Nonlocal Gravitational Models and Exact Solutions

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To specify different types of cosmic fluids one uses a relation between the pressure p and the energy density ϱ: $p = w\varrho$, where w is the state parameter.

Contemporary experiments give strong support that

$w > 0$ — **Atoms. (4%)**

$w = 0$ — **the Cold Dark Matter. (23%)**

$w < 0$ — **the Dark Energy. (73%),** $w_{DE} = -1 \pm 0.2$**.**

Different variants of the Dark Energy

- The cosmological constant,
- Scalar and k-essence fields, phantom fields and quintom models,
- Modified $f(R)$ gravity models,
- Nonlocal modified gravity and nonlocal scalar fields,
- Vectors and, in particular, Yang–Mills fields.

Model with scalar fields

- $w > -1$, is achieved in quintessence models.
- $w = -1$, is realized by means of the cosmological constant.
- $w < -1$, is called a "phantom" one and can be realized due to a scalar field with a ghost (phantom) kinetic term.

To describe the crossing of the cosmological constant barrier $w = -1$ we can use to fields: one usual scalar field and one a phantom scalar field.

$f(R)$ gravity

Let us consider a $f(R)$ gravity model

$$S_f = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}f(R), \tag{1}$$

The equation of motion are the following:

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} - D_\mu\partial_\nu f'(R) + g_{\mu\nu}\Box_g f'(R) = 0. \tag{2}$$

Equations of $f(R)$ metric gravity is equivalent to general relativity equations with an additional scalar field.

They can be rewritten as

$$\tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R} = 8\pi G_N\tilde{T}_{\mu\nu}(\varphi), \tag{3}$$

If we transforms the metric tensor

$$\tilde{g}_{\mu\nu} = F(R) g_{\mu\nu}, \tag{4}$$

then

$$\tilde{R}^{\mu}_{\nu} = \frac{R^{\mu}_{\nu}}{F} - \frac{g^{\mu\alpha} D_{\nu}(D_{\alpha}F)}{F^2} + 3\frac{g^{\mu\alpha} D_{\nu}F D_{\alpha}F}{2F^3} - \frac{g^{\alpha\beta} D_{\alpha}(D_{\beta}F)}{2F^2}\delta^{\mu}_{\nu}, \tag{5}$$

and

$$\tilde{R} = \frac{R}{F} - 3\frac{g^{\beta\alpha} D_{\beta}(D_{\alpha}F)}{F^2} + 3\frac{g^{\beta\alpha} D_{\beta}F D_{\alpha}F}{2F^3}. \tag{6}$$

So, one can obtain that

$$F(R) = f'(R), \tag{7}$$

$$V = \frac{f(R) - Rf'(R)}{16\pi G_N (f'(R))^2}, \qquad \varphi = \frac{\sqrt{3}}{4\sqrt{\pi G_N}} \ln(F). \tag{8}$$

The standard example.

For

$$f(R) = R - \frac{1}{6M^2}R^2, \tag{9}$$

one get the exponential potential:

$$V(\varphi) = \frac{3M^2}{32\pi G_N}\left(1 - e^{\frac{4\sqrt{\pi G_N}}{\sqrt{3}}\varphi}\right)^2 \tag{10}$$

and

$$\varphi = \frac{\sqrt{3}}{4\sqrt{\pi G_N}} \ln\left(1 - \frac{R}{3M^2}\right). \tag{11}$$

As known the Einstein equation are the second order differential equations in $g_{\mu\nu}$.

The $f(R)$ gravity equations are the fourth order differential equations in $g_{\mu\nu}$.

Nonlocal gravity models

There are another type of modifications that explicitly includes a function of $\Box_g$ operator, in particular, $\Box_g^{-1}$ and defines a non-local modification of gravity.

A modification that does not assume the existence of a new dimensional parameter in the action

$$S_2 = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G_N} R \left(1 + f(\Box^{-1} R) \right) + \mathcal{L}_{\rm matter} \right\} , \tag{12}$$

The action (12) can be rewritten in the following form:

$$\tilde{S}_2 = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left\{ R\left(1 + f(\phi)\right) - \partial_\mu \xi \partial^\mu \phi - \xi R \right\} + \mathcal{L}_{\rm matter} \right] .$$

By the variation over ξ, we obtain $\Box\phi = R$. Substituting $\phi = \Box^{-1} R$, we reobtain (12).

