Supersymmetric Renormalization Group Flows

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with

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ArXiv 1107.3011; Phys. Rev. **D81** (2010) 125001; Phys. Rev. **D80** (2009) 085007; Phys. Rev. **D80** (2009) 101701; JHEP **0903** (2009) 028

Supersymmetries and Quantum Symmetries – SQS 2011, Dubna

Research Training Group
Quantum and Gravitational Fields

seit 1558

Supersymmetry (susy) and ERGE

- intermediate and strong couplings, susy-breaking, condensation, ...

  non-perturbative methods
- exact solutions:

  low dimensions, extended supersymmetry, large N, ...
- lattice simulations:

  susy broken by discretisation $\Rightarrow$ fine-tuning, demanding
  $\Longrightarrow$ need complement to lattice studies
- functional renormalization group?

  successfully applied to
  phase transitions, condensed matter physics, infrared sector of gauge theories, quantum gravity, renormalizability-proofs

Functional renormalization flow[1]

Wilson, Wegner-Houghton, Polchinsky, Wetterich

add momentum- and scale-dependent regulator term to action

$$\Delta S_k[\phi] = \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\phi^*(p) R_k(p)\phi(p)$$

$\Longrightarrow$ scale-dependent generating functional

$$Z_k[j] = \int \mathcal{D}\phi\,\mathrm{e}^{-S[\phi]+(j,\phi)-\Delta S_k[\phi]}$$

$\Longrightarrow$ scale dependent Schwinger functional and effective action

$$W_k[j] = \log Z_k[j] \quad , \quad \Gamma_k[\varphi] = (\mathcal{L}W_k)[\varphi] - \Delta S_k[\varphi]$$

- Γ_k not necessarily convex

[1] *renormalization group*: Bogoliubov and Shirkov, Doklady AN SSSR, 103 (1955) 203

Flow of effective action

conditions on regulator (cutoff-function) $R_k(p)$:

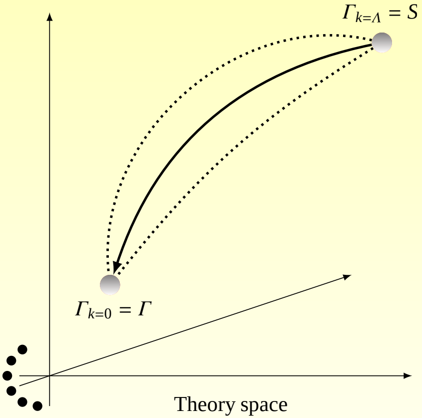
- serves as IR regularization
- recover effective action in IR: $\Gamma_{k=0}[\varphi] = \Gamma[\varphi]$
- recover classical theory at UV-cutoff Λ

exact and UV/IR finite flow equation

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{tr}\left(\frac{\partial_k R_k}{\Gamma_k^{(2)}[\varphi] + R_k}\right)$$

Wetterich

- rhs: quantum fluctuations near momentum shell $p \approx k$
- integration $\Rightarrow$ all quantum fluctuations below cutoff Λ
- new: supersymmetric flows for supersymmetric theories



$\Gamma_{k=\Lambda} = S$
$\Gamma_{k=0} = \Gamma$
Theory space

Challenges

- manifest supersymmetric renormalization flow
  mass deceneracy, Ward identities, ...
- dynamical susy breaking $\Longrightarrow$ phase transitions
  fixed-point structure, temperature effects, equation of state
- non-renormalization theorems
- switch to relevant dof at low energies?

related (mostly structural) investigations by:

Sonoda; Bonini & Vian; Falkenberg & Geyer;

Arnone & Yoshida; Arnone& Guerrieri & Yoshida;

Rosten, Sonoda & Ulker; Horiskoshi & Aoki & Taniguchi & Terao

Wess-Zumino models in two dimensions

- $\mathcal{N} = 1$ supersymmetry $\Longrightarrow$ real superfield

$$\begin{aligned} \Phi(x,\theta) &= \phi(x) + \bar{\theta}\gamma_*\psi(x) + \tfrac{1}{2}(\bar{\theta}\gamma_*\theta)F(x) \\ \mathcal{L}_\mathrm{E} &= \tfrac{1}{2}\bar{\mathcal{D}}\Phi\gamma_*\mathcal{D}\Phi + W(\Phi)\Big|_{\bar{\theta}\gamma_*\theta} \end{aligned}$$

- in components

$$\mathcal{L}_\mathrm{E} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{i}{2}\bar{\psi}\not{\partial}\psi - \tfrac{1}{2}F^2 + \tfrac{1}{2}W''(\phi)\bar{\psi}\gamma_*\psi - FW'(\phi)$$

- supersymmetry

$$\delta\phi = \bar{\varepsilon}\gamma_*\psi, \quad \delta\psi = (F + i\gamma_*\not{\partial}\phi)\varepsilon, \quad \delta F = i\bar{\varepsilon}\not{\partial}\psi$$

- eliminate dummy $F \Longrightarrow$

$$\mathcal{L}_\mathrm{E} = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{i}{2}\bar{\psi}\not{\partial}\psi + \tfrac{1}{2}W'^{\,2}(\phi) + \tfrac{1}{2}W''(\phi)\,\bar{\psi}\gamma_*\psi$$

- classical model determined by superpotential W
- $W(\phi) \sim \phi^m$: m even: susy always unbroken
  m odd: susy breaking possible
- need supersymmetric regulator for flow equation

$$\Delta\mathcal{L}_k = \frac{1}{2}\Phi K(\bar{\mathcal{D}},\mathcal{D})\Phi|_{\bar{\theta}\gamma_*\theta}$$

$$(\bar{\mathcal{D}}\gamma_*\mathcal{D})^{2n} \propto \bar{\mathcal{D}}\not{\partial}\mathcal{D}\triangle^{n-1} \quad , \quad (\bar{\mathcal{D}}\gamma_*\mathcal{D})^{2n+1} \propto \bar{\mathcal{D}}\gamma_*\mathcal{D}\triangle^{n}$$

