Unified BRST formulation of AdS (partially) massless and massive fields

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Based on

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OUTLINES

- Algebraic tools: Howe duality
- Ambient space formulation: fields, BRST operator, and constraints.
- Physical interpretation: quadratic Casimir operator and spin numbers.
- Generating BRST formulation: fields, BRST operator, and constraints.
- Q_p-cohomology analysis: a relationship to the unfolded formulation.

ALGEBRAIC TOOLS:
HOWE DUALITY

Functions in auxiliary variables

Two types of indices

$$A_I^A$$

running

$$A = 0, ..., d \quad \text{and} \quad I = 0, ..., n-1$$

Polynomials

$$\Phi = \Phi(A)$$

Expansion coefficients are covariant tensors on AdS_d

$$\Phi(A) = \sum \Phi_{A_1 \, ... \, A_{m_0}; \, ... \, ; C_1 \, ... \, C_{m_{n-1}}} A_0^{A_1} \cdots A_0^{A_{m_0}} \cdots A_{n-1}^{C_1} \cdots A_{n-1}^{C_{m_{n-1}}}$$

Howe dual pair

o(d-1,2) - sp(2n)

Orthogonal algebra $o(d-1,2)$

$$J^{AB} = A_I^A \frac{\partial}{\partial A_{BI}} - A_I^B \frac{\partial}{\partial A_{AI}}\,.$$

Rotations

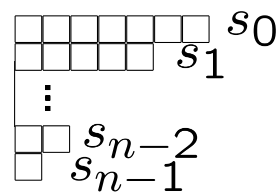
Symplectic algebra $sp(2n)$

$$T_{IJ} = A_I^A A_{AJ}\,, \qquad T_I{}^J = \frac{1}{2}\{A_I^A, \frac{\partial}{\partial A_J^A}\}\,, \qquad T^{IJ} = \frac{\partial}{\partial A_I^A}\frac{\partial}{\partial A_{AJ}}\,.$$

Trace creation **Young symmetrizer** **Trace annihilation**

Note that the algebras commute to each other $[J,T]=0$!

Finite-dimensional irrep of $o(d-1,2)$ algebra



$\Uparrow$

Howe duality

$\Downarrow$

Highest weight conditions of $sp(2n)$ algebra

$$T_I{}^I \phi = (s_I + \frac{d+1}{2})\, \phi \,.$$

$$T^{IJ} \phi = 0\,, \qquad T_I{}^J \phi = 0 \quad I < J\,,$$

Two type of constraints: from $sp(2n)$ to $sp(2n-2)$

Take a distinguished direction along A_0^A so that from now on we consider variables A_0^A and A_i^A, $i = 1, ..., n-1$ separately.

We identify $sp(2n-2) \subset sp(2n)$ subalgebra preserving the direction.

$$N_i{}^j \equiv T_i{}^j = A_i^A \frac{\partial}{\partial A_j^A} \quad i \neq j\,, \qquad N_i = N_i{}^i \equiv T_i{}^i - \frac{d+1}{2} = A_i^A \frac{\partial}{\partial A_i^A}\,,$$

which form $gl(n-1)$ subalgebra, and

$$T_{ij} = A_i^A A_{jA}\,, \qquad T^{ij} = \frac{\partial}{\partial A_i^A}\frac{\partial}{\partial A_{jA}}\,,$$

that complete above set of elements to $sp(2n-2)$ algebra.

There are two different realizations of $sp(2n)$ generators involving A_0^A and/or $\partial/\partial A_0^A$:

- realization on the space of polynomials in A_i^A with coefficients in functions on R^{d+1}. In this case

$$A_0^A = X^A, \qquad \frac{\partial}{\partial A_0^A} = \frac{\partial}{\partial X^A},$$

where X^A are Cartesian coordinates in R^{d+1}. Generators that involve X^A and/or $\partial/\partial X^A$ are denoted by

$$\mathcal{S}_i^\dagger = A_i^A \frac{\partial}{\partial X^A}, \qquad \bar{\mathcal{S}}^i = X^A \frac{\partial}{\partial A_i^A},$$

$$\mathcal{S}^i = \frac{\partial}{\partial A_i^A}\frac{\partial}{\partial X_A}, \qquad \Box_X = \frac{\partial}{\partial X^A}\frac{\partial}{\partial X_A}.$$