The reasons to consider (12) with corrections involving $\Box^{-1}R$ as an origin for dark energy is the following. This term is dimensionless and one may construct the distortion function without introducing any dimensional functions. It appears as a prefactor for the Newtonian gravitational constant, and explain weakening of gravity at cosmological scales.

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A modification that assumes the existence of a new dimensional parameter L can be of the form

$$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}R + \frac{1}{2}R\mathcal{F}(\Box/M_*^2)R - \Lambda\right) \qquad (13)$$

where M_* is the mass scale at which the higher derivative terms in the action become important.

By virtue of the field redefinition one can transform the non-local gravity action (13) as follows:

$$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}(1+\Phi)R + \frac{1}{2}\tau\mathcal{F}(\Box/M_*^2)\tau - \frac{M_P^2}{2}\Phi\tau - \Lambda\right)$$

with two scalar fields Φ and τ. Variation w.r.t. Φ gives $\tau = R$.

Biswas T., Mazumdar A., and Siegel W. 2006, *JCAP* **0603** 009 (hep-th/0508194), Biswas T., Koivisto T., and Mazumdar T. 2010, *JCAP* **1011** 008 (arXiv:1005.0590)

Gravity models with a minimally coupling nonlocal scalar field

The SFT inspired nonlocal gravitation models are introduced as a sum of the SFT action of the tachyon field ϕ plus the gravity part of the action. One cannot deduce this form of the action from the SFT.

Let us consider the $f(R)$ gravity model with a nonlocal scalar field:

$$S_f = \int d^4x\sqrt{-g}\left(\frac{f(L^2R)}{16\pi G_N L^2} + \frac{1}{\alpha' g_o^2}\left(\frac{1}{2}\phi\,\mathcal{F}\left(\alpha'\Box_g\right)\phi - V(\phi)\right) - \Lambda\right), \tag{14}$$

where $f(L^2R)$ is an arbitrary differentiable function.

We use the signature $(-,+,+,+)$, $g_{\mu\nu}$ is the metric tensor, G_N is the Newtonian constant.

The function $\mathcal{F}$ is assumed to be be an entire function.

The function $\mathcal{F}$ can be represented as the convergent series:

$$\mathcal{F}(\Box_g) = \sum_{n=0}^{\infty} f_n \Box_g^n.$$

The Weierstrass factorization theorem asserts that the function $\mathcal{F}$ can be represented as a product involving its zeroes J_k:

$$\mathcal{F}(J) = J^m e^{Y(J)} \prod_{k=1}^{\infty} \left(1 - \frac{J}{J_k}\right) e^{\frac{J}{J_k} + \frac{J^2}{2J_k^2} + \cdots + \frac{1}{p_k}\left(\frac{J}{J_k}\right)^{p_k}},$$

where m is an order of the root $J = 0$ (m can be equal to zero), $Y(J)$ is an entire function, natural numbers p_n are chosen such that the series $\sum\limits_{n=1}^{\infty} \left(\frac{J}{J_n}\right)^{p_n+1}$ is an absolutely and uniformly convergent one.

Scalar fields ϕ (associated with the open string tachyon) is dimensionless, while $[\alpha'] =$ length^2, $[L] =$ length and $[g_o] =$ length.

Let us introduce dimensionless coordinates $\bar{x}_\mu = x_\mu/\sqrt{\alpha'}$,

the dimensionless Newtonian constant $\bar{G}_N = G_N/\alpha'$,

the dimensionless parameter $\bar{L} = L/\sqrt{\alpha'}$, and

the dimensionless $\bar{g}_o = g_o/\sqrt{\alpha'}$.

The dimensionless cosmological constant $\bar{\Lambda} = \Lambda\alpha'^2$, $\bar{R}$ is the curvature scalar in the coordinates $\bar{x}_\mu$:

$$S_f = \int d^4\bar{x}\sqrt{-g}\left(\frac{f(\bar{L}^2\bar{R})}{16\pi\bar{G}_N\bar{L}^2} + \frac{1}{\bar{g}_o^2}\left(\frac{1}{2}\phi\,\mathcal{F}\left(\bar{\Box}_g\right)\phi - V(\phi)\right) - \bar{\Lambda}\right)$$

In the following formulae we omit bars, but use only dimensionless coordinates and parameters.

