On dynamical equations in non-linear models

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P-brane dynamics for any (D, p)

The Dirac action for a p-brane

$$S = T \int \sqrt{|G|} d^{p+1}\xi,$$

G is the determinant of the induced metric

$$G_{\alpha\beta} := \partial_\alpha x_m \partial_\beta x^m.$$

The Euler-Lagrange equations and $p+1$ primary constraints

$$\partial_\tau \mathcal{P}^m = -T\partial_r(\sqrt{|G|}G^{r\alpha}\partial_\alpha x^m), \quad \mathcal{P}^m = T\sqrt{|G|}G^{\tau\beta}\partial_\beta x^m, \tag{1}$$

$$\tilde{T}_r := \mathcal{P}^m \partial_r x_m \approx 0, \quad \tilde{U} := \mathcal{P}^m \mathcal{P}_m - T^2 |\det G_{rs}| \approx 0. \tag{2}$$

We use the orthogonal gauge

$$L\tau = x^0 \equiv t, \quad G_{\tau r} = -L(\dot{\vec{x}} \cdot \partial_r \vec{x}) = 0, \tag{3}$$
$$g_{rs} := \partial_r \vec{x} \cdot \partial_s \vec{x}, \quad G_{\alpha\beta} = \begin{pmatrix} L^2(1 - \dot{\vec{x}}^2) & 0 \\ 0 & -g_{rs} \end{pmatrix}$$

with $\dot{\vec{x}} := \partial_t \vec{x} = L^{-1}\partial_\tau \vec{x}$.
The solution of constraint $\tilde{U}$

$$\mathcal{P}_0 = \sqrt{\vec{\mathcal{P}}^2 + T^2|g|}, \quad g = \det(g_{rs}). \tag{4}$$

$\mathcal{P}_0$ becomes Hamiltonian density $\mathcal{H}_0$ since $\dot{\mathcal{P}}_0 = 0$.

Definition of $\mathcal{P}_0$ (1) and $G^{\tau\tau} = 1/L^2(1 - \dot{\vec{x}}^2)$ results in

$$\mathcal{P}_0 := TL\sqrt{|detG|}G^{\tau\tau} = T\sqrt{\frac{|g|}{1 - \dot{\vec{x}}^2}}. \tag{5}$$

Density $\vec{\mathcal{P}}$ and its evolution are

$$\vec{\mathcal{P}} = \mathcal{P}_0\dot{\vec{x}}, \quad \dot{\vec{\mathcal{P}}} = T^2\partial_r\left(\frac{|g|}{\mathcal{P}_0}g^{rs}\partial_s\vec{x}\right). \tag{6}$$

PDE for $\vec{x}$

$$\ddot{\vec{x}} = \frac{T}{\mathcal{P}_0}\partial_r\left(\frac{T}{\mathcal{P}_0}|g|g^{rs}\partial_s\vec{x}\right). \tag{7}$$

The canonical Hamiltonian generates

$$\dot{\vec{x}} = \{H_0, \vec{x}\}, \quad \dot{\vec{\mathcal{P}}} = \{H_0, \vec{\mathcal{P}}\}, \quad \{\mathcal{P}_i(\sigma), x_j(\tilde{\sigma})\} = \delta_{ij}\delta^{(2)}(\sigma^r - \tilde{\sigma}^r)$$

with integrated density $\mathcal{H}_0 (= \mathcal{P}_0)$

$$H_0 = \int d^p\sigma \sqrt{\vec{\mathcal{P}}^2 + T^2|g|}. \tag{8}$$

Residual symmetry of orthonormal gauge

$$\tilde{t} = t, \quad \tilde{\sigma}^r = f^r(\sigma^s). \tag{9}$$

Reduced constraints $\tilde{T}_r$

$$T_r := \vec{\mathcal{P}}\partial_r\vec{x} = 0 \quad \Leftrightarrow \quad \dot{\vec{x}}\partial_r\vec{x} = 0, \quad (r = 1, 2, .\,, p). \tag{10}$$

