag{460}$$

where first level (order) Hamiltonian is

$$H_1 = v_n(x)\psi_n \tag{461}$$

and (first level) bracket is defined as

$$\{A, B\}_1 = A(\frac{\overleftarrow{\partial}}{\partial x_n}\frac{\overrightarrow{\partial}}{\partial \psi_n} - \frac{\overleftarrow{\partial}}{\partial \psi_n}\frac{\overrightarrow{\partial}}{\partial x_n})B. \tag{462}$$

Note that when the Grassmann grading [Berezin, 1987] of the conjugated variables x_n and ψ_n are different, the bracket (462) is known as Buttin bracket[Buttin, 1996].
In the Faddeev-Jackiw formalism [Faddeev, Jackiw, 1988] for the Hamiltonian treatment of systems defined by first-order Lagrangians, i.e. by a Lagrangian of the form

$$L = f_n(x)\dot{x}_n - H(x), \tag{463}$$

motion equations

$$f_{mn}\dot{x}_n = \frac{\partial H}{\partial x_m}, \tag{464}$$

for the regular structure function f_{mn}, can be put in the explicit hamiltonian (Poisson; Dirac) form

$$\dot{x}_n = f^{-1}_{nm}\frac{\partial H}{\partial x_m} = \{x_n, x_m\}\frac{\partial H}{\partial x_m} = \{x_n, H\}, \tag{465}$$

where the fundamental Poisson (Dirac) bracket is

$$\{x_n, x_m\} = f^{-1}_{nm}, \; f_{mn} = \partial_m f_n - \partial_n f_m. \tag{466}$$

Hamiltonization of the general dynamical systems

The system (551) is an important example of the first order regular hamiltonian systems. Indeed, in the new variables,

$$y_n^1 = x_n, y_n^2 = \psi_n, \tag{467}$$

lagrangian (550) takes the following first order form

$$\begin{aligned} &L = (\dot{x}_n - v_n(x))\psi_n \Rightarrow \frac{1}{2}(\dot{x}_n\psi_n - \dot{\psi}_n x_n) - v_n(x)\psi_n \\ &= \frac{1}{2}y_n^a \varepsilon^{ab} \dot{y}_n^b - H(y) = f_n^a(y)\dot{y}_n^a - H(y), f_n^a = \frac{1}{2}y_n^b \varepsilon^{ba}, H = v_n(y^1)y_n^2, \\ &f_{nm}^{ab} = \frac{\partial f_m^b}{\partial y_n^a} - \frac{\partial f_n^a}{\partial y_m^b} = \varepsilon^{ab}\delta_{nm}; \end{aligned} \tag{468}$$

corresponding motion equations and the fundamental Poisson bracket are

$$\dot{y}_n^a = \varepsilon_{ab}\delta_{nm}\frac{\partial H}{\partial y_m^b} = \{y_n^a, H\}, \{y_n^a, y_m^b\} = \varepsilon_{ab}\delta_{nm}. \tag{469}$$

To the canonical quantization of this system corresponds

$$[\hat{y}_n^a, \hat{y}_m^b] = i\hbar\varepsilon_{ab}\delta_{nm},\ \ \hat{y}_n^1 = y_n^1,\ \ \hat{y}_n^2 = -i\hbar\frac{\partial}{\partial y_n^1} \tag{470}$$

In this quantum theory, classical part, motion equations for y_n^1, remain classical.

Now we return to our extended system (551) and formulate conditions for the integrals of motion $H(x, \psi)$

$$H = H_0(x) + H_1 + ... + H_N, \tag{471}$$

where

$$H_n = A_{k_1 k_2 ... k_n}(x)\psi_{k_1}\psi_{k_2}...\psi_{k_N}, \ 1 \leq n \leq N, \tag{472}$$

we are assuming Grassmann valued ψ_n and the tensor $A_{k_1 k_2 ... k_n}$ is skew-symmetric. For integrals (471) we have

$$\dot{H} = \{\sum_{n=0}^{N} H_n, H_1\} = \sum_{n=0}^{N} \{H_n, H_1\} = \sum_{n=0}^{N} \dot{H}_n = 0. \tag{473}$$

Now we see, that each term in the sum (471) must be conserved separately. In particular for Hamiltonian systems (455), zeroth, H_0 and first level H_1, (461), Hamiltonians are integrals of motion. For $n = 0$

$$\dot{H}_0 = H_{0,k} v_k = 0, \tag{474}$$

for $1 \leq n \leq N$ we have

$$\begin{aligned}
&\dot{H}_n = \dot{A}_{k_1k_2\ldots k_n}\psi_{k_1}\psi_{k_2}\ldots\psi_{k_N} + A_{k_1k_2\ldots k_n}\dot{\psi}_{k_1}\psi_{k_2}\ldots\psi_{k_N} + \ldots \\
&+A_{k_1k_2\ldots k_n}\psi_{k_1}\psi_{k_2}\ldots\dot{\psi}_{k_N} \\
&= (A_{k_1k_2\ldots k_n,k}v_k - A_{kk_2\ldots k_n}v_{k_1,k} - \ldots \\
&-A_{k_1\ldots k_{n-1}k}v_{k_n,k})\psi_{k_1}\psi_{k_2}\ldots\psi_{k_N} = 0,
\end{aligned} \tag{475}$$

and there is one-to-one correspondence between the existence of the integrals (472) and the existence of the nontrivial solutions of the following equations

$$\frac{D}{Dt}A_{k_1k_2\ldots k_n} = A_{k_1k_2\ldots k_n,k}v_k - A_{kk_2\ldots k_n}v_{k_1,k} - \ldots - A_{k_1\ldots k_{n-1}k}v_{k_n,k} = 0 \tag{476}$$

For $n = 1$ the system (476) gives

$$A_{k_1,k}v_k - A_kv_{k_1,k} = 0 \tag{477}$$

and this equation has at list one solution, $A_k = v_k$.

If we have two (or more) independent first order integrals

$$H_1^{(1)} = A_k^1 \Psi_k; \; H_1^{(2)} = A_k^2 \Psi_k, ... \tag{478}$$

we can construct corresponding (reducible) second (or higher)order MBKY tensor(s)

$$\begin{aligned} &H_2 = H_1^{(1)} H_1^{(2)} = A_k^1 A_l^2 \Psi_k \Psi_l = A_{kl} \Psi_k \Psi_l; \\ &H_M = H_1^{(1)} ... H_M^{(M)} = A_{k_1 ... k_M} \Psi_{k_1} ... \Psi_{k_M}, \\ &A_{k_1 ... k_M} = \{A_{k_1}^{(1)} ... A_{k_M}^{(M)}\}, \; 2 \leq M \leq N \end{aligned} \tag{479}$$

where under the bracket operation, $\{B_{k_1,...,k_N}\} = \{B\}$ we understand complete anti-symmetrization. The system (476) defines a Generalization of the Bochner-Killing-Yano structures of the geodesic motion of the point particle, for the case of the general (549) (and extended (551)) dynamical systems.

Having $A_M, 2 \leq M \leq N$ independent MBKY structures, we can construct corresponding second order Killing tensors and Nambu-Poisson dynamics. In the superintegrable case, we have maximal number of the motion integrals, N-1.
The structures defined by the system (476) we call the Modified Bochner-Killing-Yano structures or MBKY structures for short, [Makhaldiani, 1999].
The dynamics of spinning point-particles in a D-dimensional curved space-time is described by the one-dimensional supersymmetric σ-model [Berezin, Marinov, 1977].

PVD can dy defined (see e.g. [Aref, 1983, Meleshko, Konstantinov, 1993]) as the following first order system

$$\dot{z}_n = i \sum_{m \neq n}^{N} \frac{\gamma_m}{z_n^* - z_m^*}, \; z_n = x_n + i y_n, \; 1 \leq n \leq N. \tag{480}$$

Corresponding first order lagrangian, hamiltonian, momenta, Poisson brackets and commutators are

$$\begin{aligned} L &= \sum_n \frac{i}{2} \gamma_n (z_n \dot{z}_n^* - \dot{z}_n z_n^*) - \sum_{n \neq m} \gamma_n \gamma_m ln |z_n - z_m| \\ H &= \sum_{n \neq m} \gamma_n \gamma_m \ln |z_n - z_m| \\ &= \frac{1}{2} \sum_{n \neq m} \gamma_n \gamma_m (\ln(z_n - z_m) + \ln(p_n - p_m)), \\ p_n &= \frac{\partial L}{\partial \dot{z}_n} = -\frac{i}{2} \gamma_n z_n^*, \; p_n^* = \frac{\partial L}{\partial \dot{z}_n^*} = \frac{i}{2} \gamma_n z_n, \end{aligned} \tag{481}$$

$$\{p_n, z_m\} = \delta_{nm},\ \{p_n^*, z_m^*\} = \delta_{nm},\ \{x_n, y_m\} = \delta_{nm},$$
$$[p_n, z_m] = -i\hbar\delta_{nm} \Rightarrow [x_n, y_m] = -i\frac{\hbar}{\gamma_n}\delta_{nm} \quad (482)$$

So, quantum vortex dynamics corresponds to the noncommutative space. It is natural to assume that vortex parameters are quantized as

$$\gamma_n = \frac{\hbar}{a^2}n,\ n = \pm 1, \pm 2, ... \quad (483)$$

and a is a characteristic (fundamental) length.

Nabu – Babylonian God
of Wisdom and Writing.

The Hamiltonian mechanics (HM) is in the fundamentals of mathematical description of the physical theories [Faddeev, Takhtajan, 1990]. But HM is in a sense blind; e.g., it does not make a difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) [Sinai, 1993] and (super)integrable Hamiltonian systems (with maximal number of the integrals of motion).
Nabu mechanics (NM) [Nambu, 1973, Whittaker, 1927] is a proper generalization of the HM, which makes the difference between dynamical systems with different numbers of integrals of motion explicit (see, e.g.[Makhaldiani, 2007]).

In the canonical formulation, the equations of motion of a physical system are defined via a Poisson bracket and a Hamiltonian, [Arnold, 1978]. In Nambu's formulation, the Poisson bracket is replaced by the Nambu bracket with $n+1, n \geq 1$, slots. For $n = 1$, we have the canonical formalism with one Hamiltonian. For $n \geq 2$, we have Nambu-Poisson formalism, with n Hamiltonians, [Nambu, 1973], [Whittaker, 1927].

The system of N vortexes (480) for $N = 3$, and

$$\begin{aligned} u_1 &= ln|z_2 - z_3|^2, \\ u_2 &= ln|z_3 - z_1|^2, \\ u_3 &= ln|z_1 - z_2|^2 \end{aligned} \tag{484}$$

reduce to the following system

$$\begin{aligned} \dot{u}_1 &= \gamma_1(e^{u_2} - e^{u_3}), \\ \dot{u}_2 &= \gamma_2(e^{u_3} - e^{u_1}), \\ \dot{u}_3 &= \gamma_3(e^{u_1} - e^{u_2}), \end{aligned} \tag{485}$$

The system (485) has two integrals of motion

$$H_1 = \sum_{i=1}^{3} \frac{e^{u_i}}{\gamma_i}, H_2 = \sum_{i=1}^{3} \frac{u_i}{\gamma_i}$$

and can be presented in the Nambu–Poisson form, [Makhaldiani, 1997,2]

$$\dot{u}_i = \omega_{ijk} \frac{\partial H_1}{\partial u_j} \frac{\partial H_2}{\partial u_k} = \{x_i, H_1, H_2\} = \omega_{ijk} \frac{e^{u_j}}{\gamma_j} \frac{1}{\gamma_k},$$

where

$$\omega_{ijk} = \epsilon_{ijk}\rho, \rho = \gamma_1\gamma_2\gamma_3$$

and the Nambu–Poisson bracket of the functions A, B, C on the three-dimensional phase space is

$$\{A, B, C\} = \omega_{ijk} \frac{\partial A}{\partial u_i} \frac{\partial B}{\partial u_j} \frac{\partial C}{\partial u_k}. \qquad (486)$$

This system is superintegrable: for $N = 3$ degrees of freedom, we have maximal number of the integrals of motion $N - 1 = 2$.

As an example of the infinite dimensional Nambu-Poisson dynamics, let me conside the following extension of Schrödinger quantum mechanics [Makhaldiani, 2000]

$$iV_t = \Delta V - \frac{V^2}{2}, \quad (487)$$

$$i\psi_t = -\Delta\psi + V\psi. \quad (488)$$

An interesting solution to the equation for the potential (487) is

$$V = \frac{4(4-d)}{r^2}, \quad (489)$$

where d is the dimension of the spase. In the case of $d = 1$, we have the potential of conformal quantum mechanics.