In the metric variational approach the equations of gravity are as follows:

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} - D_\mu\partial_\nu f'(R) + g_{\mu\nu}\Box_g f'(R) = 8\pi G_N T_{\mu\nu},$$

$$\mathcal{F}(\Box_g)\phi = \frac{dV}{d\phi}, \tag{15}$$

where the energy–momentum (stress) tensor $T_{\mu\nu}$ is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{g_o^2}\Big(E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\rho\sigma}E_{\rho\sigma} + W\right)\Big),$$

$$E_{\mu\nu} \equiv \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\mu \Box_g^l \phi \partial_\nu \Box_g^{n-1-l}\phi,$$

$$W \equiv \frac{1}{2}\sum_{n=2}^{\infty} f_n \sum_{l=1}^{n-1} \Box_g^l \phi \Box_g^{n-l}\phi - \frac{f_0}{2}\phi^2 + V(\phi).$$

There are two different cases:

- The potential $V(\phi) = C_2\phi^2 + C_1\phi + C_0$, where C_2, C_1 and C_0 are arbitrary constants. In this case one can construct the equivalent action with local fields and quadratic potentials. Number of local fields is equal to number of roots of $\mathcal{F}(\Box)$, with a glance of order of them. It has been proved for an arbitrary analytic function $\mathcal{F}$ with simple and double roots.

  I.Ya. Aref'eva, L.V. Joukovskaya, S.Yu.V., *J. Phys. A: Math. Theor.* **41** (2008) 304003, `arXiv:0711.1364`;

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- The potential $V(\phi) \neq C_2\phi^2 + C_1\phi + C_0$. In this case situation is more difficult and exact solutions is possible to find only adding some scalar field, for example, a k-essence field.

  Numerical Solution:

  L. Joukovskaya, *JHEP* 0902 (2009) 045, `arXiv:0807.2065`

  Approximate solutions for field equation:

  G. Calcagni and G. Nardelli, *Int. J. Mod. Phys.* **D 19** (2010) 329–338, `arXiv:0904.4245`

  Exact solutions for field equation:

  S.Yu.V., Theor.Math.Phys.166 (2011) 392–402, `arXiv:1005.5007`

SOLUTIONS FOR EQUATIONS OF MOTION

(S.Yu.V., Theor.Math.Phys.166 (2011) 392-402, `arXiv:1005.5007`)

Let us consider nonlocal Klein–Gordon equation in the case of an arbitrary potential:

$$\mathcal{F}(\Box_g)\phi = V'(\phi), \tag{16}$$

where prime is a derivative with respect to ϕ. A particular solution of (16) is a solution of the following system:

$$\sum_{n=0}^{N-1} f_n \Box_g^n \phi = V'(\phi) - C, \qquad f_N \Box_g^N \phi = C, \tag{17}$$

where $N-1$ is a natural number, C is an arbitrary constant.

In the case $f_1 \neq 0$ we can choose $N = 2$.

In the spatially flat FRW metric with the interval:

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \qquad (18)$$

where $a(t)$ is the scale factor, we obtain from (17):

$$f_1\Box_g\phi = -f_1\left(\ddot{\phi} + 3H\dot{\phi}\right) = V'(\phi) - f_0\phi - C, \qquad f_2\Box_g^2\phi = C. \qquad (19)$$

The Hubble parameter

$$H = -\frac{1}{3\dot{\phi}}\left(\ddot{\phi} + \tilde{V}'(\phi) - \frac{C}{f_1}\right), \qquad (20)$$

where

$$\tilde{V}'(\phi) \equiv \frac{1}{f_1}\left(V'(\phi) - f_0\phi\right). \qquad (21)$$

Equation

$$(\partial_t^2 + 3H\partial_t)\left(\ddot{\phi} + 3H\dot{\phi}\right) = \frac{C}{f_2}, \qquad (22)$$

is as follows

$$(\partial_t^2 + 3H\partial_t)\tilde{V}' = \tilde{V}'''\dot{\phi}^2 + \tilde{V}''(\ddot{\phi} + 3H\dot{\phi}) = -\frac{C}{f_2}. \qquad (23)$$

We eliminate H and obtain

$$\dot{\phi}^2 = \frac{1}{\tilde{V}'''}\left(\tilde{V}''\tilde{V}' - \frac{C}{f_1}\tilde{V}'' - \frac{C}{f_2}\right). \qquad (24)$$

The obtained equation can be solved in quadratures. Its general solution depend on two arbitrary parameters C and t_0.