$U(1) \times U(1) \times \ldots U(1)$ invariant p-branes

We consider p-branes evolving in $D = (2p + 1)$-dimensional Minkowski space-time and assume that their shape is invariant under the direct product $\mathcal{U} := \prod_{a=1}^{p} U_a(1)$. Each of these $U(1)$ symmetries is locally isomorphic to one of the $O(2)$ subgroups of the $SO(2p, 1)$ group of rotations.
Anzats for $\vec{x}$

$$\vec{x}^T = (q_1 \cos\theta_1, q_1 \sin\theta_1, q_2 \cos\theta_2, q_2 \sin\theta_2, \ldots, q_p \cos\theta_p, q_p \sin\theta_p), \quad (11)$$
$$q_a = q_a(t), \ \ (a = 1, \ldots, p); \quad \theta_r = \theta_r(\sigma^s), \ \ (r, s = 1, \ldots, p).$$

Vector $\vec{x}$ lies in $2p$-dim subspace and satisfies $\dot{\vec{x}}\partial_r\vec{x} = 0$. Anzats (11) originates from realization of a $2p$-dimensional vector $\vec{x}$ defined by p pairs of its "polar" coordinates

$$\vec{x}^T(t, \sigma^r) = (q_1 \cos\theta_1, q_1 \sin\theta_1, \ldots, q_p \cos\theta_p, q_p \sin\theta_p) \quad (12)$$

with q_a, θ_a depending on $t, \sigma^1, ..., \sigma^p$.

Our anzats (11) is obtained by excluding the time dependence for all "polar" angles $\theta_a = \theta_a(\sigma^r)$ as well as the σ^r dependence for all "radial" coordinates $m_a = m_a(t)$.
At any moment t vector $\vec{x}^T$ (11) is produced from $\vec{x_0}^T = (q_1, 0, q_2, 0, \ldots, q_p, 0)$ by rotations of the diagonal subgroup $\mathcal{U} \in SO(2p)$, parametrized by the angles θ_a and rotating the planes x_1x_2, x_3x_4 ,...,$x_{2p-1}x_{2p}$.
World-volume metric $G_{\alpha\beta}$ becomes

$$G_{tt} = 1 - \dot{\mathbf{q}}^2, \quad \mathbf{q} := (q_1, .., q_p), \quad g_{rs} = \sum_{a=1}^{p} q_a^2 \theta_{a,r}\theta_{a,s}, \tag{13}$$

where $\theta_{s,r} := \partial_r\theta_s$ and $\dot{\mathbf{q}} \equiv \partial_t\mathbf{q}$ and yields ds^2 on Σ_{p+1}

$$ds^2_{p+1} = (1 - \dot{\mathbf{q}}^2)dt^2 - \sum_{a=1}^{p} q_a^2(t)d\theta_a d\theta_a \tag{14}$$

independent of σ^r.

Change of σ^r by new parameters $\theta_a(\sigma^r)$ makes metric independent of σ^r with Killing vectors $\frac{\partial}{\partial\theta_a}$ on Σ_p .
It shows that Σ_p has zero curvature and is isomorphic to a flat p-dimensional torus $S^1 \times S^1 \times \dots S^1$ at any moment.
Canonical components $\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{q_a}} = \vec{\mathcal{P}} \frac{\partial \dot{\vec{x}}}{\partial \dot{q_a}}$ or $\boldsymbol{\pi} := (\pi_1, ..., \pi_p)$ for $\mathbf{q}$ are

$$\boldsymbol{\pi} = \mathcal{P}_0 \dot{\mathbf{q}}, \quad \mathcal{P}_0 = T \sqrt{\frac{|g|}{1 - \dot{\mathbf{q}}^2}}, \quad g = det(\sum_{a=1}^{p} q_a^2 \theta_{a,r} \theta_{a,s}) \tag{15}$$

and $\vec{x}^2 = \mathbf{q}^2, \quad \dot{\vec{x}}^2 = \dot{\mathbf{q}}^2$.