$\Longrightarrow R_k$ depends only on two functions, here

$$\Delta S_k = \frac{1}{2}\int d^dx\,\left(\phi\, p^2 r(p^2)\,\phi - \bar{\psi}\,\not{p} r(p^2)\,\psi - r(p^2)\,F^2\right)$$

Flow of super potential

- susy relates cut-off functions for component fields ☺
- this talk: mainly local potential approximation NLO: Synatschke, Gies, Wipf

$$\Gamma_k = \int d^d x \left(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{i}{2}\bar\psi\not\partial\psi - \tfrac{1}{2}F^2 + \tfrac{1}{2}W_k''(\phi)\bar\psi\gamma_*\psi - W_k'(\phi)F\right)$$

- project flow onto $F \Longrightarrow$ flow PDE-equation for W_k

$$\partial_k W_k(\phi) = -\frac{k^{d-1}}{A_d}\frac{W_k''(\phi)}{k^2 + W_k''(\phi)^2}, \qquad A_2 = 4\pi,\ A_3 = 8\pi^2$$

$\Longrightarrow$ flow for scalar potential $V_k = \frac{1}{2}W_k'^2$

Fixed point structure in 2 dimensions

- dimensionless quantities $k \to t,\; W_k(\phi) = k w_t(\phi)$

$$\partial_t w_t(\phi) + w_t(\phi) = -\frac{1}{4\pi}\frac{w_t''(\phi)}{1 + w_t''(\phi)^2}, \qquad \text{(nonlinear PDE)}$$

- even $w_t' \Rightarrow$ susy-breaking possible, Taylor series

$$w_t'(\phi) = \lambda_t(\phi^2 - a_t^2) + b_{4,t}\phi^4 + b_{6,t}\phi^6 + \dots$$

$\Longrightarrow \infty$-system of coupled ODE's

- a_t does not enter equations for higher order couplings $\lambda_t, b_{2i,t}$
- fixed points: $\partial_t w_* = 0 \Longrightarrow (a^*)^2 = 1/2\pi$, a_t^2 : IR-unstable

fixed points continued

- keep terms up to $b_{2n,t}\phi^{2n} \Longrightarrow 2n$ non-Gaussian fixed points

$$\pm\left(\lambda_p^*, b_{4,p}^*, \ldots, b_{2n,p}^*\right), \quad p = 1, \ldots, n$$

- ordering $\lambda_n^* > \lambda_{n-1}^* > \ldots \Longrightarrow$

  λ_n^* : 1 IR-unstable direction a_t^2

  λ_{n-1}^* : 2 IR-unstable directions

  λ_{n-2}^* : 3 IR-unstable directions

  $\vdots$

- root belonging to IR-stable fixed point $\lambda_n^* \overset{n\to\infty}{\longrightarrow} \lambda_{\text{crit}} = 0.9816$

λ^*	$\Re(\theta^I)$ of non-Gaussian fixed points, truncation at 2n=16							
$\pm.9816$	-1.54	-7.43	-18.3	-37.3	-68.9	-120	-204	-351
$\pm.8813$	6.16	-1.64	-9.82	-25.6	-52.5	-96.9	-170	-300
$\pm.7131$	21.4	4.37	-1.57	-11.1	-30.1	-63.3	-120	-223
$\pm.5152$	28.7	13.3	3.33	-1.39	-11.6	-32.8	-71.7	-145
$\pm.3158$	20.0	20.0	8.40	2.57	-1.14	-11.6	-34.3	-80.4
$\pm.1437$	11.2	11.2	8.63	5.19	1.95	$-.842$	-11.1	-35.7
$\pm.0322$	4.20	4.20	2.86	2.72	2.72	1.47	$-.540$	-10.5
$\pm.0003$	1.57	1.57	1.43	1.43	1.14	.542	.542	-0.221

- odd solutions of nonlinear ODE for $u(\phi) = w_*''(\phi)$:

$$(1-u^4)u'' = 2u'^2(3-u^2)u - (1-u^2)^3 4\pi u$$

- periodic solutions for $u'(0) < 2\lambda_{\rm crit}$

  previous polynomials converge to periodic solution
- $u'_{\rm crit}(0) = 2\lambda_{\rm crit}$: IR-stable fixed point, finite for $|\phi| \le 0.5823$

next-to leading-order flows

- wave function renormalization $\Longrightarrow \eta = -\partial_t \log Z_k^2$
- θ^0 critical exponent of relevant direction a_t^2 (related to W)
- new superscaling relation (exact in NLO)

$$\nu_w \equiv \frac{1}{\theta_0} = \frac{d-\eta}{2}$$

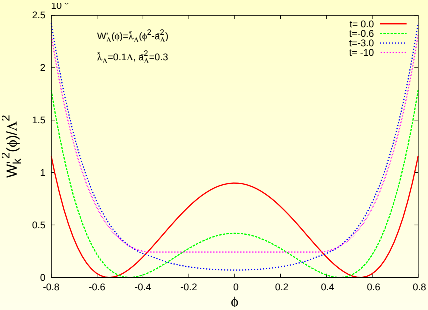
superscaling relation at maximally IR-stable fixed point (d=2)

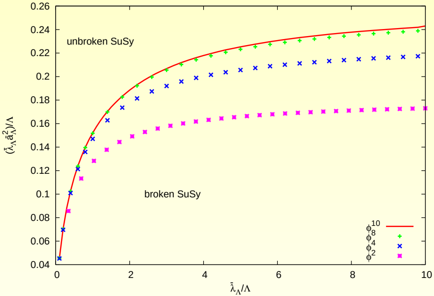
$2n$	2	4	6	8	10	12	14
η	0.3284	0.4194	0.4358	0.4386	0.4388	0.4387	0.4386
$1/\nu_W$	0.8358	0.7903	0.7821	0.7807	0.7806	0.78065	0.7807