- There exists realization on the space of polynomials in A_i^A with coefficients in formal power series in variables Y^A such that

$$A_0^A = Y'^A = Y^A + V^A\,, \qquad \frac{\partial}{\partial A_0^A} = \frac{\partial}{\partial Y^A}\,,$$

where V^A is some $o(d-1,2)$ vector $V^A V_A = -1$. Respective $sp(2n)$ generators are realized by inhomogeneous differential operators. In this case

$$S_i^\dagger = A_i^A \frac{\partial}{\partial Y^A}\,, \qquad \bar{S}^i = (Y^A + V^A)\frac{\partial}{\partial A_i^A}\,,$$

$$S^i = \frac{\partial}{\partial A_i^A}\frac{\partial}{\partial Y_A}\,, \qquad \Box_Y = \frac{\partial}{\partial Y^A}\frac{\partial}{\partial Y_A}\,.$$

This is the **twisted realization**.

THE AMBIENT SPACE FORMULATION

Ambient space fields

AdS space is a hyperboloid embedded in the ambient space $R^{d-1,2}$.

Functions on the ambient space

$$\phi = \phi(X, A)$$

- X^A, $A = 0, ..., d$ - $R^{d-1,2}$ space-time coordinates
- A_i^A, $i = 1, ..., n-1$ - target space variables (standard Howe variables)

In order to describe AdS fields some constraints and gauge equivalence are required. They are given by certain $sp(2n)$ generators or their higher powers.

Constraints and gauge symmetries

- General off-shell constraints.

$$T^{ij}\phi = 0\,, \qquad N_i{}^j\phi = 0 \quad i<j\,, \qquad N_i\phi = s_i\phi\,.$$

- The radial dependence is fixed by

$$h\phi = 0\,, \qquad h = N_X - w\,, \qquad N_X = X^A\frac{\partial}{\partial X^A}\,.$$

In proper coordinates the condition is solved by $\phi = \phi_0(x,A)\,r^w$.

- Equations of motion

$$\Box_X\phi = 0\,, \qquad \mathcal{S}^i\phi = 0\,.$$

- Tangent constraints

$$\bar{\mathcal{S}}^{\hat\alpha}\phi = 0\,, \qquad \hat\alpha = p+1,\,\ldots\,,n-1\,.$$

- Extra constraint

$$(\bar{\mathcal{S}}^p)^t\phi = 0\,, \qquad t = 1,2,\ldots,t_{\max}\,, \quad t_{\max} = s_p - s_{p+1}\,.$$

- A gauge equivalence is defined by

$$\phi \sim \phi + \mathcal{S}^\dagger_\alpha\chi^\alpha\,.$$

Functions on the ambient space-time are extended by Grassmann odd variables

$$\phi = \phi(X, A|\, b)$$

where b^α, $\alpha = 1, ..., p \leq n - 1$ - ghosts, $\mathrm{gh}\, b_\alpha = -1$.

Then the gauge symmetry defined by

$$\phi \sim \phi + \mathcal{S}^\dagger_\alpha \chi^\alpha \,.$$

takes the standard BRST form

$$\delta\phi = Q_p \chi \,, \qquad Q_p = \mathcal{S}^\dagger_\alpha \frac{\partial}{\partial b_\alpha} \,, \qquad \mathcal{S}^\dagger_\alpha = A^A_\alpha \frac{\partial}{\partial X^A}$$

where Q_p is a BRST operator, $\mathrm{gh}\, Q_p = 1$.

Comments:

- Consistency of the constraints and the gauge symmetry requires

$$w = s_p - p - t\,.$$

— massless and partially massless fields.

- Relaxing polynomial constraints $(\bar{\mathcal{S}}^p)^t\phi = 0$ makes w arbitrary — massive fields.

PHYSICAL INTERPRETATION

Parameters of the theory are

- Spins $s_1 \geq s_2 \geq \ldots \geq s_{n-1}$.
- Radial weight w, integer parameter p, depth t.