Hamiltonian density in $(\mathbf{q}, \boldsymbol{\pi})$ phase space

$$\mathcal{H}_0 = \mathcal{P}_0 = \sqrt{\boldsymbol{\pi}^2 + T^2|g|}, \quad \dot{\mathcal{P}}_0 = 0. \tag{16}$$

Hamiltonian equations

$$\dot{\mathbf{q}} = \{H_0, \mathbf{q}\} = \frac{1}{\mathcal{P}_0}\boldsymbol{\pi}, \quad \dot{\boldsymbol{\pi}} = \{H_0, \boldsymbol{\pi}\}. \tag{17}$$

Canonical PB's and Hamiltonian

$$\{\pi_a, q_b\} = \delta_{ab}, \quad H_0 = \int d^p\sigma \sqrt{\boldsymbol{\pi}^2 + T^2|g|}. \tag{18}$$

Equations of $\mathcal{U}$ invariant p-branes

To simplify equations of motion we simplify Anzats by fixing gauge for residual symmetry (9): $\theta_a = \delta_{ar}\sigma^r$

$$\theta_1(\sigma^r) = \sigma^1, \quad \theta_2(\sigma^r) = \sigma^2, \ldots, \theta_p(\sigma^r) = \sigma^p. \tag{19}$$

Then Anzats and g_{rs} are expressed

$$\vec{x}^T(t) = (q_1 \cos\sigma^1, q_1 \sin\sigma^1, \ldots, q_p \cos\sigma^p, q_p \sin\sigma^p), \tag{20}$$

$$g_{rs}(t) = q_r^2(t)\delta_{rs}, \quad g = (q_1 q_2 ... q_p)^2 \tag{21}$$

with diagonalized metric $g_{rs}(t)$ depending on t.

In gauge (19) interval and g^{rs} on Σ_p are

$$ds_p^2 = \sum_{r=1}^{p} q_r^2(t)(d\sigma^r)^2, \quad g^{rs}(t) = \frac{1}{q_r^2}\delta_{rs}. \tag{22}$$

This gauge clarifies sense of coordinates $\mathbf{q}(t) = (q_1, q_2, \ldots, q_p)$ as time dependent radii $\mathbf{R}(t) = (R_1, R_2, \ldots, R_p)$ of flat hypertorus Σ_p .
Hamiltonian density $\mathcal{H}_0$ becomes independent of σ^r

$$\mathcal{H}_0 = \mathcal{P}_0 = \sqrt{\boldsymbol{\pi}^2 + T^2 \prod_{r=1}^{p} q_r^2} = C, \quad \dot{\mathcal{P}}_0 = 0, \quad \frac{\partial}{\partial\sigma^r}\mathcal{P}_0 = 0. \tag{23}$$

This property yields condition for the initial data

$$\mathcal{P}_0 = C \quad \rightarrow \quad T\sqrt{\frac{|g|}{1-\dot{\mathbf{q}}^2}} \equiv T\sqrt{\frac{(q_1 q_2 \dots q_p)^2}{1-\dot{\mathbf{q}}^2}} = C. \tag{24}$$

Eqs. (6) for $\vec{x}$ are simplified

$$\ddot{\vec{x}} - (\frac{T}{C})^2 g g^{rs} \partial_{rs} \vec{x} = 0. \tag{25}$$

Taking into account

$$g g^{rs} = \frac{\delta_{rs}}{q_r^2} \prod_{t=1}^{p} q_t^2, \quad \partial^r \partial_r \vec{x} = -\vec{x} \tag{26}$$

we present Eqs. (25) in terms of **q**

$$\ddot{q}_r + (\frac{T}{C})^2 (q_1 \dots q_{r-1} q_{r+1} \dots q_p)^2 q_r = 0, \quad r = (1, 2, \dots, p). \tag{27}$$