- number of IR-unstable direction $=$ number of nodes of u plus 1

Supersymmetry breaking

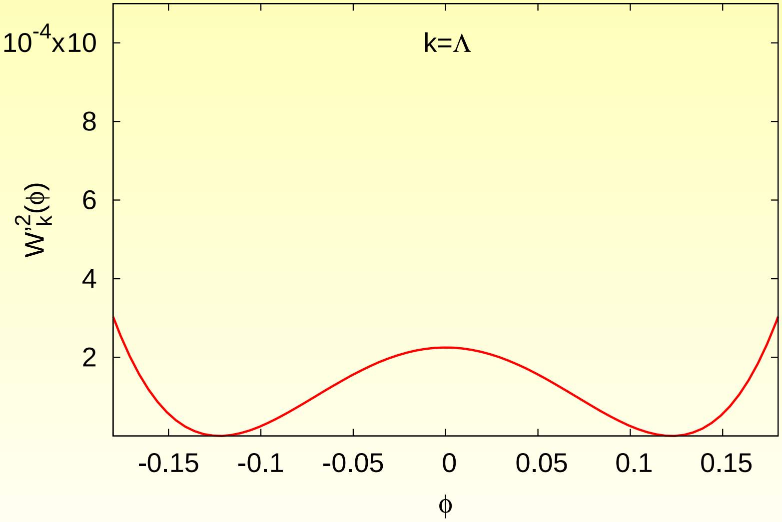
- supersymmetric phase: $\min_\phi V_{k=0}(\phi) = \min_\phi W'^2_{k=0}(\phi) = 0$
- susy broken: $W'_{k=0}(\phi)$ has no node



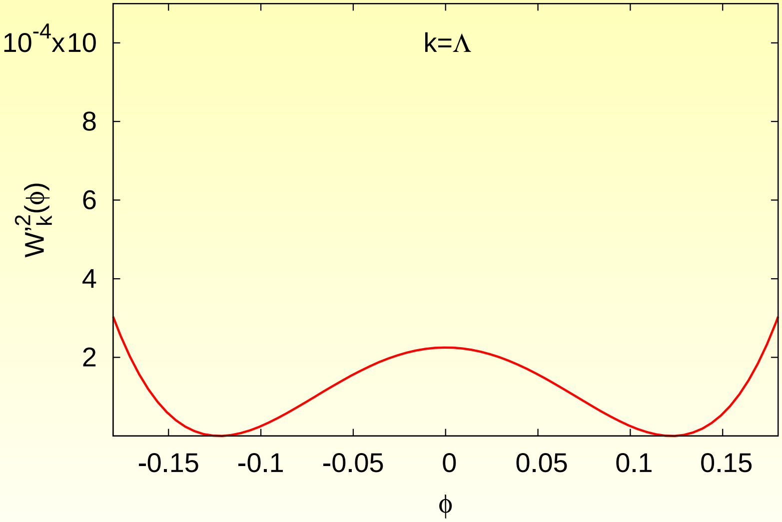


left: flow of a potential $V = W'^2$ with susy breaking, $W'_\Lambda(\phi) = \bar\lambda_\Lambda(\phi^2 - \bar a^2_\Lambda)$

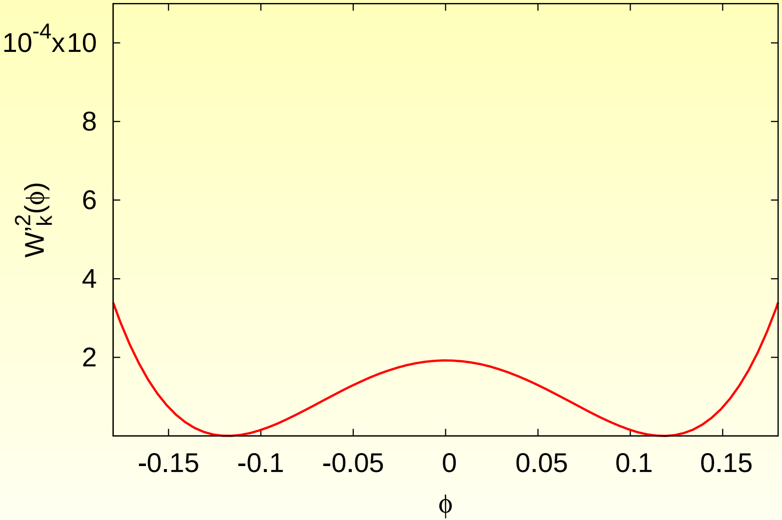
right: phase diagram for couplings specified at Λ, different truncations.



k=Λ
10^{-4}x10
8
6
4
2
$W'^2_k(\phi)$
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
ϕ



k=Λ
10-4x10
8
6
4
2
W'2k(ϕ)
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
ϕ



