Dynamical realizations of l-conformal Newton-Hooke group

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Motivations

Recently, there has been an upsurge of interest in $d = 1$ conformal mechanics

$$H = \frac{1}{2m} p_\alpha p_\alpha + V(x_1, \ldots, x_N), \quad D = tH - \frac{1}{2} x_\alpha p_\alpha,$$

$$K = -t^2 H + 2tD + \frac{1}{2} m x_\alpha x_\alpha, \qquad \boxed{x_\alpha \partial_\alpha V(x) + 2V(x) = 0}$$

$$\{H, K\} = 2D, \quad \{D, K\} = K, \quad \{H, D\} = H \qquad \boxed{\text{so(2,1)}}$$

Issues discussed:

- Relation to Lie algebras ($\{\lambda\}$ - root vectors of a simple Lie algebra)

$$V(x) = \sum_{\{\lambda\}} \frac{g}{(\lambda \cdot x)^2} \qquad \boxed{V(x) = \sum_{\beta<\gamma} \frac{g}{(x_\beta - x_\gamma)^2} \qquad (A_n\text{–series})}$$

- Integrability, superintegrability, integrable reductions
- Exact solvability in quantum case
- $N = 4$, $D(2, 1|\alpha)$ supersymmetric extensions
- Solutions to Witten-Dijkgraaf-Verlinde-Verlinde equation
- Isospin degrees of freedom

Newton-Hooke algebra

As $d > 1$ is physically more interesting, it is natural to wonder what happens beyond $d = 1$. This invokes nonrelativistic conformal algebras. Such algebras also play an important role within the context of the nonrelativistic AdS/CFT correspondence.

(Anti) de Sitter algebra$(\eta_{AB} = \text{diag}(-,+,+,+,\mp))$

$$[M_{AB}, M_{CD}] = \eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC}$$

Another basis $(M_{AB} \to (M_{\alpha\beta}, P_\alpha = M_{\alpha 4}/R), \quad \alpha, \beta = 0, 1, 2, 3)$

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\alpha\gamma} M_{\beta\delta} + \eta_{\beta\delta} M_{\alpha\gamma} - \eta_{\beta\gamma} M_{\alpha\delta} - \eta_{\alpha\delta} M_{\beta\gamma},$$

$$[M_{\alpha\beta}, P_\gamma] = \eta_{\alpha\gamma} P_\beta - \eta_{\beta\gamma} P_\alpha, \quad [P_\alpha, P_\beta] = \mp \frac{1}{R^2} M_{\alpha\beta}$$

Relativistic mechanics

$$p_\mu = (E/c, -p_i), \qquad E = \frac{Mc^2}{\sqrt{1 - \dot{x}^2/c^2}} \simeq Mc^2 + \frac{1}{2} M \dot{x}^2, \quad p_0 = cM + H/c.$$

Newton-Hooke algebra

Non–relativistic contraction $(\alpha \to (0, i),\ i = 1, 2, 3)$

$$M_{\alpha\beta} \to (M_{ij},\ M_{0i} = cK_i), \qquad P_\alpha \to (P_i,\ P_0 = cM + H/c), \qquad \boxed{R \to c\tilde{R}}$$

The limit $c \to \infty$ yields the (centrally extended) Newton-Hooke algebra

$$[M_{ij}, M_{kl}] = \delta_{ik} M_{jl} + \delta_{jl} M_{ik} - \delta_{jk} M_{il} - \delta_{il} M_{jk},$$
$$[M_{ij}, P_k] = \delta_{ik} P_j - \delta_{jk} P_i, \quad [M_{ij}, K_s] = \delta_{is} K_j - \delta_{js} K_i,$$
$$[P_i, K_j] = \delta_{ij} M, \qquad [H, K_i] = P_i, \qquad [H, P_i] = \mp \frac{1}{\tilde{R}^2} K_i$$

Remarks:

- $\tilde{R}$ is called the characteristic time
- $\Lambda = \pm \frac{1}{\tilde{R}^2}$ is interpreted as a cosmological constant
- The flat space limit $\tilde{R} \to \infty$ reproduces the Galilei algebra

Dynamical realization of Newton-Hooke algebra

$(A)dS_4$ metric (spherical coordinates)

$$ds^2 = -(1 \pm \frac{r^2}{R^2})dt^2 + \frac{1}{(1 \pm \frac{r^2}{R^2})}dr^2 + r^2 d\Omega^2, \qquad \Lambda = \mp \frac{3}{R^2}$$

Newtonian limit

$$g_{00} = -(1 + \frac{2\Phi}{c^2}) \qquad \Rightarrow \qquad \Phi = \pm \frac{r^2}{2\tilde{R}^2},$$

where $\tilde{R}$ is the characteristic time: $R = c\tilde{R}$.