Multiplication (27) by q_r results in the first integral

$$\dot{\mathbf{q}}^2 + (\frac{T}{C})^2 (q_1 q_2 ... q_p)^2 = c \tag{28}$$

which coincides with initial data (24) if $c = 1$. Eqs. (27) are presented as

$$C\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}, \tag{29}$$

where elastic energy density $V(\mathbf{q})$ turns out to be proportional to g

$$V(\mathbf{q}) := \frac{T^2}{2C} g \equiv \frac{T^2}{2C} (q_1 ... q_p)^2. \tag{30}$$

$\mathcal{U}$ invariant p-branes and p-dimensional anharmonic oscillators

General equations of elastic non relativistic media with mass density ρ

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k}. \tag{31}$$

We observe that Eqs. (29) have form of relativistic generalization (31)

$$C\ddot{q}_r = -\frac{T^2}{2C}\delta_{rs}\frac{\partial g}{\partial q_s} \tag{32}$$

with stress tensor σ_{rs}

$$\sigma_{rs} := -p\delta_{rs}, \quad p := \frac{T^2}{2C}g \equiv \frac{T^2}{2C}\prod_{s=1}^{p} q_s^2, \quad r = (1, 2, \ldots, p). \tag{33}$$

Eqs. (33) shows that p is isotropic pressure per unit area of its p-brane hypersurface, and C is a relativistic generalization of mass density ρ.
Pressure p is created by forces F_r

$$F_r(t) := -\frac{\partial V}{\partial q_r} \equiv -\frac{T^2}{C}(q_1 \dots q_{r-1} q_{r+1} \dots q_p)^2 q_r \tag{34}$$

associated with p-brane potential V (30).
It is instructive to note that the discussed equations are generated by Hamiltonian H free from the square root present in H_0 (18).
Constant C can be used to write square root free Hamiltonian density

$$\mathcal{H} = \frac{\mathcal{H}_0^2}{2C} = C/2$$

and Hamiltonian H

$$H := \int d^p\sigma \mathcal{H}, \quad \mathcal{H} = \frac{1}{2C}(\boldsymbol{\pi}^2 + T^2 \prod_{s=1}^{p} q_s^2), \tag{35}$$

with standard PB's

$$\{\pi_a, q_b\} = \delta_{ab}, \quad \{q_a, q_b\} = 0, \quad \{\pi_a, \pi_b\} = 0.$$

Hamiltonian H describes p-dimensional anharmonic oscillators without quadratic terms, and reproduces Eqs. (27).
The set of Hamiltonians (35) contains quartic potential energy for p=2 or the higher degree monomials in components of $\mathbf{q}$ for $p > 2$ respectively.

Cyclic "frequency" ω_r of a coordinate q_r is proportional to products of remaining coordinates

$$\omega_r(t) = \frac{T}{C}|q_1 \ldots q_{r-1} q_{r+1} \ldots q_p|, \quad r = (1, 2, \ldots, p). \tag{36}$$

These frequencies ω_r in Eqs. (27) can not be infinitely large because of the initial data (28) with $c = 1$

$$\sqrt{1-\dot{\mathbf{q}}^2} = \frac{T}{C}|q_1 q_2 ... q_p|, \tag{37}$$

which shows that

$$0 \leq |\dot{\mathbf{q}}| \leq 1, \quad 0 \leq \frac{T}{C}|q_1 q_2 ... q_p| \leq 1 \tag{38}$$

i.e. velocity $|\dot{\mathbf{q}}|$ grows when hypervolume $\sim |q_1 q_2 ... q_p|$ diminishes and achieves velocity of light.

If $T = 0$ nonlinear system (27) reduces to equations

$$T = 0: \quad \Rightarrow \quad \ddot{\mathbf{q}} = 0, \quad |\dot{\mathbf{q}}| = 1, \tag{39}$$

similar to equations of free massless particles in effective p-dimensional space.
In general case the system (27) is rather complicate because of monomial entanglement of coordinates q_r in interaction potential V (35).
However, there is a special solvable case that we discuss below.