The variational formulation of the extended quantum theory, is given by the following Lagrangian

$$L = (iV_t - \Delta V + \frac{1}{2}V^2)\psi. \tag{490}$$

The momentum variables are

$$P_v = \frac{\partial L}{\partial V_t} = i\psi, P_\psi = 0. \tag{491}$$

As Hamiltonians of the Nambu-theoretic formulation, we take the following integrals of motion

$$\begin{aligned} H_1 &= \int d^d x(\Delta V - \frac{1}{2}V^2)\psi, \\ H_2 &= \int d^d x(P_v - i\psi), \\ H_3 &= \int d^d x P_\psi. \end{aligned} \tag{492}$$

We invent unifying vector notation, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) = (\psi, P_\psi, V, P_v)$. Then it may be verified that the equations of the extended quantum theory can be put in the following Nambu-theoretic form

$$\phi_t(x) = \{\phi(x), H_1, H_2, H_3\}, \tag{493}$$

where the bracket is defined as

$$\{A_1, A_2, A_3, A_4\} = i\varepsilon_{ijkl} \int \frac{\delta A_1}{\delta\phi_i(y)} \frac{\delta A_2}{\delta\phi_j(y)} \frac{\delta A_3}{\delta\phi_k(y)} \frac{\delta A_4}{\delta\phi_l(y)} dy$$
$$= i \int \frac{\delta(A_1, A_2, A_3, A_4)}{\delta(\phi_1(y), \phi_2(y), \phi_3(y), \phi_4(y))} dy = i det(\frac{\delta A_k}{\delta\phi_l}). \tag{494}$$

The basic building blocks of M theory are membranes and $M5-$branes. Membranes are fundamental objects carrying electric charges with respect to the 3-form C-field, and $M5$-branes are magnetic solitons. The Nambu-Poisson 3-algebras appear as gauge symmetries of superconformal Chern-Simons nonabelian theories in 2 + 1 dimensions with the maximum allowed number of $N = 8$ linear supersymmetries.
The Bagger and Lambert [Bagger, Lambert, 2007] and, Gustavsson [Gustavsson, 2007] (BLG) model is based on a 3-algebra,

$$[T^a, T^b, T^c] = f_d^{abc} T^d \tag{495}$$

where T^a, are generators and f_{abcd} is a fully anti-symmetric tensor. Given this algebra, a maximally supersymmetric Chern-Simons lagrangian is:

$$\begin{aligned} L &= L_{CS} + L_{matter}, \\ L_{CS} &= \frac{1}{2}\varepsilon^{\mu\nu\lambda}(f_{abcd}A_\mu^{ab}\partial_\nu A_\lambda^{cd} + \frac{2}{3}f_{cdag}f_{efb}^{g}A_\mu^{ab}A_\nu^{cd}A_\lambda^{ef}), \end{aligned} \tag{496}$$

$$L_{matter} = \frac{1}{2} B_\mu^{Ia} B_a^{\mu I} - B_\mu^{Ia} D^\mu X_a^I$$
$$+ \frac{i}{2} \bar{\psi}^a \Gamma^\mu D_\mu \psi_a + \frac{i}{4} \bar{\psi}^b \Gamma_{IJ} x_c^I x_d^J \psi_a f^{abcd}$$
$$- \frac{1}{12} tr([X^I, X^J, X^K][X^I, X^J, X^K]), \ I = 1, 2, ..., 8, \qquad (497)$$

where A_μ^{ab} is gauge boson, ψ^a and $X^I = X_a^I T^a$ matter fields. If $a = 1, 2, 3, 4$, then we can obtain an $SO(4)$ gauge symmetry by choosing $f_{abcd} = f \varepsilon_{abcd}, f$ being a constant. It turns out to be the only case that gives a gauge theory with manifest unitarity and $N = 8$ supersymmetry.

The action has the first order form so we can use the formalism of the first section. The motion equations for the gauge fields

$$f^{nm}_{abcd}\dot{A}^{cd}_m(t,x) = \frac{\delta H}{\delta A^{ab}_n(t,x)}, f^{nm}_{abcd} = \varepsilon^{nm} f_{abcd} \quad (498)$$

take canonical form

$$\dot{A}^{ab}_n = f^{abcd}_{nm}\frac{\delta H}{\delta A^{cd}_m} = \{A^{ab}_n, A^{cd}_m\}\frac{\delta H}{\delta A^{cd}_m} = \{A^{ab}_n, H\},$$
$$\{A^{ab}_n(t,x), A^{cd}_m(t,y)\} = \varepsilon_{nm} f^{abcd}\delta^{(2)}(x-y) \quad (499)$$

The quasi-classical description of the motion of a relativistic (nonradiating) point particle with spin in accelerators and storage rings includes the equations of orbit motion

$$\begin{aligned}
&\dot{x}_n = f_n(x),\ f_n(x) = \varepsilon_{nm}\partial_m H,\ n,m = 1,2,...,6;\\
&x_n = q_n,\ x_{n+3} = p_n,\ \varepsilon_{n,n+3} = 1,\ n = 1,2,3;\\
&H = e\Phi + c\sqrt{\wp^2 + m^2c^2},\ \wp_n = p_n - \frac{e}{c}A_n
\end{aligned} \tag{500}$$

and Thomas-BMT equations
[Tomas, 1927, Bargmann, Michel,Telegdi, 1959] of classical spin motion

$$\begin{aligned}
&\dot{s}_n = \varepsilon_{nmk}\Omega_m s_k = \{H_1, H_2, s_n\},\ H_1 = \Omega\cdot s,\ H_2 = s^2,\\
&\{A,B,C\} = \varepsilon_{nmk}\partial_n A\partial_m B\partial_k C,
\end{aligned} \tag{501}$$

$$\Omega_n = \frac{-e}{m\gamma c}((1+k\gamma)B_n - k\frac{(B\cdot\wp)\wp_n}{m^2c^2(1+\gamma)} + \frac{1+k(1+\gamma)}{mc(1+\gamma)}\varepsilon_{nmk}E_m\wp_k) \tag{502}$$

where, parameters e and m are the charge and the rest mass of the particle, c is the velocity of light, $k=(g-2)/2$ quantifies the anomalous spin g factor, γ is the Lorentz factor, p_n are components of the kinetic momentum vector, E_n and B_n are the electric and magnetic fields, and A_n and Φ are the vector and scalar potentials;

$$B_n = \varepsilon_{nmk}\partial_m A_k,\ E_n = -\partial_n\Phi - \frac{1}{c}\dot{A}_n,$$
$$\gamma = \frac{H-e\Phi}{mc^2} = \sqrt{1+\frac{\wp^2}{m^2c^2}} \tag{503}$$

The spin motion equations we put in the Nambu-Poisson form. Hamiltonization of this dynamical system according to the general approach of the previous sections we will put in the ground of the optimal control

The general method of Hamiltonization of the dynamical systems we can use also in the spinning particle case. Let us invent unified configuration space $q = (x, p, s),\ x_n = q_n,\ p_n = q_{n+3},\ s_n = q_{n+6},\ n = 1, 2, 3$; extended phase space, (q_n, ψ_n) and hamiltonian

$$H = H(q, \psi) = v_n \psi_n,\ n = 1, 2, ...9; \tag{504}$$

motion equations

$$\begin{aligned} \dot{q}_n &= v_n(q), \\ \dot{\psi}_n &= -\frac{\partial v_m}{\partial q_n} \psi_m \end{aligned} \tag{505}$$

where the velocities v_n depends on external fields as in previous section as control parameters which can be determined according to the optimal control criterium.

We already have Nambu-Poisson formulation of the spinning part of the dynamics. Let us define the extended Hamiltonian as

$$H_1 = H(x,p) + H_1(s) = H_1(q),\ H_2 = H(x,p) + H_2(s) = H_2(q),$$
$$H_1(s) = \Omega \cdot s,\ H_2(s) = \frac{1}{2}s^2 \tag{506}$$

Than the Nambu-Poisson form of the dynamics will be

$$\dot{A}(q) = \{A(q), H_1, H_2\}, \tag{507}$$

where

$$\{A, B, C\} = f_{NMK} \frac{\partial A}{\partial q_N} \frac{\partial B}{\partial q_M} \frac{\partial C}{\partial q_K},\ N, M, K = 1, 2, ..., 9, \tag{508}$$

and the structure function f_{NMK} is defined from the comparison with the motion equations for q_N.

The structure function is antisymmetric; when $A = q_n$, we obtain the motion equation for q_n,

$$\begin{aligned} \dot{q}_n &= f_{n,m+6,k+3} \frac{\partial H_1(s)}{\partial s_m} \frac{\partial H(x,p)}{\partial p_k} = \delta_{nk} \frac{\partial H(x,p)}{\partial p_k}, \\ f_{n,m+6,k+3}\Omega_m &= \delta_{nk}, \end{aligned} \tag{509}$$

for $A = p_n$,

$$\dot{p}_n = f_{n+3,m+6,k} \frac{\partial H_1(s)}{\partial s_m} \frac{\partial H(x,p)}{\partial q_k} = -\delta_{nk} \frac{\partial H(x,p)}{\partial q_k}, \tag{510}$$

for $A = s_n$,

$$\begin{aligned} \dot{s}_n &= f_{n+6,m+6,k+6} \frac{\partial H_1(s)}{\partial s_m} \frac{\partial H_2(s)}{\partial s_k} = \varepsilon_{nmk}\Omega_m s_k, \\ f_{n+6,m+6,k+6} &= \varepsilon_{nmk}. \end{aligned} \tag{511}$$

With the Nambu-Poisson formulation, we have, as usual, two Hamiltonian reductions,

$$\dot{A} = \{A, H_1(q)\}_1 = \{A, H_2(q)\}_2 \tag{512}$$

Note that, if we take collective coordinates and Hamiltonian $H_1(q)$, the Hamiltonian motion equations will contain extra terms beyond original motion equations [Balandin, Golubeva, 1999].

Note that the procedure of reduction of the higher order dynamical system, e.g. second order Euler-Lagrange motion equations, to the first order dynamical systems, in the case to the Hamiltonian motion equations, can be continued using fractal calculus. E.g. first order system can be reduced to the half order one,

$$\begin{aligned} &D^{1/2}q = \psi, \\ &D^{1/2}\psi = p \Leftrightarrow \dot{q} = p. \end{aligned} \tag{513}$$

We define the following dynamical system [Makhaldiani, Postnov, WIP],

$$\begin{aligned} &D^{1/2}q = f(q),\ D^{1/2} = \partial_\theta + \theta\partial_t,\ q(t,\theta) = q_0(t) + \theta q_1(t), \\ &f(q) = f_0(q) + \theta f_1(q) = f_0(q_0) + \theta(f_0'(q_0)q_1 + f_1(q_0)) \end{aligned} \tag{514}$$

which is equivalent to the following dynamical system in component form

$$\begin{aligned} &q_1(t) = f_0(q_0), \\ &\dot{q}_0 = f_0'(q_0)q_1 + f_1(q_0) \end{aligned} \tag{515}$$

It always bothers me
that according to the laws
as we understand them today,
it takes a computing machine
an infinite number of logical operations
to figure out what goes on in no matter how tiny
a region of space and no matter how tiny a region of time.
R. Feynman, The Character of Physical Law (1985).

To request an answer on Feynman's paradox we may assume that Physics at a very small scale is discrete. Quantum Fields on continuous spacetime is then replaced by a lattice of quantum systems that evolve in discrete time steps.
Quantum cellular automaton (QCA) is a quantum version of the cellular automaton of von Neumann which describes a dynamics on a discrete lattice in discrete time-steps.

Computers are physical devices and their behavior is determined by physical laws. The Quantum Computations [Benenti, Casati, Strini, 2004 , Nielsen, Chuang, 2000], Quantum Computing, Quanputing [Makhaldiani, 2007.2], is a new interdisciplinary field of research, which benefits from the contributions of physicists, computer scientists, mathematicians, chemists and engineers. Contemporary digital computer and its logical elements can be considered as a spatial type of discrete dynamical systems [Makhaldiani, 2001]

$$S_n(k+1) = \Phi_n(S(k)), \tag{516}$$

where

$$S_n(k), \quad 1 \leq n \leq N(k), \tag{517}$$

is the state vector of the system at the discrete time step k. Vector S may describe the state and Φ transition rule of some Cellular Automata [Toffoli, Margolus, 1987].The systems of the type (516) appears in applied mathematics as an explicit finite difference scheme approximation of the equations of the physics [Samarskii, Gulin, 1989].

Definition: *We assume that the system (516) is time-reversible if we can define the reverse dynamical system*

$$S_n(k) = \Phi_n^{-1}(S(k+1)). \tag{518}$$

In this case the following matrix

$$M_{nm} = \frac{\partial \Phi_n(S(k))}{\partial S_m(k)}, \tag{519}$$

is regular, i.e. has an inverse. If the matrix is not regular, this is the case, for example, when $N(k+1) \neq N(k)$, we have an irreversible dynamical system (usual digital computers and/or corresponding irreversible gates).