10^-4x10
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4
2
W'^2_k(ϕ)
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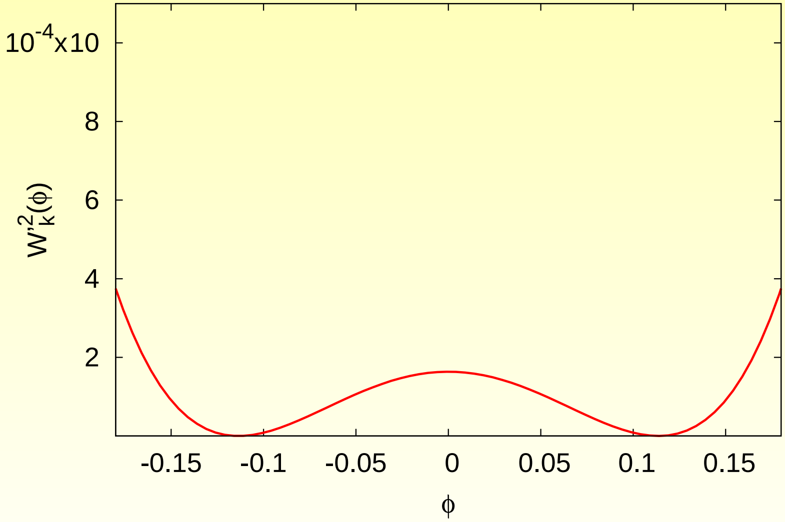
0
0.05
0.1
0.15
ϕ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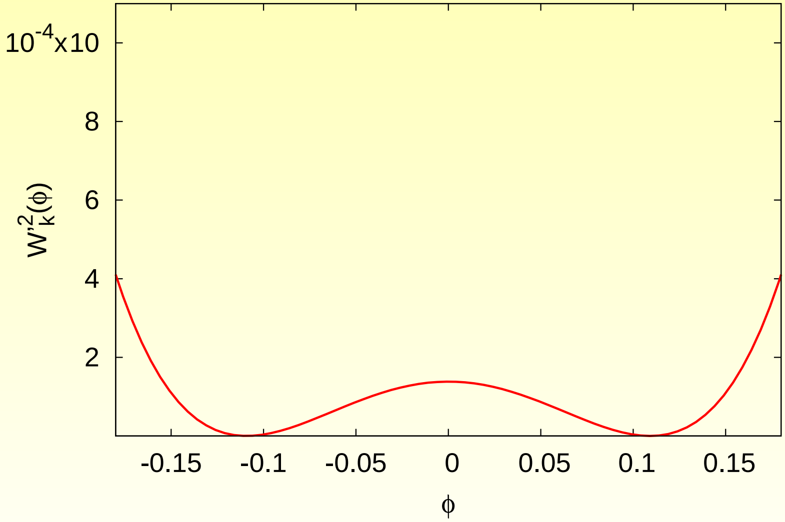

10^-4x10
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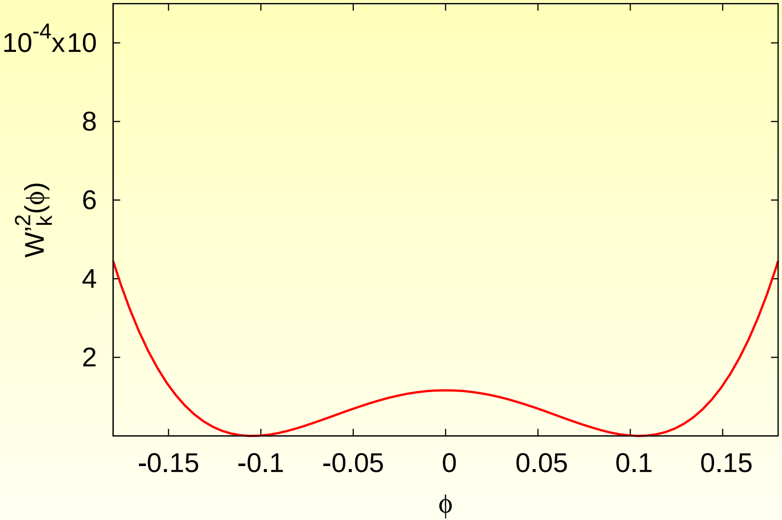

10^-4 x10
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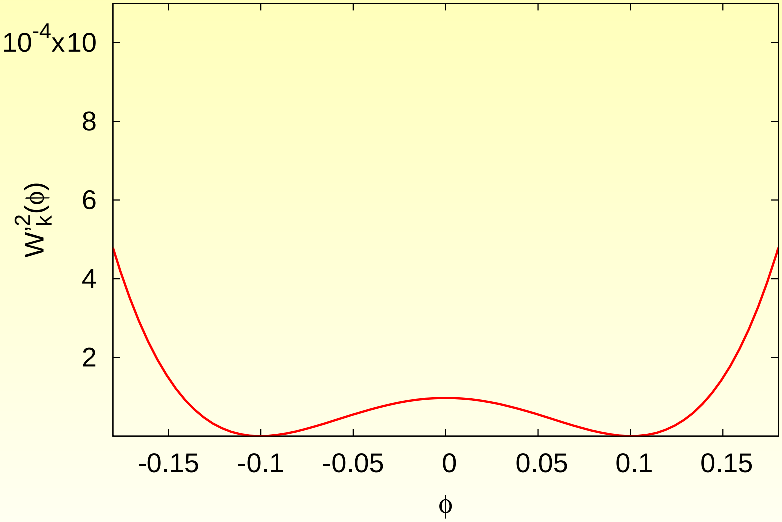
10^-4x10
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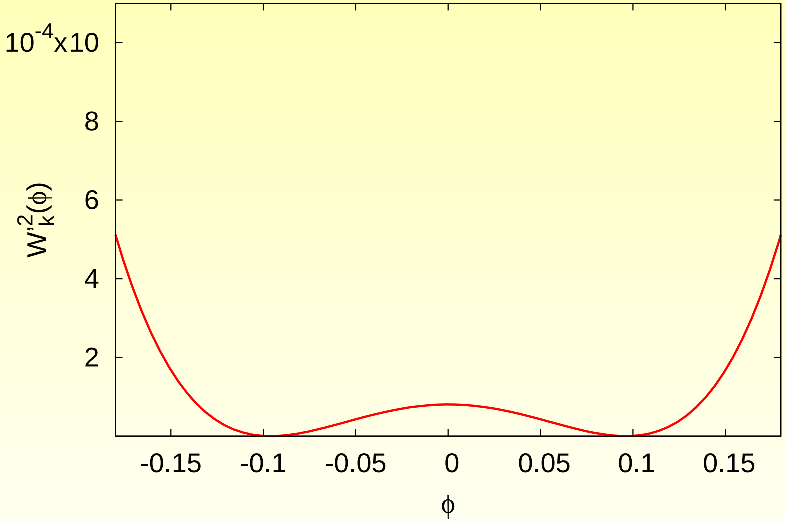