Elliptic and hyperelliptic solutions

Degenerate case is characterized by coincidence $q_1 = q_2 = ... = q_p \equiv q$. System (27) reduces to nonlinear diff. eqn.

$$\ddot{q} + (\frac{T}{C})^2 q^{(2p-1)} = 0 \tag{40}$$

equivalent to initial data constraint (37) for $q(t)$

$$p\dot{q}^2 + (\frac{T}{C})^2 q^{2p} = 1. \tag{41}$$

After change of q by $y = \Omega^{\frac{1}{p}} \sqrt{p} q$, and introduction of frequency $\Omega := \frac{T}{C} p^{-\frac{p}{2}}$, Eq. (41) takes form

$$(\frac{dy}{d\tilde{t}})^2 = \frac{1}{2}(1 - y^p)(1 + y^p), \tag{42}$$

with new variable $\tilde{t} := \sqrt{2}\Omega^{\frac{1}{p}} t$.

In membrane case ($p = 2$) Eq. (42) coincides with eqn. defining Jacobi elliptic cosine $cn(x; k)$

$$(\frac{dy}{dx})^2 = (1 - y^2)(1 - k^2 + k^2 y^2), \tag{43}$$

if elliptic modulus $k = \frac{1}{\sqrt{2}}$.
Thus, $y(t) = cn(\sqrt{2\omega}t; \frac{1}{\sqrt{2}})$ with $2\omega = T/C$.
Using relation $q \equiv y/\sqrt{2\omega}$ we obtain elliptic cosine solution

$$q(t) = \frac{1}{\sqrt{2\omega}} cn(\sqrt{2\omega}(t + t_0); \frac{1}{\sqrt{2}}) \equiv \sqrt{\frac{C}{T}} cn(\sqrt{\frac{T}{C}}(t + t_0); \frac{1}{\sqrt{2}}). \tag{44}$$

If initial data are $\dot{q}(t_0) > 0$ then solution (44) describes expanding torus which at some point reaches the maximal size $q_{max} = \sqrt{\frac{C}{T}}$ and then shrinks to a point after a finite time $\mathbf{K}(1/\sqrt{2})\sqrt{\frac{C}{T}}$ (where $\mathbf{K}(1/\sqrt{2}) = 1,8451$ is quarter period of elliptic cosine).
Equation of surface $\Sigma_2(t)$ of contracting torus (44) in $D = 5$ is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4\frac{C}{T} cn\left(\sqrt{\frac{T}{C}}(t + t_0), \frac{1}{\sqrt{2}}\right)^2, \;\; x_1 x_4 = x_2 x_3.$$

For $p > 2$ integration of Eq. (42) defines implicit dependence of q on time

$$\tilde{t} = \pm\sqrt{2}\int \frac{dy}{\sqrt{1 - y^{2p}}} + const. \qquad (45)$$

The integral (45) is an hyperelliptic integral, thus general solution of Eq. (41) is expressed in terms of hyperelliptic functions.

After change of variable $z = y^{2p}$ hyperelliptic solution takes form

$$\tilde{t} - \tilde{t_0} = \pm \frac{1}{\sqrt{2}p} \int_0^{z^{\frac{1}{2p}}} dz \;\; z^{(\frac{1}{2p}-1)} \cdot (1-z)^{-\frac{1}{2}}. \tag{46}$$