Let us consider an extension of the dynamical system (516) given by the following action function

$$A = \sum_{kn} l_n(k)(S_n(k+1) - \Phi_n(S(k))) \tag{520}$$

and corresponding motion equations

$$\begin{aligned} S_n(k+1) &= \Phi_n(S(k)) = \frac{\partial H}{\partial l_n(k)}, \\ l_n(k-1) &= l_m(k)\frac{\partial \Phi_m(S(k))}{\partial S_n(k)} = l_m(k)M_{mn}(S(k)) = \frac{\partial H}{\partial S_n(k)}, \end{aligned} \tag{521}$$

where

$$H = \sum_{kn} l_n(k)\Phi_n(S(k)), \tag{522}$$

is discrete Hamiltonian. In the regular case, we put the system (521) in an explicit form

$$\begin{aligned} S_n(k+1) &= \Phi_n(S(k)), \\ l_n(k+1) &= l_m(k)M^{-1}_{mn}(S(k+1)). \end{aligned} \tag{523}$$

From this system it is obvious that, when the initial value $l_n(k_0)$ is given, the evolution of the vector $l(k)$ is defined by evolution of the state vector $S(k)$. The equation of motion for $l_n(k)$ - Elenka is linear and has an important property that a linear superpositions of the solutions are also solutions.

Statement: *Any time-reversible dynamical system (e.g. a time-reversible computer) can be extended by corresponding linear dynamical system (quantum - like processor) which is controlled by the dynamical system and has a huge computational power,* [Makhaldiani, 2001, Makhaldiani, 2002, Makhaldiani, 2007.2, Makhaldiani, 2011.2].

For motion equations (521) in the continual approximation, we have

$$\begin{aligned} &S_n(k+1) = x_n(t_k+\tau) = x_n(t_k) + \dot{x}_n(t_k)\tau + O(\tau^2), \\ &\dot{x}_n(t_k) = v_n(x(t_k)) + O(\tau), \quad t_k = k\tau, \\ &v_n(x(t_k)) = (\Phi_n(x(t_k)) - x_n(t_k))/\tau; \\ &M_{mn}(x(t_k)) = \delta_{mn} + \tau \frac{\partial v_m(x(t_k))}{\partial x_n(t_k)}. \end{aligned} \quad (524)$$

(de)Coherence criterion: *the system is reversible, the linear (quantum, coherent, soul) subsystem exists, when the matrix M is regular,*

$$detM = 1 + \tau \sum_n \frac{\partial v_n}{\partial x_n} + O(\tau^2) \neq 0. \quad (525)$$

For the Nambu - Poisson dynamical systems (see e.g. [Makhaldiani, 2007])

$$\begin{aligned} &v_n(x) = \varepsilon_{nm_1m_2...m_p} \frac{\partial H_1}{\partial x_{m_1}} \frac{\partial H_2}{\partial x_{m_2}} ... \frac{\partial H_p}{\partial x_{m_p}}, \quad p = 1, 2, 3, ..., N-1, \\ &\sum_n \frac{\partial v_n}{\partial x_n} \equiv divv = 0. \end{aligned} \quad (526)$$

Let me motivate an idea of construction of the reversible dynamical systems by simple example from field theory. There are renormalizable models of scalar field theory of the form (see, e.g. [Makhaldiani, 1980])

$$L = \frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2) - g\varphi^n, \tag{527}$$

with the constraint

$$n = \frac{2d}{d-2}\ , \tag{528}$$

where d is dimension of the space-time and n is degree of nonlinearity. It is interesting that if we define d as a function of n, we find

$$d = \frac{2n}{n-2} \tag{529}$$

the same function !
Thing is that, the constraint can be put in the symmetric implicit form [Makhaldiani, 1980]

$$\frac{1}{n} + \frac{1}{d} = \frac{1}{2} \tag{530}$$

Now it is natural to consider the following symmetric function

$$f(y) + f(x) = c \tag{531}$$

and define its solution

$$y = f^{-1}(c - f(x)). \tag{532}$$

This is the general method, that we will use in the following construction of the reversible dynamical systems. In the simplest case,

$$f(x) = x, \tag{533}$$

we take

$$y = S(k+1), \quad x = S(k-1), \quad c = \tilde{\Phi}(S(k)) \tag{534}$$

and define our reversible dynamical system from the following symmetric, implicit form (see also [Toffoli, Margolus, 1987])

$$S(k+1) + S(k-1) = \tilde{\Phi}(S(k)), \tag{535}$$

explicit form of which is

$$\begin{aligned} S(k+1) &= \Phi(S(k), S(k-1)) \\ &= \tilde{\Phi}(S(k)) - S(k-1). \end{aligned} \tag{536}$$

This dynamical system defines given state vector by previous two state vectors. We have reversible dynamical system on the time lattice with time steps of two units,

$$
\begin{aligned}
&S(k+2,2) = \Phi(S(k,2)),\\
&S(k+2,2) \equiv (S(k+2), S(k+1)),\\
&S(k,2) \equiv (S(k), S(k-1))).
\end{aligned} \tag{537}
$$

Starting from a general discrete dynamical system, we obtained reversible dynamical system with internal(spin,bit) degrees of freedom

$$
\begin{aligned}
S_{ns}(k+2) &\equiv \begin{pmatrix} S_n(k+2) \\ S_n(k+1) \end{pmatrix} = \begin{pmatrix} \Phi_n(\Phi(S(k)) - S(k-1)) - S(k)) \\ \Phi_n(S(k)) - S_n(k-1) \end{pmatrix} \\
&\equiv \Phi_{ns}(S(k)), \quad s = 1, 2
\end{aligned}
\tag{538}
$$

where

$$
S(k) \equiv (S_{ns}(k)), \quad S_{n1}(k) \equiv S_n(k), \quad S_{n2}(k) \equiv S_n(k-1) \tag{539}
$$

For the extended system we have the following action

$$
A = \sum_{kns} l_{ns}(k)(S_{ns}(k+2) - \Phi_{ns}(S(k))) \tag{540}
$$

and corresponding motion equations

$$S_{ns}(k+2) = \Phi_{ns}(S(k)) = \frac{\partial H}{\partial l_{ns}(k)},$$
$$l_{ns}(k-2) = l_{mt}(k)\frac{\partial \Phi_{mt}(S(k))}{\partial S_{ns}(k)}$$
$$= l_{mt}(k)M_{mtns}(S(k)) = \frac{\partial H}{\partial S_{ns}(k)}, \tag{541}$$

By construction, we have the following reversible dynamical system

$$\begin{aligned} S_{ns}(k+2) &= \Phi_{ns}(S(k)), \\ l_{ns}(k+2) &= l_{mt}(k)M^{-1}_{mtns}(S(k+2)), \end{aligned} \tag{542}$$

with classical S_{ns} and quantum l_{ns}(in the external, background S) string bit dynamics.

We can also consider p-point generalization of the previous structure,

$$\begin{aligned}
& f_p(S(k+p)) + f_{p-1}(S(k+p-1)) + ... + f_1(S(k+1)) \\
& + f_1(S(k-1)) + ... + f_p(S(k-p)) = \tilde{\Phi}(S(k)), \\
& S(k+p) = \Phi(S(k), S(k+p-1), ..., S(k-p)) \\
& \equiv f_p^{-1}(\tilde{\Phi}(S(k)) - f_{p-1}(S(k+p-1)) - ... - f_p(S(k-p)))
\end{aligned} \quad (543)$$

and corresponding reversible p-oint cluster dynamical system

$$\begin{aligned}
& S(k+p,p) \equiv \Phi(S(k,p)), \\
& S(k+p,p) \equiv (S(k+p), S(k+p-1), ..., S(k+1)), \\
& S(k,p) \equiv (S(k), S(k-1), ..., S(k-p+1)), \quad S(k,1) = S(k).
\end{aligned} \quad (544)$$

So we have general method of construction of the reversible dynamical systems on the time (tame) scale p. The method of linear extension of the reversible dynamical systems (see [Makhaldiani, 2001] and previous section) defines corresponding Quanputers,

$$\begin{aligned}
& S_{ns}(k+p) = \Phi_{ns}(S(k)), \\
& l_{ns}(k+p) = l_{mt}(k) M^{-1}_{mtns}(S(k+p)),
\end{aligned} \quad (545)$$

This case the quantum state function $l_{ns},\ s = 1, 2, ...p$ will describes the state with spin $(p-1)/2$.
Note that, in this formalism for reversible dynamics minimal value of the spin is $1/2$. There is not a place for a scalar dynamics, or the scalar dynamics is not reversible. In the Standard model (SM) of particle physics, [Beringer et al, 2012], all of the fundamental particles, leptons, quarks and gauge bosons have spin. Only scalar particles of the SM are the Higgs bosons. Perhaps the scalar particles are composed systems or quasiparticles like phonon, or Higgs dynamics is not reversible (a mechanism for 'time arrow').

The $NP \stackrel{?}{=} P$ problem will be solved if for some $NP-$ complete problem, e.g. TSP, a polynomial algorithm find; or show that there is not such an algorithm; or show that it is impossible to find definite answer to that question.
TSP means to find minimal length path between N fixed points on a surface, which attends any point ones. We consider a system where N points with quenched positions $x_1, x_2, ..., x_N$ are independently distributed on a finite domain D with a probability density function $p(x)$. In general, the domain D is multidimensional and the points x_n are vectors in the corresponding Euclidean space. Inside the domain D we consider a polymer chain composed of N monomers whose positions are denoted by $y_1, y_2, ..., y_N$. Each monomer y_n is attached to one of the quenched sites x_m and only one monomer can be attached to each site. The state of the polymer is described by a permutation $\sigma \in \Sigma_N$ where Σ_N is the group of permutations of N objects.

The Hamiltonian for the system is given by

$$H = \sum_{n=1}^{N} V(|y_n - y_{n-1}|) \qquad (546)$$

Here V is the interaction between neighboring monomers on the polymer chain. For convenience the chain is taken to be closed, thus we take the periodic boundary condition $x_0 = x_N$. A physical realization of this system is one where the x_n are impurities where the monomers of a polymer loop are pinned. In combinatorial optimization, if one takes $V(x)$ to be the norm, or distance, of the vector x then $H(\sigma)$ is the total distance covered by a path which visits each site x_n exactly once. The problem of finding σ_0 which minimizes $H(\sigma)$ is known as the traveling salesman problem (TSP) [Gutin, Pannen, 2002].

In field theory language to the TSP we correspond the calculation of the following correlator

$$
\begin{aligned}
&G_{2N}(x_1, x_2, ..., x_N) = Z_0^{-1} \int d\varphi(x) \varphi^2(x_1) \varphi^2(x_2) ... \varphi^2(x_N) e^{-S(\varphi)} \\
&= \frac{\delta^{2N} F(J)}{\delta J(x_1)^2 ... \delta J(x_N)^2}, \; F(J) = \ln Z(J), \\
&Z(J) = \int d\varphi e^{-\frac{1}{2}\varphi \cdot A \cdot \varphi + J \cdot \varphi} = e^{\frac{1}{2} J \cdot A^{-1} \cdot J}, \; A^{-1}(x, y; m) = e^{-m|x-y|}, \\
&L_{min}(x_1, ..., x_N) = -\frac{d}{dm} \ln G_{2Ns} + O(e^{-am}) \\
&< A^{-1} > \equiv \frac{1}{\Gamma(s)} \int_0^\infty dm m^{s-1} A^{-1}(x, y; m) = \frac{1}{|x-y|^s} \\
&= L_s A^{-1}(x - y; s) \\
&k(d) \Delta_d L_s A^{-1}(x; s) = \delta^d(x) \Rightarrow A(x; s) = k(d) \Delta_d L_s, \\
&s = d - 2; \varphi = \varphi(x, m).
\end{aligned}
\qquad (547)
$$

If we take relativistic massive scalar field, then $A = \Delta_d + m^2$,

$$A^{-1}(x) \sim |x|^{2-d} e^{-m|x|}, \tag{548}$$

and for $d = 2$, we also have the needed behaviour. Note that G_{2N} is symmetric with respect to its arguments and contains any paths including minimal length one.

Hamiltonization of dynamical systems. Let us consider the following system of the ordinary differential equations [Arnold, 1978].

$$\dot{x}_n = v_n(x) + j_n(t),\ 1 \leq n \leq N, \tag{549}$$

Lagrangian,

$$L = (\dot{x}_n - v_n(x) - j_n(t))\psi_n \tag{550}$$

and the corresponding motion equations

$$\dot{x}_n = v_n(x) + j_n(t), \dot{\psi}_n = -\frac{\partial v_m}{\partial x_n}\psi_m. \tag{551}$$

The system (551) extends the system (549) by linear equation for the ψ. The extended system can be put in the Hamiltonian form [Makhaldiani, Voskresenskaya, 1997].

Quanputing. The idea of computations on quanputers is in finding of the needed (value of the) state (wave function $\psi(t,x)$) from the initial, easy constructible, state ($\psi(0,x)$,) which is superposition of different states, including interesting one, with the same weight. During the computation the weight of the interesting state is growing till the value when we can guess the solution of the problem and then test it, which is much more easier then to find it [Kitaev, Shen, Vyalyi, 2002], [Benenti, Casati, Strini, 2004], [Giorgadze, 2013].