10^-4 x10
8
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2
W'^2_k(ϕ)
-0.15
-0.1
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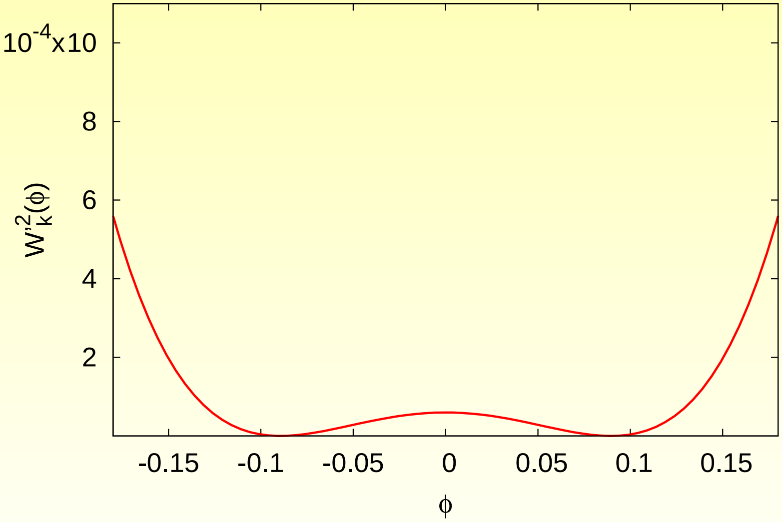
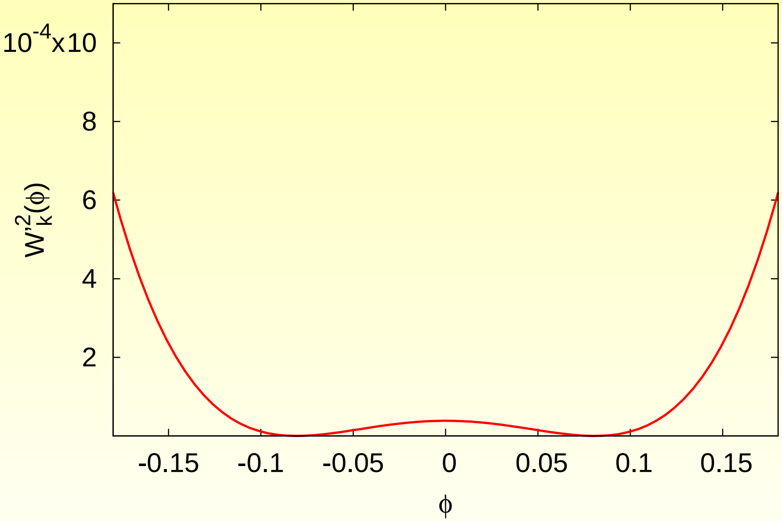
0.15
ϕ


10^-4x10
8
6
4
2
W'^2_k(φ)
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
φ


10^-4x10
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2
W'^2_k(ϕ)
-0.15
-0.1
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0
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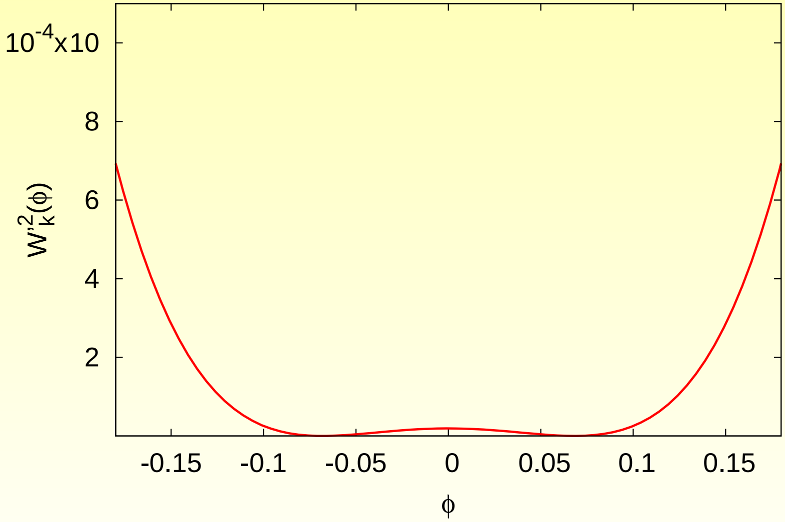
0.15
ϕ


10^-4x10
8
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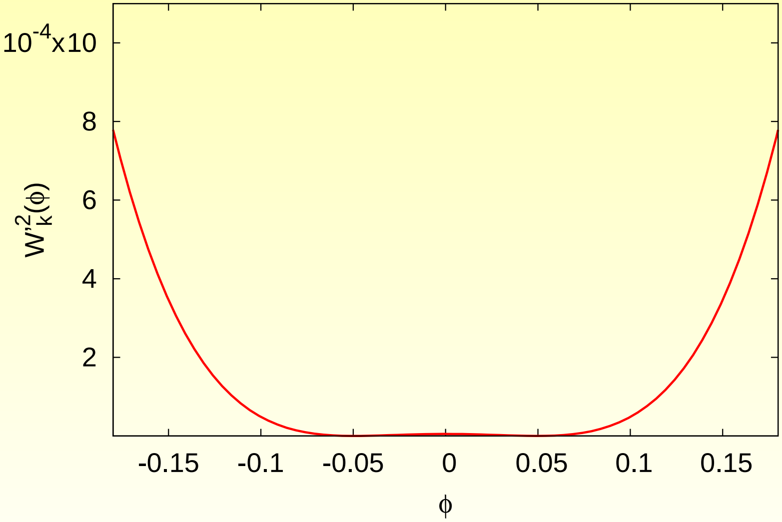
W'^2_k(ϕ)
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
ϕ