Dynamical realization (Cartesian coordinates, $i = 1, 2, 3$)

$$S = \int dt \Big(\frac{1}{2}\dot{x}^i \dot{x}^i + V(x) \mp \frac{1}{2\tilde{R}^2} x^i x^i \Big)$$

Consider a non-relativistic spacetime parametrized by (t, x^i). Apart from time translations, consider the dilatations and the special conformal transformations

$$H = \partial_t, \qquad D = t\partial_t + l x_i \partial_i, \qquad K = t^2\partial_t + 2l t x_i \partial_i.$$

These form $so(2,1)$ for arbitrary l.

l-conformal Galilei algebra (J. Negro, M. del Olmo, A. Rodriguez-Marco, 1997)

$$[H, D] = H, \qquad [H, C_i^{(n)}] = nC_i^{(n-1)},$$

$$[H, K] = 2D, \qquad [D, K] = K,$$

$$[D, C_i^{(n)}] = (n-l)C_i^{(n)}, \qquad [K, C_i^{(n)}] = \boxed{(n-2l)}\, C_i^{(n+1)},$$

$$[M_{ij}, C_k^{(n)}] = -\delta_{ik}C_j^{(n)} + \delta_{jk}C_i^{(n)}, \qquad [M_{ij}, M_{kl}] = -\delta_{ik}M_{jl} - \delta_{jl}M_{ik} + \dots$$

Realization in spacetime

$$C_i^{(0)} = P_i = \partial_i, \qquad C_i^{(1)} = K_i = t\partial_i, \qquad \dots \qquad , \qquad C_i^{(n)} = t^n\partial_i$$

The algebra is finite–dimensional provided $n = 0, 1, \dots, 2l$ which means that l is half–integer. $1/l$ is called the dynamical exponent. $C_i^{(n)} = t^n\partial_i$ are called the generators of accelerations.

l-conformal Newton-Hooke algebra (A.G., I. Masterov, 2011)

$$
\begin{aligned}
&[H,D]=H\mp\frac{2}{\tilde{R}^2}K, && [H,C_i^{(n)}]=nC_i^{(n-1)}\pm\frac{(n-2l)}{\tilde{R}^2}C_i^{(n+1)},\\
&[H,K]=2D, && [D,K]=K,\\
&[D,C_i^{(n)}]=(n-l)C_i^{(n)}, && [K,C_i^{(n)}]=(n-2l)C_i^{(n+1)},\\
&[M_{ij},C_k^{(n)}]=-\delta_{ik}C_j^{(n)}+\delta_{jk}C_i^{(n)}, && [M_{ij},M_{kl}]=-\delta_{ik}M_{jl}-\delta_{jl}M_{ik}+\ldots
\end{aligned}
$$

Remarks:

- l–conformal Newton–Hooke algebra and l–conformal Galilei algebra are isomorphic

$$
H \quad \to \quad H\mp\frac{1}{\tilde{R}^2}K
$$

Warning: in dynamical realizations this implies change of the Hamiltonian

- The flat space limit $\tilde{R}\to\infty$ yields the l–conformal Galilei algebra

Conformal extension of Newton-Hooke algebra

- Realization in spacetime $(\Lambda < 0)$

$$H = \partial_t, \qquad D = \frac{1}{2}\tilde{R}\sin(2t/\tilde{R})\partial_t + l\cos(2t/\tilde{R})x_i\partial_i$$

$$K = -\frac{1}{2}\tilde{R}^2(\cos(2t/\tilde{R}) - 1)\partial_t + l\tilde{R}\sin(2t/\tilde{R})x_i\partial_i$$

$$C_i^{(n)} = \tilde{R}^n\big(\tan(t/\tilde{R})\big)^n\big(\cos(t/\tilde{R})\big)^{2l}\partial_i$$