Let us consider the following nonlinear evolution equation

$$iV_t = \Delta V - \frac{1}{2}V^2 + J, \tag{552}$$

extended Lagrangian and Hamiltonian

$$L = \int dx^D (iV_t - \Delta V + \frac{1}{2}V^2 - J)\psi, \;\; H = \int dx^D (\Delta V - \frac{1}{2}V^2 + J)\psi \tag{553}$$

and corresponding Hamiltonian motion equations [Makhaldiani, 2000].

$$\begin{aligned} &iV_t = \Delta V - \frac{1}{2}V^2 + J = \{V, H\}, \\ &i\psi_t = -\Delta\psi + V\psi = \{\psi, H\}, \\ &\{V(t,x), \psi(t,y)\} = \delta^D(x-y) \end{aligned} \tag{554}$$

The solution of the problem is given in the form

$$|T) = U(T)|0), \;\; \psi(t,x) =< x|t), \;\; U(T) = Texp(-i\int_0^T H(t)) \tag{555}$$

Under the programming of the quanputer we understand construction of the potential V, or the corresponding Hamiltonian. For the given potential, we calculate corresponding source J.

The discrete version of the system can be put in the form [Makhaldiani, 2007.2].

$$S_m(n+1) = \Phi_n(S(n)) + J_m(n),$$
$$\Psi_m(n-1) = A_{mk}(S(n))\Psi_k(n),\ A_{mk}(S(n)) = \frac{\partial \Phi_k(S(n))}{\partial S_m(n)} \quad (556)$$

when the matrix A is regular, we obtain explicit form of the corresponding discrete dynamics

$$S_m(n+1) = \Phi_n(S(n)) + J_m(n),$$
$$\Psi_m(n) = A^{-1}_{mk}(S(n+1))\Psi_k(n), \quad (557)$$

Now the state vector $S(n)$ and wave vector $\Psi_m(n)$ may correspond not only to the discrete values of the potential $V(n,m) = S_m(n)$, and wave function $\psi(n,m) = \Psi_m(n)$ but also any representation of the computing process from theoretical to experimental realization on a quanputer, including algorithm of solution, higher level programm realization of the algorithm [Makhaldiani, 2011.2].

Complex Polynomial Equations and Nambu-poisson Dynamics

We consider the following polynomial equation

$$P_N(z) - tz^{N+1} = 0,\ z \in \mathcal{C},\ t \in (0,\infty) \quad (558)$$

For small times t all zeros but one of this polynomial are near the zeros of the polynomial $P_N(z)$. The extra zero z_{N+1} is far from other zeros, for small t,

$$z_{N+1} = \frac{a_N}{t} + ... \tag{559}$$

In regular case main zeros are linear functions of t, for small t.
For large times all $n+1$ zeros are near the zeros of the equation

$$a_0 - tz^{N+1} = 0,\ z_n = \sqrt[N+1]{a_0/t}\exp(2\pi i\frac{n}{N+1}),\ n = 0, 1, ..., N \tag{560}$$

At a root x_c of multiplicity k we have

$$\frac{P_N^{(k)}(x_c)}{n!}(x - x_c)^k + ... = tx_c^{N+1},$$
$$x_n(t) = x_c + c_{n,k}t^{1/k},\ c_{n,k} = (\frac{x_c^{N+1}n!}{P_N^{(k)}(x_c)})^{\frac{1}{k}}\exp(2\pi i\frac{n}{k}),\ 0 \leq n \leq k-1 \tag{561}$$

So we can define the multiplicity of the root k from the time dependence of the roots. It is interesting to know how extra zero approach with time to the other zeros and then all of them organized as sites of symmetric polygon on the circle with decreasing radius. Note that coefficients

$a_n,\ 1 \leq n \leq N$ are known functions of zeros but do not depend on t - are invariants - integrals of motion. Having N integrals of motion $H_n,\ 1 \leq n \leq N$ we construct Nambu-Poisson dynamics for the roots [Nambu, 1973], [Makhaldiani, 2007], [Makhaldiani, 1988, **?**].

$$\dot{x}_n = \{x_n, H_1, H_2, ..., H_N\},\ 1 \leq n \leq N \tag{562}$$

As an example we consider quadratic deformation of the linear equation

$$\begin{aligned} &a_0 + a_1 z - tz^2 = -t(z - z_1)(z - z_2) = 0, \\ &a_0 = -tz_1 z_2,\ a_1 = t(z_1 + z_2) \end{aligned} \tag{563}$$

As a 'time independent' Hamiltonian we take

$$H = -a_0/a_1 = \frac{z_1 z_2}{z_1 + z_2} \tag{564}$$

the motion equations we find from the time independence of a_0 and a_1

$$\begin{aligned} &\dot{a}_0 = -z_1 z_2 - t(\dot{z}_1 z_2 + z_1 \dot{z}_2) = 0, \\ &\dot{a}_1 = z_1 + z_2 + t(\dot{z}_1 + \dot{z}_2) = 0, \\ &\dot{z}_1 = \frac{z_1^3 z_2}{a_0(z_1 - z_2)} = \{z_1, H\} = f_{12} \frac{\partial H}{\partial z_2}, \\ &\dot{z}_2 = \frac{z_2^3 z_1}{a_0(z_2 - z_1)} = \{z_2, H\} = f_{21} \frac{\partial H}{\partial z_1}, \end{aligned}$$

$$f_{12} = \frac{z_1 z_2 (z_1 + z_2)^2}{a_0(z_1 - z_2)} = \frac{a_1^2}{t^3(z_2 - z_1)} \quad (565)$$

In the cubic deformation of the quadratic equation

$$a_0 + a_1 z + a_2 z^2 - tz^3 = -t(z - z_1)(z - z_2)(z - z_3) = 0 \quad (566)$$

we have

$$a_0 = tz_1 z_2 z_3,\ a_1 = -t(z_1 z_2 + z_2 z_3 + z_3 z_1),\ a_2 = t(z_1 + z_2 + z_3),$$
$$\dot{z}_1 = \frac{z_1^4 z_2 z_3}{a_0(z_2 - z_1)(z_1 - z_3)} = \{z_1, H_1, H_2\} = f_{1nm} \frac{\partial H_1}{\partial z_n} \frac{\partial H_2}{\partial z_m},$$
$$f_{123} = \frac{z_1 z_2 z_3 (z_1 z_2 + z_2 z_3 + z_3 z_1)(z_1 + z_2 + z_3)}{a_0(z_2 - z_1)(z_3 - z_2)(z_1 - z_3)}$$
$$= \frac{a_1 a_2}{t^3(z_1 - z_2)(z_1 - z_3)(z_3 - z_2)},$$
$$H_1 = \frac{z_1 z_2 z_3}{z_1 z_2 + z_2 z_3 + z_3 z_1},\ H_2 = \frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{z_1 + z_2 + z_3} \quad (567)$$

Introducing new time variable $\tau = a_1 a_2 t^{-2}/2$ we put the equation in the form

$$\frac{dz_1}{d\tau} = \{z_1, H_1, H_2\} = f_{1nm} \frac{\partial H_1}{\partial z_n} \frac{\partial H_2}{\partial z_m},$$

$$f_{123} = \frac{1}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)} \tag{568}$$

For the following generalization of the Weierstrass function $V_n(z)$

$$\int_{V_n(z)}^{\infty} \frac{dV}{\sqrt{P_n(V)}} = z,$$
$$P_n(V) = \frac{4}{(n-2)^2} V^n + C_{n-2} V^{n-2} + ... + C_0, \tag{569}$$

we have the following series (re)presentation

$$V_n(z) = \wp_n(z, C_{n-2}, ..., C_0) = \frac{1}{z^{2/(n-2)}} - \frac{(n-2)^2}{4(n+2)} C_{n-2} z^{2/(n-2)} + .. \tag{570}$$

K. Aamodt et al. [ALICE collaboration] Eur. Phys. J. C65, 111 (2010); arXiv:1004.3034; arXiv:1004.3514.

O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large N field theories, string theory and gravity, *Phys. Rept.* **323** (2000) 183, hep-th/9905111.

H. Aref, Ann. Rev. Fluid Mech. **15** (1983) 345.

V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York 1978.

Aad, G. et al. , 2011, Phys Lett, B 701, 398

G. Aad et al. [ATLAS Collaboration], Phys. Lett. B 716, 1 (2012)

Abrikosov, A. A. (1957) On the Magnetic properties of superconductors of the second group, Sov.Phys.JETP 5:1174-1182 and Zh.Eksp.Teor.Fiz.32:1442-1452.

Ackermann, M.; Ajello, M.; Allafort, A.; Baldini, L.; Ballet, J.; Barbiellini, G.; Baring, M. G.; Bastieri, D. et al. (2013-02-15). Detection of the Characteristic Pion-Decay Signature in Supernova Remnants. Science (American Association for the Advancement of Science) 339 (6424): 807-811.

A. Adare et al. (PHENIX Collaboration), Phys. Rev. Lett. 98, 162301 (2007).

S. L. Adler, *Phys. Rev.* **140** (1965) B736.

S.L. Adler, R. Dashen, *Current Algebra and Applications to Particle Physics*, Benjamin, New York 1965.

K. Agashe, R. Contino and A. Pomarol, The Minimal composite Higgs model, Nucl. Phys. B719, 165 (2005) http://arxiv.org/pdf/hep-ph/0412089.pdf

O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large N field theories, string theory and gravity, *Phys. Rept.* **323** (2000) 183, hep-th/9905111.

J. Alexandre, Int. J. Mod. Phys. A 26, 4523 (2011) [arXiv:1109.5629 [hep-ph]].

K. Aamodt et al. [ALICE collaboration] *Eur. Phys. J.* **C65** (2010) 111 [arXiv:1004.3034], [arXiv:1004.3514].

Al Khawaja, Usama; Stoof, Henk (2001). "Skyrmions in a ferromagnetic BoseEinstein condensate". Nature 411 (6840): 91820.

M. Anselmino, B.L. Ioffe and E. Leader, Sov. J. Nucl. 49, 136, (1989)

Aprile E. et al , 2012, Phys Rev Lett, 109, 181301

H. Aref, Ann. Rev. Fluid Mech. **15** (1983) 345.

V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York 1978.

M.Axenides,E.G.Floratos,L.Perivolaropoulos, Dynamical Effects of the Cosmological Constant (astro-ph/0004080) Modern Physics Letters A 15, 1541-1550

J. Bagger, N. Lambert, *Modeling multiple M2's*, Phys. Rev. D **75** (2007) 045020; [arXiv:hep-th/0611108].

V.V. Balandin, N.I. Golubeva, *Hamiltonian Methods for the Study of Polarized Proton Beam Dynamics in Accelerators and Storage Rings*, [arXiv://arxiv.org/abs/physics/9903032v1].

V. Bargmann, L. Michel, V.L. Telegdi, *Precession of the polarization of particles moving in a homogeneous electromagnetic field*, Phys. Rev. Lett., **2(10)** (1959) 435.

C. Beck, E.G.D. Cohen,Superstatistics, *Physica A* **322** (2003) 267-275

G. Benenti, G. Casati, G. Strini, *Principles of quantum computation and information*, Vol. I: Basic concepts, World Scientific, Singapore 2004; Vol. II: *Basic tools and special topics* World Scientific, Singapore 2007.

J. Beringer et al. (Particle Data Group) *Review of Particle Physics*, Phys. Rev. D **86** (2012) 010001 [1528 pages]

F.A. Berezin, *Introduction to Superanalysis*, Reidel, Dordrecht 1987.

F.A. Berezin, M.S. Marinov, Ann. Phys. (N.Y.) **104** (1977) 336.

M. Berry, Speculations on the Riemann operator, Symposium on Supersymmetry and Trace formulae, Cambridge, 1997.

C. Buttin, C.R. Acad. Sci. Paris. **269** (1969) 87.

M. Biyajima, M. Kaneyama, T. Mizoguchi and G. Wilk, *Eur. Phys. J.* **C 40** (2005) 243.

N.N.Bogoliubov and D.V.Shirkov, Introduction to the Theory of Quantized Fields, New York, 1959.

I.L.Bogolubsky, E.M. Ilgenfritz, M. M$\ddot{u}$ller-Preussker, A. Sternbeck, Phys.Lett.B **676** 69 (2009).

L.Brekke, P.G.O.Freund, Phys. Rep. **233** 1 (1993).

W. Buchmuller and S. H. H. Tye, Phys. Rev. D 24, 132 (1981).

D. Lopez-Val and J. Sola, Phys. Lett. B 702, 246 (2011) [arXiv:1106.3226 [hep-ph]]; P. Niezurawski, A. F. Zarnecki and M. Krawczyk, eConf C 050318, 0112 (2005) [hep-ph/0507006]; M. M. Muhlleitner et al., Phys. Lett. B 508, 311 (2001) [hep-ph/0101083]. S. Bae, B. Chung and P. Ko, Eur. Phys. J. C 54, 601 (2008) [hep-ph/0205212]; S. Y. Choi and J. S. Lee, Phys. Rev. D 62, 036005 (2000) [hep-ph/9912330].

I. Bakas, D. Orlando and P.M. Petropoulos, Ricci flows and expansion in axiondilaton cosmology, JHEP 0701 (2007) 040 [arXiv:hep-th/0610281].