Evaluating the quadratic Casimir operator

$$C_2 = -\frac{1}{2} J_{AB} J^{AB} \,, \qquad J_{AB} = L_{AB} + M_{AB} \,,$$

$$L_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} \,, \qquad M_{AB} = \sum_{i=1}^{n-1} \Big(A_{Ai} \frac{\partial}{\partial A_i^B} - A_{Bi} \frac{\partial}{\partial A_i^A} \Big) \,.$$

gives on the Q_p equivalences classes

$$C_2 \phi \cong \Big(w(d-1+w) + \sum_{l=1}^{n-1} s_l (s_l - 2l + d - 1) \Big) \phi \,.$$

The standard expression gives $E_0(E_0 - d + 1) = w(w + d - 1)$ and hence we find for correct values (Fronsdal'79, Metsaev'95, Skvortsov'09)

$$E_0 = w + d - 1 = (\text{special } w = s_p - p - t) = s_p - p - t + d - 1 \,.$$

GENERATING BRST FORMULATION

Space of functions

Functions on anti-de Sitter space

$$\Phi = \Phi(x, Y, A|\, b, \theta)$$

- x^m, $m = 0, ..., d-1$ - AdS space-time coordinates
- Y^A, A_i^A, $i = 1, ..., n-1$ - target space variables (twisted Howe variables)
- b^α, $\alpha = 1, ..., p \leq n-1$ - target space ghosts, $gh\, b_\alpha = -1$
- θ^m, $m = 0, ..., d-1$ - space-time ghosts, $gh\, \theta^m = 1$

Covariant background derivative

BRST operator

$$\nabla = d + \frac{1}{2}\theta^m \omega_m^{AB} J_{AB} , \qquad d = \theta^m \frac{\partial}{\partial x^m}$$

$$\nabla^2 = 0 , \quad \mathrm{gh}\, \nabla = 1$$

AdS background connection ω_m^{AB} satisfies

$$d\omega^{AB} + \omega^A{}_C \wedge \omega^{CB} = 0$$

Here $V^A = const$. Basis differential forms dx^m are replaced with extra Grassmann odd ghost variables θ^m, $m = 0, ..., d-1$ because ∇ is interpreted as a part of BRST operator.

Operator Q_p and off-shell constraints

BRST operator

$$Q_p = S^{\dagger}_{\alpha}\frac{\partial}{\partial b_{\alpha}}\,, \qquad Q_p^2 = 0\,, \qquad \text{gh}\, Q_p = 1$$

Off-shell constraints along with Q_p form some algebra:

1) Massless and massive fields: constraints **linear** in $sp(2n)$ generators.

2) Partially massless fields: some constraints are **polynomial** in $sp(2n)$ generators. The respective polynomial order is proportional to the depth of partial masslessness.

Remark: anti-de Sitter algebra acts in the Q_p –cohomology

$$[J^{AB}, Q_p] = 0$$

Total BRST operator: generating formulation

BRST operator

$$\Omega = \nabla + Q_p \,, \qquad \Omega^2 = 0 \,,$$

$$\nabla^2 = Q_p^2 = 0 \,, \qquad [Q_p, \nabla] = 0 \,.$$

Plus appropriate off-shell (BRST extended) constraints.

BRST operator is a sum of the term associated to the spacetime isometry algebra and the term associated to the Howe dual symplectic algebra.

Part of algebraic constraints can be relaxed = infinitely reducible string-like system.

Q_p - cohomology

THEOREM: The Q_p - cohomology evaluated in the subspace singled out by off-shell constraints is non-empty only for

- $gh = 0$: infinite-dimensional AdS Weyl module (0-form gauge invariant quantities).

- $gh = -p$: finite-dimensional AdS gauge module (p-form gauge potentials) For massive fields it is zero.

Remark: AdS Weyl module confirms the BMV conjecture (**Brink-Metsaev-Vasiliev'00**).

CONCLUSION

- Uniform and concise description of AdS fields: massless, partially massless, and massive.
- Howe duality makes a constraint structure of the theory manifest:

  AdS_d algebra $o(d-1,2)$ — symplectic algebra $sp(2n)$

- A chain of cohomological resolutions:

  Metric-like fields $\overset{\sigma_-\text{-cohomology}}{-}$ "unfolded" fields

  "Unfolded" fields $\overset{Q_p\text{-cohomology}}{-}$ "parent" fields

- Off-shell formulations of nonlinear HS theories (**Vasiliev'95, Grigoriev'06**)
- Our main believe: the approach is crucial for understanding of the searched-for geometry underlying HS interactions.