- Realization in spacetime $(\Lambda > 0)$

$$H = \partial_t, \qquad D = \frac{1}{2}\tilde{R}\sinh(2t/\tilde{R})\partial_t + l\cosh(2t/\tilde{R})x_i\partial_i$$

$$K = \frac{1}{2}\tilde{R}^2(\cosh(2t/\tilde{R}) - 1)\partial_t + l\tilde{R}\sinh(2t/\tilde{R})x_i\partial_i$$

$$C_i^{(n)} = \tilde{R}^n\big(\tanh(t/\tilde{R})\big)^n\big(\cosh(t/\tilde{R})\big)^{2l}\partial_i$$

- The flat space limit $\tilde{R} \to \infty$ reproduces the l–conformal Galilei algebra.

Example: $l = \frac{1}{2}$ ($\Lambda < 0$)

	Flat spacetime	Newton-Hooke spacetime
translations		
	$\delta x^i = a^i$	$\delta x^i = a^i \cos(t/\tilde{R})$
boosts		
	$\delta x^i = t v^i$	$\delta x^i = v^i \tilde{R} \sin(t/\tilde{R})$
dilatations		
	$\delta t = 2t\lambda$	$\delta t = \lambda \tilde{R} \sin(2t/\tilde{R})$
	$\delta x^i = \lambda x^i$	$\delta x^i = \lambda x^i \cos(2t/\tilde{R})$
special conformal transformations		
	$\delta t = 2\sigma t^2$	$\delta t = -\sigma \tilde{R}^2 (\cos(2t/\tilde{R}) - 1)$
	$\delta x^i = 2\sigma t x^i$	$\delta x^i = \sigma x^i \tilde{R} \sin(2t/\tilde{R})$

Symmetries are used to integrate equations of motion. The number of functionally independent integrals of motion needed to integrate a differential equation correlates with its order. The presence of the generators of accelerations in the nonrelativistic conformal algebras in general leads to higher derivative formulations (J. Lukierski, P.C. Stichel, W.J. Zakrzewski, 2006; J. Gomis, K. Kamimura, 2011; K. Andrzejewski, J. Gonera, P. Maslanka, 2012)

The objective: to construct a dynamical realization of the l-conformal Newton-Hooke algebra without higher derivatives

Previous work on l-conformal Galilei algebra (S. Fedoruk, E. Ivanov, J. Lukierski, 2011; A.G., I. Masterov, 2013)

$$\ddot{\rho} = \frac{\gamma^2}{\rho^3}, \qquad \rho^2 \frac{d}{dt}\left(\rho^2 \frac{d}{dt} x^i\right) + \lambda(l)\gamma^2 x^i = 0$$

where $\lambda(l)$ is a specific positive number coefficient.

The method of nonlinear realizations

- The coset space

$$G = e^{itH} e^{izK} e^{iuD} e^{ix_i^{(n)} C_i^{(n)}} \times SO(n)$$

- Left multiplication by the group element $g = e^{iaH} e^{ibK} e^{icD} e^{i\lambda_i^{(n)} C_i^{(n)}} e^{\frac{i}{2}\omega_{ij} M_{ij}}$ determines the action of the group on the coset $(\Lambda < 0)$

$$\tilde{H} = \frac{\partial}{\partial t}, \qquad \tilde{D} = \frac{\tilde{R}}{2} \sin \frac{2t}{\tilde{R}} \frac{\partial}{\partial t} + \cos \frac{2t}{\tilde{R}} \frac{\partial}{\partial u} - \left(\frac{1}{\tilde{R}} \sin \frac{2t}{\tilde{R}} + z \cos \frac{2t}{\tilde{R}} \right) \frac{\partial}{\partial z}$$

$$\tilde{K} = \frac{\tilde{R}^2}{2} \left(1 - \cos \frac{2t}{\tilde{R}} \right) \frac{\partial}{\partial t} + \tilde{R} \sin \frac{2t}{\tilde{R}} \frac{\partial}{\partial u} + \left(\cos \frac{2t}{\tilde{R}} - z\tilde{R} \sin \frac{2t}{\tilde{R}} \right) \frac{\partial}{\partial z}$$