A. P. Balachandran, S. Kurkcuoglu and E. Rojas, The Star Product on the Fuzzy Supersphere, JHEP 0207, 056 (2002) [arXiv:hep-th/0204170];

V.V. Balandin, N.I. Golubeva, *Hamiltonian Methods for the Study of Polarized Proton Beam Dynamics in Accelerators and Storage Rings*, [`arXiv://arxiv.org/abs/physics/9903032v1`].

V. Bargmann, L. Michel, V.L. Telegdi, *Precession of the polarization of particles moving in a homogeneous electromagnetic field*, Phys. Rev. Lett., **2(10)** (1959) 435.

John D. Barrow, Douglas J. Shaw, The Value of the Cosmological Constant, General Relativity and Gravitation 43, 2555-2560 (2011), arXiv:1105.3105.

Baskaran, G. (2011). "Possibility of Skyrmion Superconductivity in Doped Antiferromagnet $K_2Fe_4Se_5$". arXiv:1108.3562

G. Benenti, G. Casati, G. Strini, *Principles of quantum computation and information*, Vol. I: Basic concepts, World Scientific, Singapore 2004; Vol. II: *Basic tools and special topics* World Scientific, Singapore 2007.

J. Beringer et al. (Particle Data Group) *Review of Particle Physics*, Phys. Rev. D **86** (2012) 010001 [1528 pages]

M. Berry, *Speculations on the Riemann operator*, in proceedings of *Symposium on Supersymmetry and Trace formulae*, Cambridge 1997.

J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

J. D. Bjorken, S. J. Brodsky and A. Scharff Goldhaber, Phys. Lett. B 726, 344 (2013) [arXiv:1308.1435 [hep-ph]].

F.A. Berezin, *Introduction to Superanalysis*, Reidel, Dordrecht 1987.

F.A. Berezin, M.S. Marinov, Ann. Phys. (N.Y.) **104** (1977) 336.

J. Beringer et al. (Particle Data Group), PR D86, 010001 (2012)

A. Blondel et al., arXiv:1302.3318 [physics.acc-ph]; R. Belusevic and T. Higo, arXiv:1208.4956 [physics.acc-ph].

A. Bodmer "Collapsed Nuclei" Phys. Rev. D4, 1601 (1971)

W. Broer et al, Europ. Phys. Lett. 95 (2011) 30001.

N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, New York 1959.

I.L. Bogolubsky, E.M. Ilgenfritz, M. M$\ddot{u}$ller-Preussker, A. Sternbeck, *Phys.Lett.B* **676** (2009) 69.

R. H. Brandenberger, Lect. Notes Phys. 646:127-167 (2004), arXiv:hep-th/0306071.

L. Brekke, P.G.O. Freund, *Phys. Rep.* **233** (1993) 1.

S. Brodsky, G. Farrar, Phys. Rev. Lett. **31** 1153 (1973).

Stanley J. Brodsky, Guy F. de Tèramond, Alexandre Deur, Nonperturbative QCD Coupling and its β function from Light-Front Holography, PRD 81,096010 (2010), arXiv:1002.3948

S. J. Brodsky, M. Mojaza and X. G. Wu, Phys. Rev. D 89, 014027 (2014) [arXiv:1304.4631 [hep-ph]].

S. J. Brodsky and L. Di Giustino, Phys. Rev. D 86, 085026 (2012) [arXiv:1107.0338 [hep-ph]].

M. Bronshtein, K voprosu o vozmozhnoy teorii mira kak tselogo (On a possible theory of the world as a whole), in Uspekhi Astronomicheskikh Nauk, vypusk III (1933) pp. 3-30

G. E. Brown and M. Rho, Phys. Rept. 269 (1996) 333 [hep-ph/9504250].

V. D. Burkert, Phys. Rev. D 63, 097904 (2001).

C. Buttin, C.R. Acad. Sci. Paris. **269** (1969) 87.

P.Carruthers, C.C.Shih, Int. J. Mod. Phys. **A4** 5587 (1989).

G. Calcagni, Fractal universe and quantum gravity, Phys. Rev. Lett. 104 (2010) 251301 [arXiv:0912.3142].

S. Carlip, Spontaneous dimensional reduction in short-distance quantum gravity?, AIP Conf. Proc. 1196 (2009) 72 [arXiv:0909.3329].

A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov and A. K. Geim, The electronic properties of graphene, Rev. Mod. Phys. 81 , 109-162 (2009).

W. E. Caswell, Phys. Rev. Lett. 33 (1974) 244.

S. Chatrchyan et al. [CMS Collaboration], Phys. Lett. B 716, 30 (2012).

J.C. Collins, *Renormalization*, Cambrige Univ. Press, London 1984.

David Cox, John Little, Donal O'Shea, Ideals, Varieties, and Algorithms,
An Introduction to Computational Algebraic Geometry and Commutative Algebra, Springer 1998.

Ya.Z.Darbaidze, N.V.Makhaldiani, A.N.Sisakian, L.A.Slepchenko, TMF **34** 303 (1978).

M.Davis, Applied nonstandard analysis, New York, 1977.

Ya.Z. Darbaidze, N.V. Makhaldiani, A.N. Sisakian, L.A. Slepchenko, *TMF* **34** (1978) 303.

M. Davis, *Applied nonstandard analysis*, New York 1977.

D. Diakonov, Prog. Par. Nucl. Phys. 51 (2003) 173.

A. Di Giacomo, G. Paffuti, P. Rossi, *Selected Problems in Theoretical Physics [With Solutions]*, World Scientific 1992.

P.A.M. Dirac, *Proc. Roy, Soc.* **A167** (1938) 148.

P.A.M. Dirac. Generalized Hamiltonian dynamics. Can. J. Math., 2:129, 148, 1950.
P.A.M. Dirac. Generalized Hamiltonian dynamics. Proc. Roy. Soc. A, 246:326-332, 1958
P.A.M. Dirac. Lectures on Quantum Mechanics. Belfer Graduate School Monograph Series 3. Yeshiva University, New York, 1964.

P.A.M. Dirac, *Proc. Roy, Soc.* **A167** (1938) 148.

D. P. DiVincenzo and A. Peres, Quantum Codewords Contradict Local Realism, Physical Review A 55, pp. 4089-4092 (1997).

D. Djukanovic, M. R. Schindler, J. Gegelia, G. Japaridze, S. Scherer, *Phys. Rev. Lett.* **93** (2004) 122002.

S. D. Drell, A. C. Hearn, *Exact Sum Rule For Nucleon Magnetic Moments*, *Phys. Rev. Lett.* **16** (1966) 908.

M. Duff, L. Okun and G. Veneziano, Trialogue on the number of fundamental constants, arXiv:physics/0110060.

M. J. Duff, String triality, black hole entropy and Cayleys hyperdeterminant, Phys. Rev. D76 (2007) 025017.
R. Kallosh and A. Linde, Strings, black holes, and quantum information, Phys. Rev. D73 (2006) 104033.
P. Levay, Stringy black holes and the geometry of entanglement, Phys. Rev. D 74 (2006) 024030.

G. Dufour, R. Guérout, A. Lambrecht, V.V. Nesvizhevsky, S. Reynaud, A.Yu. Voronin, Quantum Reflection of Antihydrogen from Nanoporous Media, Physical Review A 87 (2013) 022506

Lisa Dyson, Matthew Kleban, Leonard Susskind, Disturbing Implications of a Cosmological Constant, JHEP 0210:011, 2002, arXiv:hep-th/0208013

E. Egorian and O. V. Tarasov, Theor. Math. Phys. 41 (1979) 863 [Teor. Mat. Fiz. 41 (1979) 26]

A. Einstein *Sittzungsber.preuss. Akad. Wiss.* **1** (1917) 142.

A. Einstein, B. Podolsky, and N. Rosen, Can Quantum Mechanical Description of Physical Reality Be Considered Complete?, Physical Review 47, pp. 777-800 (1935).

V. Elias, J. Phys. G. 27, 217 (2001).

M. Levin and X.-G. Wen, Detecting Topological Order in a Ground State Wave Function, Phys. Rev. Lett. 96, 110405 (2006) [arXiv:cond-mat/0510613];
A. Kitaev and J. Preskill, Topological entanglement entropy, Phys. Rev. Lett. 96, 110404 (2006) [arXiv:hep-th/0510092];
A. Hamma, R. Ionicioiu and P. Zanardi, Ground state entanglement and geometric entropy in the Kitaev's model, Phys. Lett. A 337, 22 (2005) [arXiv:quant-ph/0406202].

P. Calabrese and J. L. Cardy, Entanglement entropy and quantum field theory, J. Stat. Mech. 0406, P002 (2004) [arXiv:hep-th/0405152];
P. Calabrese and J. L. Cardy, Entanglement entropy and quantum field theory: A non-technical introduction, Int. J. Quant. Inf. 4, 429 (2006) [arXiv:quant-ph/0505193];
P. Calabrese and J. Cardy, Entanglement entropy and conformal field theory, J. Phys. A 42 (2009) 504005 [arXiv:0905.4013 [cond-mat.stat-mech]].

W. Ernst, I. Schmitt, *Nuovo Cim.* **A33** (1976) 195.

L.D. Faddeev, R. Jackiw, Phys.Rev.Lett. **60** (1988) 1692.

L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer, Berlin 1990.

E. Farhi and R. Jaffe, "Strange Matter", Phys. Rev. D30, 2379 (1984)

E. Fermi, Ricerca Scientifica 7 (1936) 13

E. Fermi, W.H. Zinn, Phys. Rev. 70 (1946) 103

E. Fermi, L. Marshall, Phys. Rev. 71 (1947) 666

E. Fermi, *Prog.Theor. Phys.* **5** (1951) 570.

S. Ferrara, M. Porrati, V.L. Telegdi, *Phys. Rev. D, $g = 2$ as the natural value of the tree level gyromagnetic ratio of elementary particles*, **46** (1992) 3529.

Feynman, R. P. (1955). Application of quantum mechanics to liquid helium. Progress in Low Temperature Physics 1: 1753.

I. V. Fialkovsky, D. V. Vassilevich , Quantum Field Theory in Graphene, arXiv:1111.3017

C. S. Fischer and H. Gies, JHEP 0410 (2004) 048.

V. Flambaum, I. Khriplovich and O. Sushkov, Nucl.Phys. A449, p. 750 (1986).

D.H. Friedan, Nonlinear models in two + epsilon dimensions, Phys. Rev. Lett. 45 (1980) 1057; Nonlinear models in two + epsilon dimensions, Ann. Phys. 163 (1985) 318.

Fukuda, J.-I.; Žumer, S. (2011). "Quasi-two-dimensional Skyrmion lattices in a chiral nematic liquid crystal". Nature Communications 2: 246.

G. Gamov, D. Ivanenko, L. Landau, Mirovye postoyannye i predel'ny perekhod [World constants and limiting transition], Zhurnal Russkogo Fiz.-Chim. Obschestva, chast' Fiz. 60 (1928) 13-17 (in Russian). Reprinted in Yad. Fiz. 65 (2002) No.7 [Physics of Atom. Nucl. 65 (2002) No.7].

G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge Univ. Press, Cambridge 1990.

I.M. Gel'fand, M.I. Graev and I.I. Piatetskii-Shapiro, *Representation Theory and Automorphic Functions*, Saunders, London 1966.

I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, Funck. Anal. i Priloz. 23 (1989) 94;

I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, Adv. Math. 84 (1990) 255;

I.M. Gelfand, M.I. Graev, V.S. Retakh, Russian Math. Surveys 47 (1992) 1.

S.B. Gerasimov, *A Sum Rule for Magnetic Moments and Damping of the Nucleon Magnetic Moment in Nuclei*, *J.Nucl.Phys.(USSR)* **2** (1965) 598.

Giorgadze G. Geometry of Quantum Computation,
Nova Publishers, (N.Y.), 2013.

M.L. Goldberger and S.B. Treiman, Phys. Rev. 110 (1958) 1178.

D.J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343;

V. L. Ginzburg, *Theoretical Physics and Astrophysics*, Pergamon, New York 1979.

A. Gustavsson, *Algebraic structures on parallel M2-branes*,
Nucl. Phys. B.**811** (2009) 66; [arXiv:0709.1260 [hep-th]].

G. Gutin, A.P. Punnen (Eds), *The Traveling Salesman Problem and Its Variations*, Combinatorial Optimization Series, Kluwer, Boston 2002.

Gibbons, G.W. and Hawking, S.W. (1977) 'Cosmological event horizons, thermodynamics, and particle creation' Physical Review D 15, 2738-2751

V. L. Ginzburg, *Theoretical Physics and Astrophysics*, Pergamon, New York 1979.

M.L. Goldberger, S.B. Treiman, Phys. Rev. **110** 1178 (1958).

Y.A. Gol'fand and E.P. Likhtman, JETP Lett., 13, 323, 1971.