Using (46) one can find the time of contraction $\Delta\tilde{t}_c$ of p-torus from the maximal value of the radii $q_{max} = (\frac{C}{T})^{\frac{1}{p}}$ to $q_{min} = 0$

$$\Delta\tilde{t}_c = \frac{1}{\sqrt{2}p} \int_0^1 dz \;\; z^{(\frac{1}{2p}-1)} \cdot (1-z)^{\frac{1}{2}-1} = \frac{1}{\sqrt{2}p} B(\frac{1}{2p}, \frac{1}{2}), \tag{47}$$

where $B(\frac{1}{2p}, \sqrt{2})$ is the Euler beta function $B(\alpha, \beta)$.
Coming back to the real time t we obtain

$$\Delta t_c \equiv \frac{1}{\sqrt{2}} \Omega^{-\frac{1}{p}} \Delta\tilde{t}_c = \frac{1}{2\sqrt{p}} (\frac{2E}{T})^{\frac{1}{p}} B(\frac{1}{2p}, \frac{1}{2}).$$

So, the case of degenerate p-torus with coinciding radii q turns out to be integrable and reveals connection of p-brane equations with (hyper)elliptic curves and Euler beta function.

Summary

Exact solvability of brane equations is studied, and a new $U(1) \times U(1) \times \ldots U(1)$ invariant anzats for the solution of p-brane equations in $D = (2p + 1)$-dimensional Minkowski space is proposed.

The reduction of the p-brane Hamiltonian to the Hamiltonian of p-dimensional relativistic anharmonic oscillator with the monomial potential of the degree equal to $2p$ is revealed.

It is shown that the case of degenerate p-torus with equal radii is integrable and that the p-brane solutions are expressed in terms of elliptic ($p = 2$) or hyperelliptic ($p > 2$) functions.

- P. K. Townsend, *The eleven-dimensional supermembrane revisited*, Phys. Lett. B350 (1995) 184; E. Witten, *String Theory Dynamics In Various Dimensions*, Nucl. Phys. B443 (1995) 85; T. Banks, W. Fishler, S.H. Shenker, L. Susskind, *M theory as a Matrix Mode: A conjecture* Phys. Rev. D55(8):5112; M. J. Duff, *The world in Eleven Dimensions: Supergravity, Supermembranes and M-theory* (IOP, Bristol 1999).
- C. Rebbi, *Dual models and relativistic strings*, Phys. Rep. 12 (1974) 1.
- R. W. Tucker, *Extended Particles and the Exterior Calculus*, Lectures given at the Rutherford Laboratory, Feb 1976.
- J. Hoppe, MIT PhD Thesis,1982, and in Proc. Int. Workshop on Constraints Theory and Relativistic Dynamics. G. Longhi, L. Lusanna (eds), Singapore: World Scientific 1987.
- E. Berqshoeff, E. Sezgin, P.K. Townsend, Phys. Lett. B189 (1987) 75.
- M. Duff, P. Howe, T. Inami, K. Stelle, Phys. Lett. B191 (1987) 70.
- E. Floratos, J. Illipoulos, Phys. Lett. B201 (1988) 237.

- B. de Witt, J. Hoppe, G. Nicolai, Nucl. Phys. B305 [FS23] (1988) 545.
- A.A. Zheltukhin, Sov. J. Nucl. Phys. 48 (1988) 375.
- B. de Witt, M. Lusher, G. Nicolai, Nucl. Phys. B320 (1989) 135.
- I.A. Bandos and A.A. Zheltukhin, Fortsch. Phys. 41 (1993) 619.
- M. Bordemann, J. Hoppe, Phys. Lett. B317 (1993) 315 ; ibid. B325 (1994) 359.
- J. Polchinski, Phys. Rev. Lett. 75 (1995) 4729.
- D.V. Uvarov and A.A. Zheltukhin, Phys. Lett. B545 (2002) 183.
- J. Arnlind, J. Hoppe, S. Theisen, Phys. Lett. B 599 (2004) 118.
- J. Hoppe, Compl. Anal. Oper. Theory 3 (2009) 419.
- M. Trzetrzelewski and A.A. Zheltukhin, Phys. Lett. B679 (2009) 523.
- A.A. Zheltukhin and M. Trzetrzelewski, J. Math. Phys. 51 (2010) 062303.

L.D. Landau and E.M. Lifshits, *Theory of Elasticity* Moscow, ”Science”, 1987.