$$\tilde{C}_i^{(0)} = \sum_{n=0}^{2l} \sum_{m=0}^{n} \frac{(-1)^n}{\tilde{R}^m} \frac{(2l)!}{m!(n-m)!(2l-n)!} \left(\cos \frac{t}{\tilde{R}} \right)^{2l-m} \left(\sin \frac{t}{\tilde{R}} \right)^m \times$$

$$\times z^{n-m} e^{u(n-l)} \frac{\partial}{\partial x_i^{(n)}}, \qquad \tilde{C}_i^{(n)} = \frac{(-1)^n (2l-n)!}{(2l)!} \underbrace{[\tilde{K}, [\tilde{K}, ..[\tilde{K}, \tilde{C}_i^{(0)}]..]]}_{n\, times}$$

- Maurer-Cartan one-forms

$$G^{-1}dG = i\left(w_H H + w_D D + w_K K + w_i^{(n)} C_i^{(n)}\right)$$

$$w_H = e^{-u}dt, \quad w_D = du - 2zdt, \quad w_K = e^{u}\left(dz + z^2 dt\right) + \frac{2}{\tilde{R}^2}\sinh u dt$$

$$w_i^{(n)} = dx_i^{(n)} - (n+1)x_i^{(n+1)}w_H - (n-l)x_i^{(n)}w_D - \\ -(n-2l-1)x_i^{(n-1)}\left(w_K + \frac{1}{\tilde{R}^2}w_H\right)$$

where $x_i^{(-1)} = x_i^{(2l+1)} = 0$.

- Constraints (E. Ivanov, S. Krivonos, V. Leviant, 1989; A.G., I.M., 2013)

$$w_D = 0, \qquad \tilde{\gamma}^{-1}w_K - \tilde{\gamma}w_H = 0, \qquad w_i^{(n)} = 0$$

where $\tilde{\gamma}$ is a coupling constant.

Dynamical realizations of l-conformal Newton-Hooke algebra ($\Lambda < 0$)

- Solution of constraints (introduce new variable $\rho = e^{\frac{u}{2}}$ which implies $z = \frac{\dot{\rho}}{\rho}$)

$$\ddot{\rho} = \frac{\gamma^2}{\rho^3} - \frac{\rho}{\tilde{R}^2}$$

$$\rho^2 \dot{x}_i^{(n)} = (n+1) x_i^{(n+1)} + (n - 2l - 1)\gamma^2 x_i^{(n-1)} := x_i^{(m)} A^{mn}$$

where $\gamma^2 = \tilde{\gamma}^2 + \frac{1}{\tilde{R}^2}$ and $n = 0, \ldots, 2l$.

In general, the system of ordinary differential equations for $x_i^{(n)}$ can be reduced to a single equation for $x_i^{(0)}$ of the order $2l+1$. The equations of motion can be integrated by purely algebraic means with the use of the integrals of motion corresponding to the l-conformal Newton-Hooke algebra.

An alternative possibility is to split $\rho^2 \dot{x}_i^{(n)} = x_i^{(m)} A^{mn}$ into invariant subsystems of order not greater than two and treat them separately.

Eigenvalues of A^{mn}

$$\pm ip\gamma, \qquad p = 2, 4, \ldots, 2l \text{ (integer } l) \\ p = 1, 3, \ldots, 2l \text{ (half-integer } l)$$

Denote the corresponding eigenvectors by $v^p_{(n)}$ and $\bar{v}^p_{(n)}$

Dynamical realization without higher derivatives

$$\ddot{\rho} = \frac{\gamma^2}{\rho^3} - \frac{\rho}{\tilde{R}^2}, \qquad \rho^2 \frac{d}{dt}\left(\rho^2 \frac{d}{dt}\chi^p_i\right) + (p\gamma)^2 \chi^p_i = 0, \qquad \chi^p_i = x^{(n)}_i (v^p_{(n)} + \bar{v}^p_{(n)})$$

describes a set of decoupled generalized oscillators moving in an external field provided by the conformal mode