R. Golub, W. Mampe, J. M. Pendelbury and P. Ageron, Scientific American, June 1979

Vasco Gonsalves, Joao Penedones, Emilio Trevisani, Factorization of Mellin amplitudes, arXiv:1410.4185v1 [hep-th] 15 Oct 2014.

M.B. Green, J.H. Schwarz and E. Witten. Super String Theory. Vols 1,2. Cambridge University Press 1987.

D. M. Greenberger, M. Horne, and A. Zeilinger, Going beyond Bell's Theorem in Bell's Theorem, Quantum Theory, and Conceptions of the Universe, edited by M. Kafatos, pp. 73-76 (Kluwer, Dordrecht, 1989).

Griffiths and Harris, Principles of Algebraic Geometry, [Wiley Classics Library Edition Published 1994]

D.J. Gross, F. Wilczek, Phys. Rev. Lett. **30** 1343 (1973).

H. Grosse, C. Klimcik, P. Presnajder, Field Theory on a Supersymmetric Lattice, Commun.Math. Phys., 185 (1997) 155-175 and hep-th/9507074;H. Grosse, C. Klimcik, P. Presnajder, N=2 Superalgebra and Non-Commutative Geometry,hep-th/9603071;H. Grosse, G. Reiter, The Fuzzy Supersphere, J. Geom. and Phys., 28 (1998) 349-383 and math-ph/9804013.

S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109];

A. Gustavsson, *Algebraic structures on parallel M2-branes*, Nucl. Phys. B.**811** (2009) 66; [arXiv:0709.1260 [hep-th]].

G. Gutin, A.P. Punnen (Eds), *The Traveling Salesman Problem and Its Variations*, Combinatorial Optimization Series, Kluwer, Boston 2002.

Hagedorn, Nuovo Cim. Suppl. 3, (1965) 147 and Nuovo Cim. A 52, 64 (1967), R. Hagedorn, CERN 71-12, 1971, and CERN-TH.7190/94, 1994, and references therein; K. Huang and S.Weinberg, Phys. Rev. Lett. 25, 895 (1970); W. Nahm, Nucl. Phys. B 45, 525 (1972). For the most recent analysis see W. Broniowski, W. Florkowski and L. Ya. Glozman, Phys. Rev. 70, 117503 (2004).

R. Hamilton, Three-manifolds with positive Ricci curvature, J. Diff. Geom. 17 (1982) 255.

Heinze, Stefan; Von Bergmann, Kirsten; Menzel, Matthias; Brede, Jens; Kubetzka, André; Wiesendanger, Roland; Bihlmayer, Gustav; Blügel, Stefan (2011). "Spontaneous atomic-scale magnetic skyrmion lattice in two dimensions". Nature Physics 7 (9): 713718.

W. Heisenberg, *Z. Phys.* **126** (1949) 569.

W. Heisenberg, *Introduction to the Unified field Theory of Elementary particles,* Interscience Publishers, London 1966.

G. 't Hooft, Nucl. Phys. B 33 (1971) 173; Nucl. Phys. B 35 (1971) 167.

G. 't Hooft, *Nucl.Phys.* **B 61** (1973) 455.
G. 't Hooft, Erice Lectures 1977 (unpublished).

G. 't Hooft, Dimensional reduction in quantum gravity, in Salamfestschrift, A. Ali, J. Ellis, and S. Randjbar-Daemi (eds.), World Scientific, Singapore (1993) [gr-qc/9310026].

G. 't Hooft, report at the *Marseille Conference on Yang-Mills Fields*, 1972.

G. 't Hooft, *Nucl.Phys.* **B 61** (1973) 455.

P. Horava, Phys. Lett. B 694,172 (2010) [arXiv:0811.2217 [hep-th]].
P. Horava, Phys. Rev. D79, 084008 (2009) [arXiv:0901.3775]; JHEP 03, 020 (2009) [arXiv:0812.4287];
P. Horava, Phys. Lett. B 694, 172 (2010) [arXiv:0811.2217]; Phys. Rev. Lett. 102, 161301 (2009) [arXiv:0902.3657].

R. M. Hornreich, M. Luban and S. Shtrikman, Phys. Rev. Lett. 35, 1678 (1975).
G. Grinstein, Phys. Rev. B23, 4615 (1981) .
E. Ardonne, P. Fendley and E. Fradkin, Annals Phys. 310, 493 (2004) [cond-mat/0311466].
E. Fradkin, J. of Phys. A42, 504011 (2009).

P. Howe, K. Stelle and P. Townsend, Nucl. Phys. B214, 519 (1983).

P. Howe, K.S. Stelle and P. West, Phys. Lett. 124B, 55 (1983).

Ernst-Michael Ilgenfritz, Axel Maas, Topological aspects of G2 Yang-Mills theory, arXiv:1210.5963.

A. Izergin and V. Korepin, Letter in Mathematical Physics vol 6, page 283, 1982.

I. Jack, D.R.T. Jones and C.G. North, Nucl. Phys. B486, 479(1997).

I. Jack. D.R.T. Jones and A. Pickering, Phys. Lett. B435, 61 (1998).

J.D. Jackson, *Classical Electrodynamics, 3rd ed.* JohnWiley & Sons, Inc. New York 1999.

Jackson, J.D. (1957). "Catalysis of Nuclear Reactions between hydrogen isotopes by –Mesons". Physical Review 106 (2): 330.

X. Ji, J.Osborne, J.Phys. G27 127 (2001).

D. R. T. Jones, Nucl. Phys. B 75 (1974) 531;

D.R.T. Jones, Phys. Lett. 123B, 45 (1983).

M. Kaku, *Strings, Conformal Fields, and M-Theory*, Springer, New York 2000.

D. B. Kaplan and H. Georgi, Phys. Lett. B 136 (1984) 183

A. L. Kataev, S. A. Larin, Analytical five-loop expressions for the renormalization group QED β-function in different renormalization schemes, Pisma v ZhETF, vol.96, iss. 1, (2012), pp.64-67

K. Kawarabayashi, M. Suzuki, *Phys. Rev. Lett.* **16** (1966) 255

D.I.Kazakov, L.R.Lomidze, N.V.Makhaldiani, A.A.Vladimirov, Ultraviolet Asymptotics in Renormalizable Scalar Theories, Communications of JINR, **E2-8085**, Dubna, 1974.

D.I.Kazakov, *Supersymmetric Generalization of the Standard Model of Fundamental Interactions*, Textbook, (Dubna, JINR, 2004).

D.I.Kazakov, D.V.Shirkov, Fortschr. d. Phys. **28** 447 (1980).

D.I. Kazakov, L.R. Lomidze, N.V. Makhaldiani, A.A. Vladimirov, *Ultraviolet Asymptotics in Renormalizable Scalar Theories*, JINR Communications, **E2-8085**, Dubna 1974.

V. Khachatryan et al. [CMS Collaboration], JHEP 1009, 091 (2010) [arXiv:1009.4122 [hep-ex]].

Valentin V. Khoze, Inflation and Dark Matter in the Higgs Portal of Classically Scale Invariant Standard Model, arXiv:1308.6338v1 [hep-ph] 29 Aug 2013,

I. B. Khriplovich, Yad. Fiz. 10 (1969) 409 [Sov. J. Nucl. Phys. 10 (1970) 235]

Kiselev, N. S.; Bogdanov, A. N.; Schäfer, R.; Rößler, U. K. (2011). "Chiral skyrmions in thin magnetic films: New objects for magnetic storage technologies?". Journal of Physics D: Applied Physics 44 (39): 392001. arXiv:1102.2726

A.Yu. Kitaev, A. Shen, M.N. Vyalyi, *Classical and Quantum Computation*, American Mathematical Society, 2002.

Takuya Kitamoto, On the Computation of the Determinant of a Generalized Vandermonde Matrix, in Computer Algebra, in Proceedings of "Scientific Computing, 16th International Workshop, CASC 2014" Warsaw, Poland, September 8-12, 2014, Vladimir P. Gerdt Wolfram Koepf Werner M. Seiler Evgenii V. Vorozhtsov (Eds.), Lecture Notes in Computer Science 8660, p.p. 242-255.

Z. Koba, H.B. Nielsen, P. Olesen, *Nucl. Phys.* **B40** (1972) 317.

Neal Koblitz, p-adic numbers, p-adic analysis, and Zeta-functions, Springer-Verlag, New York Heidelberg Berlin, 1977.

I.G. Koh and S. Rajpoot, Phys. Lett. 135B, 397 (1983).

P. Kovtun, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 94 (2005) 111606 [arXiv:hep-th/0405231]

Kulish, P.P. and Reshetikhin, N.Yu., Zap. Sem. LOMI 101 (1981) 101 [J. Soviet. Math. 23 (1983) 2435].
Sklyanin, E.K., Funkt. Anal. Pril. 16 (1982) 27 [Funct. Anal. Appl. 16 (1982) 262].
Drinfel'd, V.G., DAN SSSR 283 (1985) 1060 [Soviet Math. Dokl. 32 (1985) 254] ; in Proc. Int. Congr. of Mathematics, ed. A.M. Gleason, AMS : Providence (1986).
Jimbo, M., Lett. Math. Phys. 10 (1985) 63 ; 11 (1986) 247 ; Commun. Math. Phys. 102 (1986) 537.
Woronowicz, S.L., Publ. RIMS-Kyoto 23 (1987) 117 ; Commun. Math. Phys. 111 (1987) 613.

M. Kuster, G. Raffelt and B. Beltran (eds), Lecture Notes in Physics 741 (2008).

A.Yu. Kitaev, A. Shen, M.N. Vyalyi, *Classical and Quantum Computation*, American Mathematical Society, 2002.

Laughlin, R. (1981). "Quantized Hall conductivity in two dimensions". Physical Review B 23 (10): 56325633.

S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303 (1993) 334.

C. Lerche and L. von Smekal, Phys. Rev. D 65 (2002) 125006.

I. Lomidze, *On Some Generalizations of the Vandermonde Matrix and their Relations with the Eiler Beta-function*, *Georgian Math. Journal* **1** (1994) 405.

I. R. Lomidze and N. V. Makhaldiani, SOME PROPERTIES OF THE GENERALIZED EULER BETA FUNCTION, *Memoirs on Differential Equations and Mathematical Physics* **56** (2012)

C. Lucchesi, O. Piguet and K. Sibold, Helv. Phys. Acta 61 (1988) 321

C. Lucchesi and G. Zoupanos, Fortschr. Phys. 45 (1997) 129, hep-ph/9604216

V.I. Lushikov et al., Sov. Phys. JETP Lett. 9 (1969) 23

S. Moch, QCD studies and Higgs searches at the LHC, Lectures at the International Achool-Workshop "Calculations for Modern and Future Colliders", July23-August 2, 2012, Dubna, Russian Federation.

N.V. Makhaldiani, *Approximate methods of the field theory and their applications in physics of high energy, condensed matter, plasma and hydrodynamics*, Dubna 1980.

N.M. Makhaldiani, *Computational Quantum Field Theory*, JINR Communication, **P2-86-849**, Dubna 1986.

N.V. Makhaldiani, *Number Fields Dynamics and the Compactification of Space Problem in the Unified Theories of Fields and Strings*, JINR Communications, **P2-88-916**, Dubna 1988.

N. Makhaldiani, O. Voskresenskaya, *On the correspondence between the dynamics with odd and even brackets and generalized Numbu's mechanics,* JINR Communications, **E2-97-418**, Dubna 1997.

N. Makhaldiani, *The System of Three Vortexes of Two-Dimensional Ideal Hydrodynamics as a New Example of the (Integrable) Nambu-Poisson Mechanics,* JINR Communications **E2-97-407**, Dubna 1997; [arXiv:solv-int/9804002].

N. Makhaldiani, *The Algebras of the Integrals of Motion and Modified Bochner-Killing-Yano Structures of the Point particle Dynamics*,
JINR Communications **E2-99-337**, Dubna 1999.

N. Makhaldiani, *New Hamiltonization of the Schrödinger Equation by Corresponding Nonlinear Equation for the Potential,*
JINR Communications, **E2-2000-179**, Dubna 2000.

N. Makhaldiani, *How to Solve the Classical Problems on Quantum Computers,*
JINR Communications, **E2-2001-137**, Dubna 2001.

N. Makhaldiani, *Classical and Quantum Problems for Quanputers,* [arXiv:quant-ph/0210184].

N. Makhaldiani, *Adelic Universe and Cosmological Constant,*
JINR Communications, **E2-2003-215** Dubna 2003, [arXiv:hep-th/0312291].

N. Makhaldiani, *Nambu-Poisson dynamics of superintegrable systems*, *Atomic Nuclei*, **70** (2007) 564.

N. Makhaldiani, *Theory of Quanputers*, Sovremennaia Matematica i ee Prilozhenia, **44** (2007) 113; Journal of Mathematical Sciences, **153** (2008) 159.

N.V. Makhaldiani, *Renormdynamics and Scaling Functions*, in Proc. of the *XIX International Baldin Seminar on High Energy Physics Problems* eds. A.N.Sissakian, V.V.Burov, A.I.Malakhov, S.G.Bondartenko, E.B.Plekhanov, Vol.II, p. 175, Dubna 2008.