10^{-4}x10
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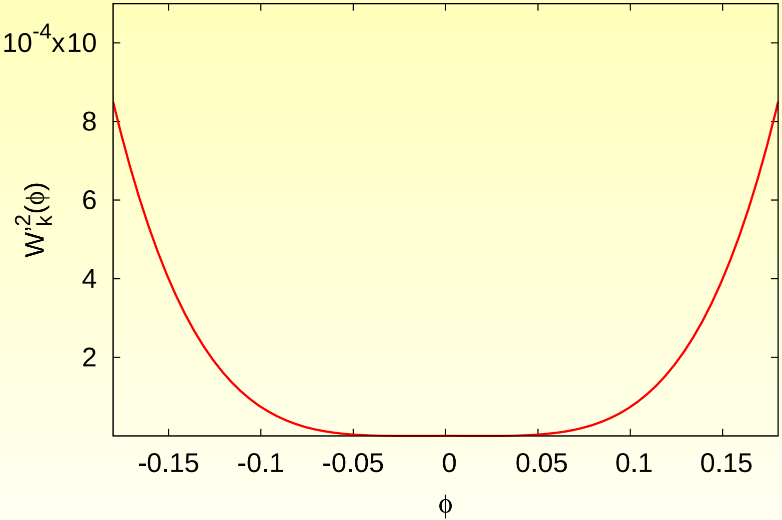
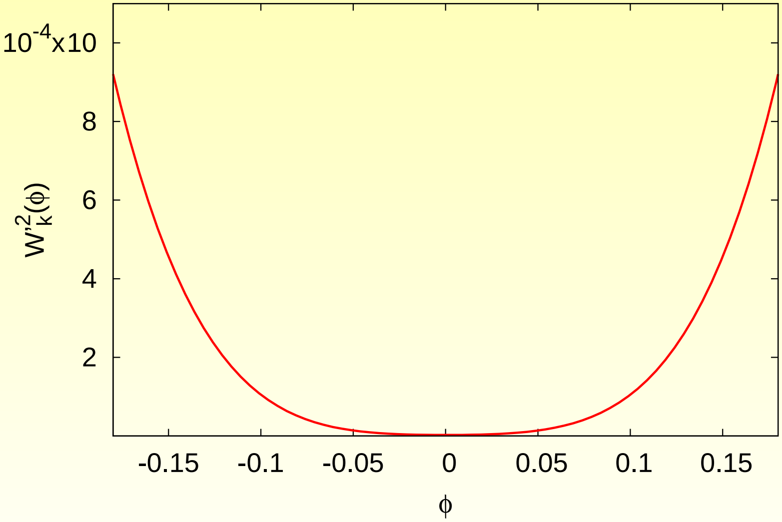
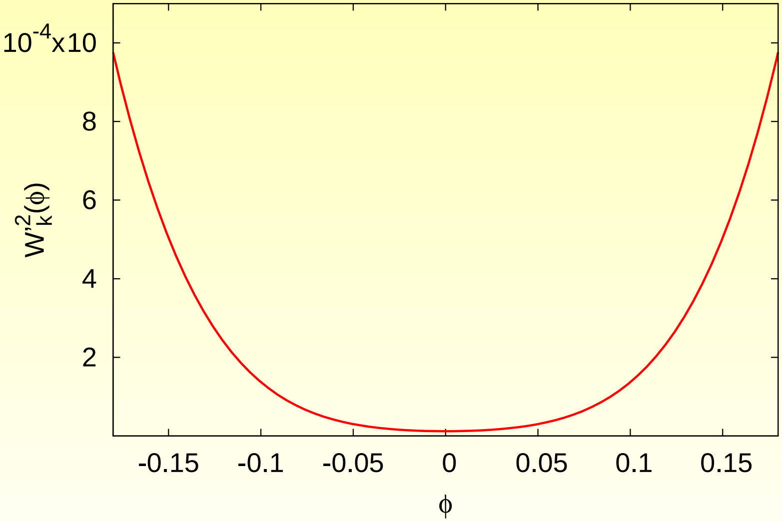
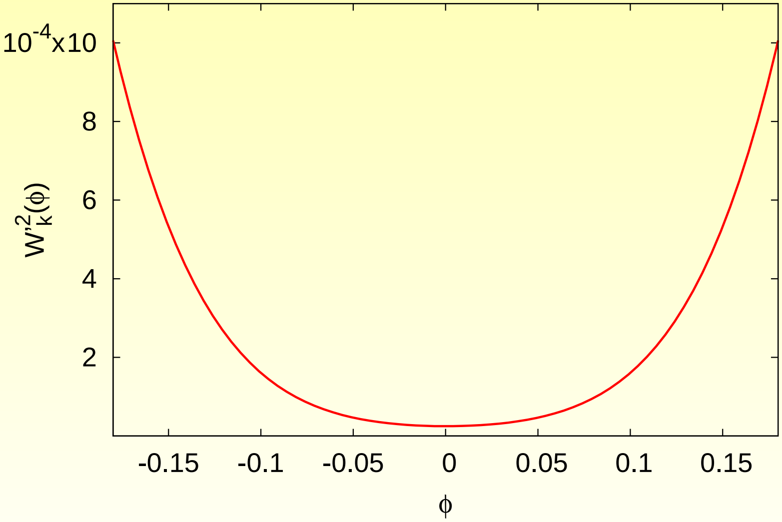
6
4
2
$W'^2_k(\phi)$
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
ϕ


10^-4 x10
8
6
4
2
W'k2(ϕ)
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
ϕ

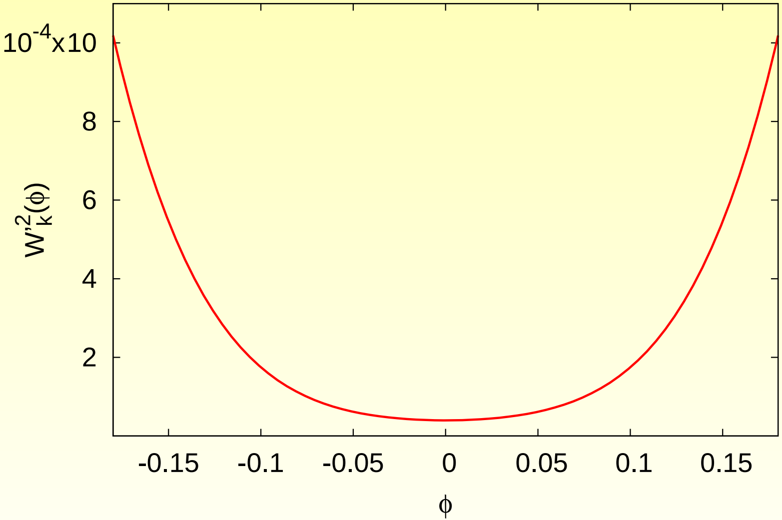

10^-4x10
8
6
4
2
W'^2_k(ϕ)
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
ϕ