$$\rho(t) = \sqrt{\frac{\left(\mathcal{D}\tilde{R}\sin\frac{t}{\tilde{R}} + \mathcal{K}\cos\frac{t}{\tilde{R}}\right)^2 + \left(\gamma\tilde{R}\sin\frac{t}{\tilde{R}}\right)^2}{\mathcal{K}}}$$

$$\chi^p_i = \alpha^p_i \cos\left(p\gamma s(t)\right) + \beta^p_i \sin\left(p\gamma s(t)\right)$$

where α^p_i, β^p_i, $\mathcal{D}$, $\mathcal{K}$ are constants of integration. The subsidiary function $s(t)$

$$s(t) = \frac{1}{\gamma}\arctan\frac{\mathcal{D}\mathcal{K} + (\mathcal{D}^2 + \gamma^2)\tilde{R}\tan\frac{t}{\tilde{R}}}{\gamma\mathcal{K}}, \qquad \frac{ds}{dt} = \frac{1}{\rho^2}$$

Remarks:

- The particle orbit is an ellipse with one point removed $(-\frac{\pi \tilde{R}}{2} < t < \frac{\pi \tilde{R}}{2})$.
- The angular velocity is determined by the conformal mode $\frac{ds}{dt} = \frac{1}{\rho^2}$.
- There are regimes of accelerated/decelerated motion in the ellipse which correlate with the motion of the conformal mode in the harmonic trap.
- In the dynamical realization the vector generators $C_i^{(n)}$ with $n > 1$ turn out to be functionally dependent on $C_i^{(0)}$, $C_i^{(1)}$ and the conformal generators H, D, K. Their explicit forms involve the conformal mode $\rho(t)$.
- The latter fact also explains why one can realize different l-conformal Newton-Hooke groups in one and the same system of ordinary differential equations

$$\ddot{\rho} = \frac{\gamma^2}{\rho^3} - \frac{\rho}{\tilde{R}^2}, \qquad \rho^2 \frac{d}{dt}\left(\rho^2 \frac{d}{dt}\chi_i^p\right) + (p\gamma)^2 \chi_i^p = 0$$

with $p = 2, 4, \ldots, 2l$ for integer l and $p = 1, 3, \ldots, 2l$ for half-integer l.

Consider the master equations for $l=\frac{3}{2}$

$$\ddot{\rho}=\frac{\gamma^2}{\rho^3}-\frac{\rho}{\tilde{R}^2}$$

$$\rho^2\dot{x}_i^{(n)}=(n+1)x_i^{(n+1)}+(n-2l-1)\gamma^2 x_i^{(n-1)}$$

Algebraically express $x_i^{(1)}$, $x_i^{(2)}$, $x_i^{(3)}$ in terms of $x_i^{(0)}:=x_i$ which obeys

$$\rho^2\frac{d}{dt}\left(\rho^2\frac{d}{dt}\left(\rho^2\frac{d}{dt}\left(\rho^2\frac{d}{dt}x_i\right)\right)\right)+10\gamma^2\rho^2\frac{d}{dt}\left(\rho^2\frac{d}{dt}x_i\right)+9\gamma^4 x_i=0$$

The static solution for the conformal mode $\rho=\sqrt{\gamma\tilde{R}}$ yields

$$x_i^{(4)}+\frac{10}{\tilde{R}^2}\ddot{x}_i+\frac{9}{\tilde{R}^4}x_i=0$$

which is a particular case of the multi-dimensional Pais–Uhlenbeck oscillator
One can demonstrate that this equation exhibits $l=\frac{3}{2}$ conformal Newton-Hooke symmetry.
Within the method of nonlinear realizations one has to factorize out dilatations along with rotations.

Open problems

- But for $l = 1$, it is not known whether l–conformal Newton-Hooke algebra can be obtained by a nonrelativistic contraction from a relativistic conformal algebra $so(p+1, q+1)$.
- Interacting models with l–conformal Newton-Hooke symmetry are unknown.
- Can l describe spin?
- Gravitational backgrounds with l–conformal Newton-Hooke symmetry?
- l–conformal Newton-Hooke symmetry in a generic multi–dimensional Pais–Uhlenbeck oscillator?
- Supersymmetric extensions.