N.V. Makhaldiani, *Renormdynamics, multiparticle production, negative binomial distribution and Riemann zeta function*, [arXiv:1012.5939v1 [math-ph]] 24 Dec 2010.

N. Makhaldiani, *Fractal Calculus (H) and some Applications*, *Physics of Particles and Nuclei Letters*, **8** (2011) 325.

N. Makhaldiani, *Regular method of construction of the reversible dynamical systems and their linear extensions - Quanputers*, Atomic Nuclei, **74** (2011) 1040.

Nugzar Makhaldiani, *Nambu-Poisson Dynamics with Some Applications*,
Physics of Particles and Nuclei, **43** (2012) 703.

Nugzar Makhaldiani, *Renormdynamics, coupling constant unification and universal properties of the multiparticle production*, XXI International Baldin Seminar on High Energy Physics Problems, September 10-15, 2012, PoS(Baldin ISHEPP XXI)068.

J. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231, http://arxiv.org/abs/hep-th/9711200

V.A.Matveev, A.N.Sisakian, L.A.Slepchenko, Nucl. Phys. **23** 432 (1976).

A.V. Meleshko, N.N. Konstantinov, *Dynamics of vortex systems*,
Naukova Dumka, Kiev 1993.

W.Miller, Jr. Symmetry and Separation of Variables, Addison-Wesley PC, London, 1977.

M.Muehlleitner, Composite Higgs and SUSY Physics at the LHC, Lectures at the International Achool-Workshop "Calculations for Modern and Future Colliders", July23-August 2, 2012, Dubna, Russian Federation.

Y. Nambu, Phys.Rev. D **7** (1973) 2405.

M.A. Nielsen, I.L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge 2000.

L.B.Okun, Leptons and Quarks, North Holland, 1982.

I. Pomeranchuk, *Dokl. Akad. Nauk SSSR* **78** (1951) 889.

H.D. Politzer, Phys. Rev. Lett. 30 (1973) 1346;

L.S.Pontriagin et al., Mathematical Theory of Optimal Processes, Nauka, Moscow, 1983.

M.C.M. Rentmeester, R.G.E. Timmermans, J.L. Friar and J.J. de Swart, *Phys. Rev. Lett.* **82** (1999) 4992.

A. Rényi, Introduction to information theory, North-Holland, Amsterdam, 1970.

J.J. Sakurai, *Currents and mesons*, The University of Chicago Press, Chicago 1969.

T. Schaefer and D. Teaney, Rep. Prog. Phys. 72 (2009) 126001.

V.Smirnov, Evaluating five-loop Konishi in N=4 SYM, Talk at the International Achool-Workshop "Calculations for Modern and Future Colliders", July23-August 2, 2012, Dubna, Russian Federation.

A. Samarskii, A. Gulin, *Numerical Methods*, Nauka, Moscow 1989.

Ya.G. Sinai, *Topics in Ergodic Theory*, Princeton University Press, Princeton NJ 1993.

G. 't Hooft, report at the Marseille Conference on Yang-Mills Fields, 1972.

G. 't Hooft, Nucl.Phys. B 61 (1973) 455.

E.C. Titchmarsh, The Theory of the Riemann zeta-function (Clarendon Press, Oxford, 1986).

M.V.Tokarev, I.Zborovský, Z-Scaling in the Proton-Proton Collisions at RHIC, in Investigations of Properties of Nuclear Matter at High Temperature and Densities, Editid by A.N. Sisakian, F.A. Soifer, Dubna, 2007.

M.V.Tokarev, I.Zborovský, T.G.Dedovich, Z-Scaling at RHIC and Tevatron, hep-ph, 0708.246, 2007.

C. Tsallis, *J. Stat. Phys.* **52** (1988) 479;
C. Tsallis, *Brazillian Journal of Physics* **29(1)** (1999);
Murray Gell-Mann and Constantino Tsallis eds., Nonextensive Entropy: Interdisciplinary Applications, Oxford (2004).

Norman Margolus, Tommaso Toffoli, *Cellular Automaton Machines*,
MIT Press, Cambridge 1987.

L.H. Thomas, Philos. Mag. **3** (1927) 1.

V.S. Vladimirov, Russian Math. Surveys **43** 19 (1988).

M.B.Voloshin, K.A.Ter-Martyrosian, Gauge Theory of Elementary Particles, Atomizdat, Moscow 1984.

M.Watkins at http://secamlocal.ex.ac.uk/∼ mwatkins/zeta/physics.htm.

S. Weinberg, *Phys. Rev.* **166** (1968) 1568

S. Weinberg, Gravitation and Cosmology, New York, 1972.

S.Weinberg, The Quantum Theory of Fields, Volum I - Foundations, Cambridge Univ. Press, 1995.

S.Weinberg, The Quantum Theory of Fields, Volum II - Modern Applications, Cambridge Univ. Press, 1996.

S.Weinberg, The Quantum Theory of Fields, Volum III - Supersymmetry, Cambridge Univ. Press, 2000.

Cheuk-Yin Wong, Grzegorz Wilk, Tsallis Fits to p_T Spectra for pp Collisions at LHC, *Acta Physica Polonica* **B42** (2012) 2047, arXiv:1210.3661.

K.G.Wilson and J.Kogut, Phys. Rep. **12C** 75 (1974).

E.T. Whittaker, *A Treatise on the Analytical Dynamics*, Cambridge 1927.

M. E. Machacek and M. T. Vaughn, Nucl. Phys. B 222, 83 (1983).

N.V. Makhaldiani, *Approximate methods of the field theory and their applications in physics of high energy, condensed matter, plasma and hydrodynamics*, Dubna 1980.

N.M. Makhaldiani, *Computational Quantum Field Theory*, JINR Communication, **P2-86-849**, Dubna 1986.

N.V. Makhaldiani, *A New Approach to the Problem of Space Compactification,*
JINR Communications, **P2-87-306**, Dubna 1987.

N.V. Makhaldiani, *Number Fields Dynamics and the*
Compactification of Space Problem in the Unified Theories of Fields and Strings,
JINR Communications, **P2-88-916**, Dubna 1988.

N. Makhaldiani, *The System of Three Vortexes of Two-Dimensional Ideal Hydrodynamics as a New Example of the (Integrable) Nambu-Poisson Mechanics*,
JINR Communications **E2-97-407**, Dubna 1997; [arXiv:solv-int/9804002].

N. Makhaldiani, *The Algebras of the Integrals of Motion and Modified Bochner-Killing-Yano Structures of the Point particle Dynamics*, JINR Communications **E2-99-337**, Dubna 1999.

N. Makhaldiani, *New Hamiltonization of the Schrödinger Equation by Corresponding Nonlinear Equation for the Potential,* JINR Communications, **E2-2000-179**, Dubna 2000.

N. Makhaldiani, *How to Solve the Classical Problems on Quantum Computers,* JINR Communications, **E2-2001-137**, Dubna 2001.

N. Makhaldiani, *Classical and Quantum Problems for Quanputers,* [arXiv:quant-ph/0210184].

N. Makhaldiani, *Adelic Universe and Cosmological Constant,* JINR Communications, **E2-2003-215** Dubna 2003, [arXiv:hep-th/0312291].

N. Makhaldiani, *Nambu-Poisson dynamics of superintegrable systems*, *Atomic Nuclei*, **70** (2007) 564.

N. Makhaldiani, *Theory of Quanputers*, Sovremennaia Matematica i ee Prilozhenia, **44** (2007) 113; Journal of Mathematical Sciences, **153** (2008) 159.

N.V. Makhaldiani, *Renormdynamics and Scaling Functions*, in Proc. of the *XIX International Baldin Seminar on High Energy Physics Problems* eds. A.N.Sissakian, V.V.Burov, A.I.Malakhov, S.G.Bondartenko, E.B.Plekhanov, Vol.II, p. 175, Dubna 2008.

N.V. Makhaldiani, *Renormdynamics, multiparticle production, negative binomial distribution and Riemann zeta function*, [arXiv:1012.5939v1 [math-ph]] 24 Dec 2010.

N. Makhaldiani, *Fractal Calculus (H) and some Applications*, *Physics of Particles and Nuclei Letters*, **8** 325 (2011).

N. Makhaldiani, *Regular method of construction of the reversible dynamical systems and their linear extensions - Quanputers*, Atomic Nuclei, **74** (2011) 1040.

Nugzar Makhaldiani, *Nambu-Poisson Dynamics with Some Applications,* Physics of Particles and Nuclei, **43** (2012) 703.

Nugzar Makhaldiani, *Renormdynamics, coupling constant unification and universal properties of the multiparticle production*, XXI International Baldin Seminar on High Energy Physics Problems, September 10-15, 2012, PoS(Baldin ISHEPP XXI)068.

N. V. Makhaldiani, *Renormdynamics, Multiparticle Production, Negative Binomial Distribution, and Riemann Zeta Function*, Physics of Atomic Nuclei, **76** 1169 (2013).

N.V.Makhaldiani, S.S.Postnov, work in progress

N. Makhaldiani, O. Voskresenskaya, *On the correspondence between the dynamics with odd and even brackets and generalized Numbu's mechanics,*
JINR Communications, **E2-97-418**, Dubna 1997.

J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

S. Mandelstam, Nucl. Phys. B213, 149 (1983).

V. Matveev, R. Muradyan, A. Tavkhelidze, Lett. Nuovo Cimento **7** 719 (1973).

V.A. Matveev, A.N. Sisakian, L.A. Slepchenko, *Nucl. Phys.* **23** (1976) 432.

A.V. Meleshko, N.N. Konstantinov, *Dynamics of vortex systems*,
Naukova Dumka, Kiev 1993.

W. Miller, Jr. *Symmetry and Separation of Variables*, Addison-Wesley PC, London 1977.

C.W. Misner, K.S. Thorne, J.A. Wheeler. Gravitation. W.H.Freeman and Company, twenty third printing 2000.

J.W. Morgan and G. Tian, Ricci Flow and the Poincar Conjecture, Clay Mathematics Monographs, Amer. Math. Soc., Cambridge, 2007 [math.DG/0607607].

J. Ellis, G. Ridolfi and F. Zwirner, Phys. Lett. 262B (1991) 477; R. Barbieri and M. Frigeni, Phys. Lett. 258B (1991) 395; A. Brignole et al., Phys. Lett. 271B (1991) 123; M. Carena, K. Sasaki and C.E.M. Wagner, Nucl. Phys. 381B (1992) 66; H.E. Haber, R. Hempfling, Phys. Rev. D48 (1993) 4280; P.H. Chankowski, S. Pokorski and J. Rosiek, Phys. Lett. 281B (1992) 100

D. Mumford, *Tata Lectures on Theta,* Birkhäuser 1983.

Y. Nambu, Phys.Rev. D **7** (1973) 2405.

- A. Neveu and J.H. Schwarz, Nucl. Phys., B31, 86, 1971.
- M.A. Nielsen, I.L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge 2000.
- H.P. Nilles, Phys. Rept. 110 (1984) 1; H.E. Haber and G.L. Kane, Phys. Rept. 117 (1985) 75; R. Barbieri, Riv. Nuovo Cim. 11 (1988) 1.
- V. Novikov, V. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. B229, 381 (1983).
- S. P. Novikov, S. V.Manakov, L. B. Pitaevskii, and V. E. Zakharov, Theory of solitons. The inverse scattering method, Plenum Press, New York, 1984.
- R. Oehme and W. Zimmermann, Commun. Math. Phys. 97 (1985) 569.
- L. Okun, The fundamental constants of physics, Sov. Phys. Usp. 34 (1991) 818.
- W. Pauli, Z. Physik 31 765 (1925).
- W. Pauli, Phys. Rev. 58 716 (1940).
- J. C. Pati and A. Salam, Physical Review D10, 275 (1974).
- J. M. Pawlowski, D. F. Litim, S. Nedelko and L. von Smekal, Phys. Rev. Lett. 93 (2004) 152002.
- G. Perelman, The entropy formula for the Ricci flow and its geometric applications [math.DG/0211159]; Ricci flow with surgery on three-manifolds [math.DG/0303109]; Finite extinction time for the solutions to the Ricci flow on certain three-manifolds [math.DG/0307245].
- S. Perlmutter et al., Ap.J. 517, 565 (1999).
- M. E. Peskin, arXiv:1207.2516 [hep-ph].
- O. Piguet and K. Sibold, Int. J. Mod. Phys. A1 (1986) 913;

M. Planck, $\ddot{U}$ber irrevesible Strahlungsvorg$\ddot{a}$nge, S.-B. Preuss Akad. Wiss. (1899) 440-480; Ann. d. Phys. 1 (1900) 69 reprinted in Max Planck, Physikalische Abhandlungen und Vortr$\ddot{a}$ge, Band I. Friedr. Vieweg. 1958, pp. 560-600, pp. 614-667.

H.D. Politzer, Phys. Rev. Lett. **30** 1346 (1973).

L.S. Pontriagin et al., *Mathematical Theory of Optimal Processes*, Nauka, Moscow 1983.