10⁻⁴x10
8
6
4
2
W'²ₖ(ϕ)
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
ϕ



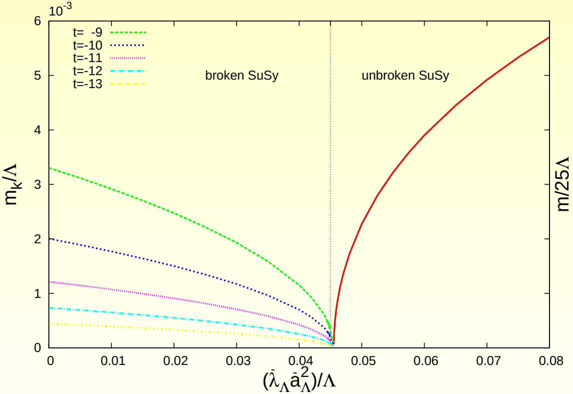
10^-4x10
8
6
4
2
W'^2_k(ϕ)
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
ϕ



10^-4x10
8
6
4
2
W'^2_k(ϕ)
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
ϕ

masses of bosons and fermions

- supersymmetric phase: $Z_k^4 m^2_{k,\mathrm{boson}} = W_k''^{\,2}(\chi_{\min}/Z_k) = Z_k^4 m^2_{k,\mathrm{fermion}}$
- broken phase:(superscaling) $Z_k^4 m^2_{k,\mathrm{boson}} = W_k'(0)\,W_k'''(0) \sim k^{1+\eta/2}$



Wess-Zumino model in 3 dimensions

with J. Braun and F. Synatschke-Czerwonka

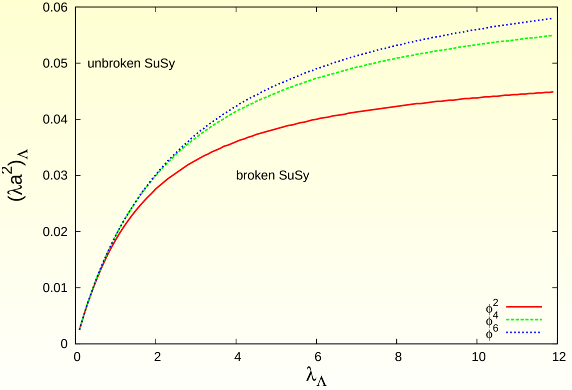
- one Wilson-Fisher fixed point for Yukawa-model
- a_t^2 defines the only IR-unstable direction
- LPA, polynomial expansion
- rapid convergence

Wilson-Fisher fixed point from polynomial expansion

$2n$	$\pm\lambda^*$	$\pm b_4^*$	$\pm b_6^*$	$\pm b_8^*$	$\pm b_{10}^*$	$\pm b_{12}^*$
4	1.546	2.305				
6	1.590	2.808	6.286			
8	1.595	2.873	7.150	13.41		
10	1.595	2.873	7.155	13.48	1.212	
12	1.595	2.870	7.118	12.90	-8.895	-183.3

$2n$	critical exponents for different truncations							
6	−0.799	−5.92	−20.9					
8	−0.767	−4.83	−14.4	−38.2				
10	−0.757	−4.35	−11.5	−26.9	−60.8			
12	−0.756	−4.16	−9.94	−21.4	−43.8	−89.0		
14	−0.756	−4.10	−9.13	−18.3	−35.1	−65.4	−123	
16	−0.756	−4.08	−8.72	−16.4	−29.9	−52.9	−91.9	−163
18	−0.756	−4.08	−8.54	−15.2	−26.4	−45.0	−75.0	−124	−209

- phase diagram from parameter study of $W'_{k\to 0}$



Finite temperature

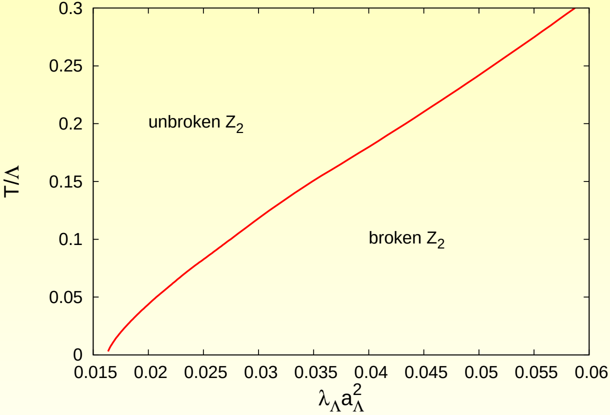
- $\int dp_0 \longrightarrow$ summation over Matsubara frequencies
- sums can be calculated explicitly $\Longrightarrow$ two flow equations

$$\partial_k W_k'^{\,\mathrm{bos}} = -\frac{k^2}{8\pi^2} W_k''' \frac{k^2 - W_k''^2}{(k^2 + W_k''^2)^2} \times F_{\mathrm{bos}}(T,k)$$

$$\partial_k W_k'^{\,\mathrm{ferm}} = -\frac{k^2}{8\pi^2} W_k''' \frac{k^2 - W_k''^2}{(k^2 + W_k''^2)^2} \times F_{\mathrm{ferm}}(T,k)$$

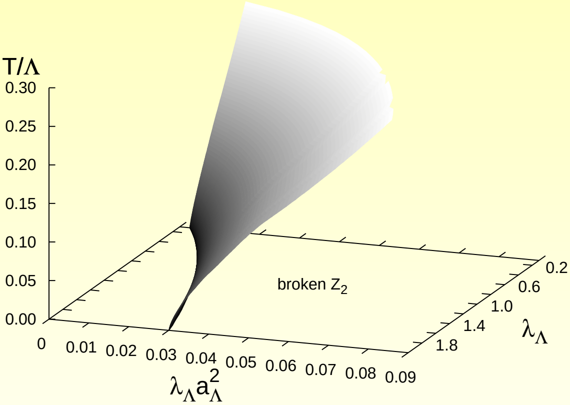
- susy breaking by thermal fluctuations (bosons $\neq$ fermions)
- $T = 0$: susy broken $\longleftrightarrow$ $\mathbb{Z}_2$ unbroken
  $\Longrightarrow$ study $\mathbb{Z}_2$ breaking at finite T