It allows to find solutions for an arbitrary potential $V(\phi)$, with the exception of linear and quadratic potentials.

Note that we do not consider other Einstein equations. In distinguish to the localization method, which allows to localize all Einstein equations, this method solves only the field equation, whereas the obtained solutions maybe do not solve other equations.

CUBIC POTENTIAL

The case of cubic potential is is connected with the bosonic string field theory. Let us find solutions (16) for

$$V(\phi) = B_3\phi^3 + B_2\phi^2 + B_1\phi + B_0, \tag{25}$$

where B_0, B_1, B_2, and B_3 are arbitrary constants, but $B_3 \neq 0$. For this potential we get (24) in the following form

$$\dot{\phi}^2 = 4C_3\phi^3 + 6C_2\phi^2 + 4C_1\phi + C_0, \tag{26}$$

where

$$C_0 = \frac{(B_1 - C)(2B_2 - f_0)}{6f_1B_3} - \frac{Cf_1^2}{6f_1f_2B_3}, \qquad C_2 = \frac{2B_2 - f_0}{4f_1}, \tag{27}$$

$$C_1 = \frac{6B_3(B_1 - C) + (2B_2 - f_0)^2}{24f_1B_3}, \qquad C_3 = \frac{3B_3}{4f_1}. \tag{28}$$

Note, that $C_3 \neq 0$ since $B_3 \neq 0$. Using the transformation

$$\phi = \frac{1}{2C_3}(2\xi - C_2), \tag{29}$$

we get the following equation

$$\dot{\xi}^2 = 4\xi^3 - g_2\xi - g_3, \tag{30}$$

where

$$g_2 = \frac{(2B_2 - f_0)^2 - 12B_3(B_1 - C)}{16f_1^2},$$

$$g_3 = 2C_1C_2C_3 - C_2^3 - C_0C_3^2 = -\frac{3B_3C}{32f_2f_1}.$$

A solution of equation (30) is either the Weierstrass elliptic function

$$\xi(t) = \wp(t - t_0, g_2, g_3), \tag{31}$$

or a degenerate elliptic function.

Let us consider degenerated cases. At $g_2 = 0$ and $g_3 = 0$

$$\phi_1 = \frac{1}{C_3(t-t_0)^2} - \frac{C_2}{2C_3} = \frac{4f_1}{3B_3(t-t_0)^2} - \frac{2B_2 - f_0}{6B_3}. \qquad (32)$$

$$H_1 = \frac{5}{3(t-t_0)}. \qquad (33)$$

We are of interest to obtain a bounded solution, which tends to a finite limit at $t \to \infty$. We have obtained such solutions in the following form

$$\phi_2 = D_2 \tanh(\beta(t-t_0))^2 + D_0, \qquad (34)$$

$$D_2 = \frac{4}{3B_3} f_1 \beta^2, \qquad D_0 = \frac{1}{18B_3}\left(3(f_0 - 2B_2) - 16 f_1 \beta^2\right), \qquad (35)$$

where β is a root of the following equation

$$1024 f_2 f_1 \beta^6 + 576 f_1^2 \beta^4 + 324 B_3 B_1 - 27(2B_2 - f_0)^2 = 0. \qquad (36)$$

Bounded real solutions for equation (26) correspond to real root of equations (36). Pure image root of (36) correspond to unbounded real solutions for equation (26), because $\tanh(\beta t)^2 = -\tan(\mathrm{i}\beta t)^2$. The solution ϕ_2 exists at

$$C = \frac{1}{36B_3}\left(64f_1^2\beta^4 - 3(2B_2 - f_0)^2 + 36B_3B_1\right). \tag{37}$$

$$H_2 = \frac{\beta(2\cosh(\beta t)^2 - 3)}{3\cosh(\beta t)\sinh(\beta t)} -$$

$$-\frac{3B_3(D_2\tanh(\beta t)^2 + D_0)^2 + (2B_2 - f_0)(D_2\tanh(\beta t)^2 + D_0) + B_1}{6f_1D_2\beta\tanh(\beta t)(1 - \tanh(\beta t)^2)}.$$

Cosmological model with a nonlocal scalar field and a k-essence field

Let us consider the k-essence cosmological model with a non-local scalar field:

$$S_2=\int d^4x\sqrt{-g}\alpha'\left(\frac{R}{16\pi G_N}+\frac{1}{g_o^2}\left(\frac{1}{2}\phi\mathcal{F}(\Box_g)\phi-V(\phi)\right)-\mathcal{P}-\Lambda\right), \tag{38}$$

where

$$X\equiv\;-g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi. \tag{39}$$

In the FRW metric $X=\dot{\Psi}^2$.