P. Ramond, Phys. Rev., D3, 2415, 1971.

M.C.M. Rentmeester, R.G.E. Timmermans, J.L. Friar, J.J. de Swart, Phys. Rev. Lett. **82** 4992 (1999).

Riazuddin, Fayyazuddin, *Phys. Rev.* **147** (1966) 1071.

A.G. Riess et al., Ap.J. 116, 1009 (1998);

B. L. Roberts and W. J. Marciano (eds.), Lepton Dipole Moments, Advanced Series on Directions in High Energy Physics, Vol. 20 (World Scientific, 2010).

Romming, N.; Hanneken, C.; Menzel, M.; Bickel, J. E.; Wolter, B.; Von Bergmann, K.; Kubetzka, A.; Wiesendanger, R. (2013). "Writing and Deleting Single Magnetic Skyrmions". Science 341 (6146): 6369

A. Sakharov, Pisma Zh.Eksp.Teor.Fiz. 5, 32 (1967), Reprinted in Sov. Phys. Usp. 34 (1991) 392-393 [Usp. Fiz. Nauk 161 (1991) No. 5 61-64].

J.J. Sakurai, *Currents and mesons*, The University of Chicago Press, Chicago 1969.

A. Samarskii, A. Gulin, *Numerical Methods*, Nauka, Moscow 1989.

C. Schmidhuber and A.A. Tseytlin, On string cosmology and the RG flow in 2-d field theory, Nucl. Phys. B426 (1994) 187 [arXiv:hep-th/9404180].

J. Schlienz and G. Mahler, Description of Entanglement, Physical Review A 52, pp. 4396-4404 (1995).

J. Schlienz and G. Mahler, The maximal entangled threeparticle state is unique, Physics Letters A 224, pp. 39-44 (1996).

J. Schwinger, *Magnetic Charge and Quantum Field Theory, Phys. Rev.* **144** (1966) 1087.

Schwinger, J. Electromagnetic mass revisited. Found. Phys. 13, 373 (1983).

Werner M. Seiler, Robin W. Tucker, Involution and Constrained Dynamics I: The Dirac Approach, in Proceedings of "Scientific Computing, 16th International Workshop, CASC 2014" Warsaw, Poland, September 8-12, 2014, Vladimir P. Gerdt Wolfram Koepf Werner M. Seiler Evgenii V. Vorozhtsov (Eds.), Lecture Notes in Computer Science 8660, p.p. 242-255.

Ya.G. Sinai, *Topics in Ergodic Theory*, Princeton University Press, Princeton NJ 1993.

A. Steyerl, Phys. Lett. B29 (1969) 33

G.J. Stoney, The philosophical magazine and journal of science, 11 (1881) 381.

K. Oura, V.G. Lifshits, A.A. Saranin, Surface science: An introduction. Bergin-Verlag, Berlin-Heidelberg : s.n., 2003.
K.W. Kolasinski, Surface science: foundations of catalysis and nanoscience. Chichester, West Sussex : John Wiley and Sons, 2008.
G.Antczak, G. Ehrlich, Surf. Sci. Rep. 62 (2003) 39.
E.Shustorovich, Metal-surface reaction energetics: theory and applications to heterogeneous analysis, chemisorption, and surface diffusion. s.l. : VCN Publishers, Inc., 1991

Y. A. Golfand and E. P. Likhtman, JETP Letters 13 (1971) 452; D. V. Volkov and V. P. Akulov, JETP Letters 16 (1972) 621; J. Wess and B. Zumino, Phys. Lett. B49 (1974) 52.
P. Fayet and S. Ferrara, Phys. Rep. 32 (1977) 249; M. F. Sohnius, Phys. Rep. 128 (1985) 41; H. P. Nilles, Phys. Rep. 110 (1984) 1; H. E. Haber and G. L. Kane, Phys. Rep. 117 (1985) 75; A. B. Lahanas and D. V. Nanopoulos, Phys. Rep. 145 (1987) 1.
J. Wess and J. Bagger, Supersymmetry and Supergravity, Princeton Univ. Press, 1983.
A. Salam, J. Strathdee, Nucl. Phys. B76 (1974) 477; S. Ferrara, J. Wess, B. Zumino, Phys. Lett. BS1 (1974) 239.
S. J. Gates, M. Grisaru, M. Rocek and W. Siegel, Superspace or One Thousand and One Lessons in Supersymmetry, Benjamin Cummings, 1983; P. West, Introduction to supersymmetry and supergravity, World Scientific, 1990; S. Weinberg,The quantum theory of fields, Vol. 3, Cambridge, UK: Univ. Press, 2000.

O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. B 93 (1980) 429.

T.R. Taylor, and G. Veneziano, Dilaton Couplings at Large Distance, Phys. Lett. B 213, 450 (1988).

Tegmark, Max; et al. (2004). Cosmological parameters from SDSS and WMAP. Physical Review D 69 (103501): 103501. arXiv:astro-ph/0310723.

E.C. Titchmarsh, *The Theory of the Riemann zeta-function,* Clarendon Press, Oxford 1986.

Norman Margolus, Tommaso Toffoli, *Cellular Automaton Machines*,
MIT Press, Cambridge 1987.

M.V. Tokarev, I. Zborovský, *Z-Scaling in the Proton-Proton Collisions at RHIC*, in *Investigations of Properties of Nuclear Matter at High Temperature and Densities, Editid by A.N. Sisakian, F.A. Soifer*, Dubna 2007

L.H. Thomas, Philos. Mag. **3** (1927) 1.

S. B. Treiman, E. Witten, R. Jackiw, B. Zumino, *Current Algebra and Anomalies*, World Scientific, Singapore 1985).

Alexander Tzalenchuk et al, Quantum Resistance Standard Based on Epitaxial Graphene, Nature Nanotechnology 5, 186 - 189 (2010).

V. S. Vanyashin and M. T. Terentyev, Sov. Phys. JETP 21 (1965) 375.

T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B 400 (1997)379.

P.G. van Zwol et al, Phys. Rev. B. 80 (2009) 235401.

B.V. Vasilishin et al., JINR Preprint 9-86-512 (Dubna, 1986).

G. Veneziano, A stringy nature needs just two constants, Europhys. Lett. 2 (1986) 199.

G. Veneziano, A Simple/Short Introduction to Pre-Big-Bang Physics/Cosmology, arXiv:hep-th/9802057

A. Vilenkin, Phys. Rev. Lett. 74, 864 (1995).

V.S. Vladimirov, *Russian Math. Surveys* **43** (1988) 19.

M.B. Voloshin, K.A. Ter-Martyrosian, *Gauge Theory of Elementary Particles*, Atomizdat, Moscow 1984.

J. von Neumann, Theory of self-reproducing automata (University of Illinois Press, Urbana and London, 1966).

M. Watkins at http://secamlocal.ex.ac.uk/∼ mwatkins/zeta/physics.htm.

S. Weinberg, Phys. Rev. 166 (1968) 1568

S. Weinberg, *Gravitation and Cosmology*, New York 1972.

S. Weinberg (1987). "Anthropic Bound on the Cosmological Constant". Phys. Rev. Lett. 59 (22): 2607-2610

S. Weinberg, *Rev. Mod. Phys.* **49** (1989) 1.

S. Weinberg, *The Quantum Theory of Fields*, Volum I - Foundations, Cambridge Univ. Press 1995.

S. Weinberg, *The Quantum Theory of Fields*, Volum II - Modern Applications, Cambridge Univ. Press 1996.

S. Weinberg, *The Quantum Theory of Fields*, Volum III - Supersymmetry, Cambridge Univ. Press 2000.

S. Weinberg. Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. John Wiley and Sons, Inc. 1 edition, 1972;

S. Weinberg, AIP Conf.Proc. 272, 346 (1993).

W. I. Weisberger, *Phys. Rev.* **143** (1966) 1302.

J. Wess and B. Zumino, Nucl. Phys., B70, 39, 1974

E.T. Whittaker, *A Treatise on the Analytical Dynamics*, Cambridge 1927.

C. M. Will, The Confrontation between General Relativity and Experiment, Living Rev. Relativity 9, (2006), http://www.livingreviews.org/lrr-2006-3.

E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

E. Witten, "Cosmic Separation Of Phases" Phys. Rev. D30, 272 (1984)

E. Witten, Some Properties of O(32) Superstrings, Phys. Lett. B 149, 351 (1984).

Ya.B. Zeldovich, Sov. Phys. JETP-9 (1959) 1389

Zelikin, Mikhail I. (2000), Homogeneous Spaces and the Riccati Equation in the Calculus of Variations, Berlin: Springer-Verlag

A. Zelmanov, Kosmologia (Cosmology), in: Razvitie astronomii v SSSR [Development of astronomy in USSR], Nauka, Moscow, 1967, pp. 320-390, in particular p. 323 (in Russian).

W. Zimmermann, The Renormalization Group of the Model of A4-Coupling in the Abstract Approach of Quantum Field Theory, Commun. Math. Phys. 76 (1980) 3964.

W. Zimmermann, Commun. Math. Phys. 97 (1985) 211

D. Zwanziger, Phys. Rev. D 65 (2002) 094039.

Sinai Ya. G. Topics in Ergodic Theory. Princeton, NJ: Princeton University Press, 1993.

Nambu Y.// Phys.Rev. D. 1973. V. 7. P. 2405.

Whittaker E.T. A Treatise on the Analytical Dynamics. Cambridge, 1927.

Makhaldiani N. Nambu-Poisson dynamics of superintegrable systems.// Atomic Nuclei. 2007. V. 70. P. 564.

Arnold V.I. Mathematical Methods of Classical Mechanics. New York: Springer, 1978.

Makhaldiani N., Voskresenskaya O. On the correspondence between the dynamics with odd and even brackets and generalized Nambu's mechanics. JINR Commun. E2-97-418. Dubna, 1997.

Berezin F.A. Introduction to Superanalysis. Dordrecht: Reidel, 1987.

Buttin C.// C.R. Acad. Sci. Paris. 1969. V. 269. P. 87.

L.D.Faddeev, Jackiw R.// Phys.Rev.Lett. 1988. V.60. P. 1692.

Makhaldiani N. The Algebras of the Integrals of Motion and Modified Bochner-Killing-Yano Structures of the Point particle Dynamics. JINR Commun. E2-99-337. Dubna, 1999.

Aref H.// Ann.Rev.Fluid Mech. 1983. V. 15. P. 345.

Meleshko A.V., Konstantinov N.N. Dynamics of vortex systems. Kiev: Naukova Dumka, 1993.

Makhaldiani N. The System of Three Vortexes of Two-Dimensional Ideal Hydrodynamics as a New Example of the (Integrable) Nambu-Poisson Mechanics. JINR Commun. E2-97-407. Dubna, 1997; solv-int/9804002.

Makhaldiani N. New Hamiltonization of the Schrödinger Equation by Corresponding Nonlinear Equation for the Potential. JINR Commun. E2-2000-179. Dubna, 2000.

Bagger J. and Lambert N. Modeling multiple M2's// Phys. Rev. D. 2007. V. 75. P. 045020. [arXiv:hep-th/0611108].

Gustavsson A. Algebraic structures on parallel M2-branes// Nucl. Phys. B. 2009. V. 811. P. 66. arXiv:0709.1260 [hep-th].

Benenti G., Casati G., Strini G. Principles of quantum computation and information, Vol. I: Basic concepts. Singapore: World Scientific, 2004;
Vol. II: Basic tools and special topics. Singapore:World Scientific, 2007.

Nielsen M.A. and Chuang I.L. Quantum computation and quantum information. Cambridge: Cambridge University Press, 2000.

Makhaldiani N. Theory of Quanputers // Sovremennaia Matematica i ee Prilozhenia. 2007. V. 44. P.113; Journal of Mathematical Sciences. 2008. V. 153. P. 159.

Makhaldiani N. How to Solve the Classical Problems on Quantum Computers. JINR Commun. E2-2001-137. Dubna, 2001.

Toffoli T., Margolus N. Cellular Automata. Machines: MIT Press, 1987.

Samarskii A., Gulin A. Numerical Methods. M.: Nauka 1989.

Makhaldiani N. Classical and Quantum Problems for Quanputers, quant-ph/0210184, 2002.

Makhaldiani N. Regular method of construction of the reversible dynamical systems and their linear extensions - Quanputers // Atomic Nuclei. 2011. V. 74. P. 1040.

P. M. Ho and Y. Matsuo, M5 from M2, JHEP 0806, 105 (2008) [arXiv:0804.3629 [hep-th]]. P. M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, M5-brane in three-form flux and multiple M2-branes, JHEP 0808, 014 (2008) [arXiv:0805.2898 [hep-th]]. For a short review, see P. -M. Ho, A Concise Review on M5-brane in Large C-Field Background, Chin. J. Phys. 48, 1 (2010) [arXiv:0912.0445 [hep-th]].

P. K. Townsend, The eleven-dimensional supermembrane revisited, Phys. Lett. B 350, 184 (1995) [arXiv:9501068 [hep-th]].