Phase diagram



finite-temperature phase diagram for fixed $\lambda_\Lambda = 0.8$

Phase diagram, continued



finite-temperature phase diagram

Linear sigma-models

- real superfield, N components

$$\Phi^i(x,\theta) = \phi^i + \bar{\theta}\psi^i(x) + \frac{1}{2}\bar{\theta}\theta F^i(x)$$

- $O(N)$-invariant composite superfield $R \equiv \frac{1}{2}\Phi^i\Phi_i$

$$R = \bar{\varrho} + (\bar{\theta}\psi_i)\phi^i + \frac{1}{2}\,\bar{\theta}\theta\left(\phi^i F_i - \frac{1}{2}\bar{\psi}^i\psi_i\right)$$

- $O(N)$-invariant supersymmetric action

$$\mathcal{S} = \int d^3x\,\left[-\frac{1}{2}\Phi^i\bar{\mathcal{D}}\mathcal{D}\Phi_i + 2N\,W\left(\frac{R}{N}\right)\right]\Big|_{\bar{\theta}\theta}$$

- rescaled dimensionless quantities

$$\rho = \frac{8\pi^2}{N}\frac{\bar{\varrho}}{k} \quad \text{and} \quad w(\rho) = 8\pi^2\,\frac{W(\frac{\bar{\varrho}}{N})}{k^2}$$

- flow equation: optimized cutoff-function
- contribution from Goldstone modes and radial mode

$$\partial_t w - \rho w' + 2w = -\frac{(1-\frac{1}{N})w'}{1+w'^2} - \frac{\frac{1}{N}(w'+2\rho w'')}{1+(w'+2\rho w'')^2}$$

- large-N limit: radial mode decouples $\Longrightarrow$

$$\partial_t u + \partial_\rho u\left[1-\rho-u^2\,\frac{3+u^2}{(1+u^2)^2}\right] = -u \qquad (u = w')$$

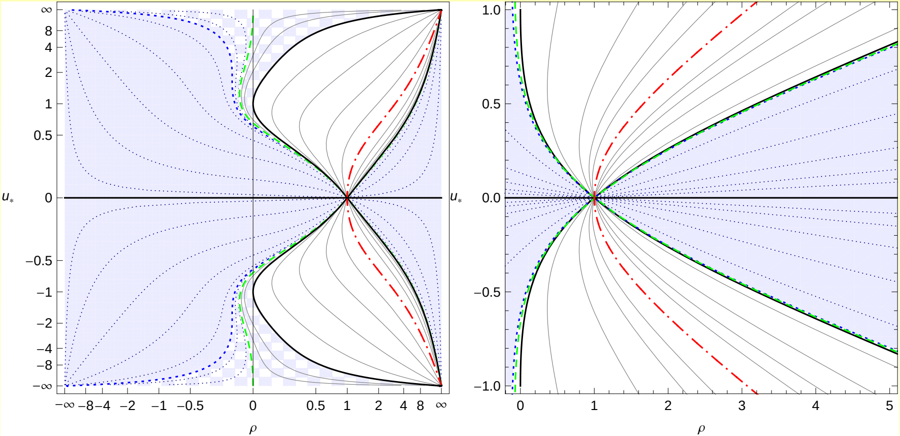
- methods of characteristics $\Longrightarrow$ exact solution

$$\frac{\rho-1}{u} - F(u) = G(ue^t), \qquad F(u) = \frac{u}{1+u^2} + 2\arctan u$$

- $u(\rho)$ at cutoff $\Lambda \Longrightarrow G(ue^t)$
- fixed point solutions depend on one real parameter c

$$\rho = 1 + H(u_*) + c\,u_*, \quad H(u_*) = u_*\,F(u_*)$$

- one-parameter family of fixed point solutions
- all solutions have node at $\rho = 1$
- two families: global vs. non-global solutions



- 1-parameter family of fixed point solutions: one marginal coupling $\sim 1/c$

  weak coupling: $c \to \infty \Rightarrow u_* = 0$ horizontal line
  intermediate coupling: two fixed point solution
  strong coupling: $c \to 0 \Rightarrow$ not globally defined

eigenperturbation

- flow in vicinity of fixed points

$$u(t,\rho) = u_*(\rho) + \delta u(t,\rho)$$

- linearize flow equation

$$\partial_t\, \delta u = \frac{u_*}{u_*'}\left(\partial_\rho - \frac{(u_* u_*')'}{u_* u_*'}\right)\delta u,$$

- explicit solution

$$\delta u(t,u) = C\, e^{\theta t}\, u_*^{\theta+1}\, u_*'\,.$$

- regularity at $\rho = 1 \Longrightarrow$
- eigenvalues of stability matrix $\Rightarrow$ all critical exponents

$$\theta = -1, 0, 1, 2, 3, \cdots$$

summary and next

- manifest supersymmetric FRG for scalar multiplet
- masses and coupling in infrared
- supersymmetry breaking, phase transitions
- critical phenomena: fixed points, universal exponents
- non-renormalization theorems
- exact solution for W_k in linear susy $O(N \to \infty)$ sigma model
- supersymmetric $O(n)$ and $CP(n)$ models
  first lattice results for supersymmetric $CP(1)$ — R. Flore, D. Körner, C. Wozar
  flow equations: how does spectrum vary with θ — R. Flore
- supersymmetric gauge theories
  first lattice-results in $d = 1, 2, 3$ — B. Wellegehausen
  study of flow equations — F. Synatschke-Czerwonka, M. Mastaler

☺ thank you ☺