The standard variant of the k-essence field Lagrangian

$$\mathcal{P}(\Psi,X)=\frac{1}{2}(p_q(\Psi)-\varrho_q(\Psi))+\frac{1}{2}(p_q(\Psi)+\varrho_q(\Psi))X+\frac{1}{2}M^4(\Psi)(X-1)^2.$$

Here $p_q(\Phi)$, $\varrho_q(\Phi)$, and $M^4(\Phi)$ are arbitrary differentiable functions. The energy density is

$$\mathcal{E}(\Psi, X) = (p_q(\Psi) + \varrho_q(\Psi))X + 2M^4(\Psi)(X^2 - X) - \mathcal{P}(\Psi, X).$$

The Einstein equations are

$$3H^2 = 8\pi G_N(\varrho + \mathcal{E} + \Lambda), \qquad (40)$$

$$2\dot{H} + 3H^2 = -8\pi G_N(p + \mathcal{P} - \Lambda). \qquad (41)$$

From S_2 we also have equation

$$\mathcal{F}(\Box_g)\phi = V'(\phi), \qquad (42)$$

and

$$\dot{\mathcal{E}} = -3H\left(\mathcal{E} + \mathcal{P}\right). \qquad (43)$$

For any real differentiable function $H_0(t)$, there exist such real differentiable functions $\varrho_q(\Phi)$ and $p_q(\Phi)$ that the functions $H_0(t)$ and $\Psi(t) = t$ are a particular solution for the system of the Einstein equations.

This property can be generalized on the model with the action S_2.

If $\Psi(t) = t$, then

$$\mathcal{E} = \varrho_q(\Psi) = \varrho_q(t), \qquad \mathcal{P} = p_q(\Psi) = p_q(t). \tag{44}$$

Substituting ϱ_q p_q in (40)–(43), we get

$$\varrho_q(\Psi) = \varrho_q(t) = \frac{3}{8\pi G_N} H_0^2(t) - \varrho(t) - \Lambda, \tag{45}$$

$$p_q(\Psi) = p_q(t) = -\varrho_q(t) - \varrho(t) - p(t) - \frac{1}{4\pi G_N}\dot{H}(t). \tag{46}$$

Let ϕ_2 is a solution to system (17) at $N = 2$.
Using $\Box_g^2 \phi_2 = C/f_2$, we get

$$\varrho(\phi_2) = E_{00}(\phi_2) + W(\phi_2), \qquad p(\phi_2) = E_{00}(\phi_2) - W(\phi_2),$$

where

$$E_{00}(\phi_2) = \frac{1}{2g_{\rm o}^2}\left(f_1(\partial_t\phi_2)^2 + 2f_2\partial_t\phi_2\partial_t\Box_g\phi_2 + f_3(\partial_t\Box_g\phi_2)^2\right),$$

$$W(\phi_2) = \frac{1}{g_{\rm o}^2}\left(\frac{f_2}{2}(\Box_g\phi_2)^2 + \frac{f_3 C}{f_2}\Box_g\phi_2 + \frac{f_4 C^2}{2f_2^2} - \frac{f_0}{2}\phi_2^2 + V(\phi_2)\right).$$

Conclusion

We can propose the following algorithm to construct exact solvable k-essence cosmological models with a nonlocal scalar fields:

- **For given potential $V(\phi)$ find $H(t)$ and $\phi(t)$ as a particular solution for**

$$\mathcal{F}(\Box_g)\phi = V'(\phi), \qquad (47)$$

- **Calculate p and ϱ.**

- **Using the Einstein equations, calculate $\varrho_q(\Psi)$ and $p_q(\Psi)$.**

  The exact solution corresponds to $\Psi(t) = t$.