Cubic Interactions of Symmetric Higher-Spin Gauge Fields in AdS_d

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Higher-Spin interactions

Higher spins: $s > 2$

Higher-spin interactions

A.Bengtsson, I.Bengtsson, Brink (1983), Berends, Burgers, van Dam (1984)

$$S = S^2 + S^3 + \dots$$

$$S^3 = \sum_{p,q,r} (D^p\varphi)(D^q\varphi)(D^r\varphi)\rho^{p+q+r+\frac{1}{2}d-3}$$

s **derivatives in interactions.**

String: $\rho \sim \sqrt{\alpha'}$

HS Gauge Theories $(m = 0)$: Fradkin, M.V. (1987)

$$AdS_4: \ (X^0)^2 + (X^4)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 = \rho^2\,, \qquad \rho = \lambda^{-1}$$

The $\rho \to \infty$ **limit is ill-defined at the interaction level**

Cubic vertices in Minkowski space of any dimension Metsaev (2005)

$$s_1 + s_2 - s_3 \leq 2N \leq s_1 + s_2 + s_3$$

Manvelyan, Mkrtchyan, Ruhl; Sagnotti, Taronna; Fotopoulos, Tsulaya; (2010)

Plan

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Cartan gravity in the AdS_d covariant formalism

$w^{AB} = -w^{BA} = dx^{\underline{n}} w_{\underline{n}}^{AB}$ **connection of** AdS_d

$o(d-1,2)$**–curvature**

$$r^{AB} = dw^{AB} + w^{AC} \wedge w_C{}^B \,,$$

$$w = w^{AB} t_{AB} = \omega^{L\,ab} L_{ab} + \lambda e^a P_a$$

Provided that e^a **is nondegenerate**

$$r^{AB}(w) = 0$$

implies that $\omega^{L\,ab}$ **and** e^a **describe** AdS_d. λ^{-1} **is radius of** AdS_d.
Covariant definitions: compensator field $V^A(x)$ **normalized to** $V^A V_A = \lambda^{-2}$. **Lorentz algebra is the stability subalgebra of** V^A.
Convenient choice is the standard gauge $V^A = \lambda^{-1}\delta^A_d$. **It leads to standard formulae from covariant definitions**

$$E^A = DV^A \equiv dV^A + w^{AB} V_B \,, \qquad \omega^{L\,AB} = w^{AB} - \lambda^2 (E^A V^B - E^B V^A) \,.$$

Action

$$S = \frac{(-1)^{d+1}}{4\lambda^2\kappa^{d-2}} \int_{M^d} G_{A_1A_2A_3A_4} \wedge r^{A_1A_2} \wedge r^{A_3A_4}\,,$$

where

$$G^{A_1\dots A_q} = \epsilon^{A_0\dots A_d} V_{A_0} E_{A_{q+1}} \wedge \dots \wedge E_{A_d}$$

$$DG^{A_1\dots A_q} \simeq (-1)^q q\lambda^2 V^{[A_1} G^{A_2\dots A_q]}\,, \qquad t^A := DE^A \equiv r^{AB} V_B = 0$$

$$G^{A_1\dots A_q} \wedge E^C = \frac{q}{d+1-q} G^{[A_1\dots A_{q-1}}\Big(\eta^{A_q]C} - \lambda^2 V^C V^{A_q]}\Big)\,.$$

Generalized Einstein equations

$$\left(G_{A_1A_2A_3} - \frac{(d-4)}{4\lambda^2} G_{A_1A_2A_3A_4A_5} \wedge r^{A_4A_5}\right) \wedge r^{A_1A_2} = 0\,.$$

Symmetric higher-spin gauge fields

A spin $s \geq 2$ massless field can be described (2001) by a one-form $dx^{\underline{n}}\omega_{\underline{n}}{}^{A_1\dots A_{s-1},B_1\dots B_{s-1}}$ carrying the irreducible representation of $o(d-1,2)$ described by the traceless two-row rectangular Young diagram of length $s-1$

$$\omega^{(A_1\dots A_{s-1},A_s)B_2\dots B_{s-1}} = 0\,, \qquad \omega^{A_1\dots A_{s-3}C}{}_{C,}{}^{B_1\dots B_{s-1}} = 0\,.$$

Linearized HS curvature R_1 has simple form

$$\begin{aligned} R_1^{A_1\dots A_{s-1},B_1\dots B_{s-1}} &= D_0(\omega^{A_1\dots A_{s-1},B_1\dots B_{s-1}}) = d\omega_1^{A_1\dots A_{s-1},B_1\dots B_{s-1}} \\ &+(s-1)\Big(w_0^{(A_1}{}_C\wedge\omega_1^{CA_2\dots A_{s-1}),B_1\dots B_{s-1}} + w_0^{(B_1}{}_C\wedge\omega_1^{A_1\dots A_{s-1},CB_2\dots B_{s-1})}\Big), \end{aligned}$$

AdS_d background gauge field w_0^{AB} satisfies the zero curvature condition $D_0^2 = 0\,.$

Bianchi identities: $D_0 R_1 = 0$.

Lorentz covariant irreducible fields $dx^{\underline{n}}\omega_{\underline{n}}{}^{a_1\dots a_{s-1},b_1\dots b_t}$ identify with those components of $dx^{\underline{n}}\omega_{\underline{n}}{}^{A_1\dots A_{s-1},B_1\dots B_{s-1}}$, that are parallel to V^A in some $s-t-1$ indices and transversal in the rest t indices. The dynamical frame-like and auxiliary Lorentz-like fields are those with $t=0$ and $t=1$

$$e^{A_1\dots A_{s-1}} = \omega^{A_1\dots A_{s-1},B_1\dots B_{s-1}}V_{B_1}\dots V_{B_{s-1}}$$

$$\omega^{A_1\dots A_{s-1},C} = \Pi_V\omega^{A_1\dots A_{s-1},CB_2\dots B_{s-1}}V_{B_2}\dots V_{B_{s-1}}$$

The HS gauge fields with $t>1$ are called extra fields.

The free action that describes properly HS gauge fields is

$$\begin{aligned} S_2 \;=\; & \frac{1}{2}\int_{M^d}\sum_{p=0}^{s-2} a(s,p)V_{C_1}\dots V_{C_{2(s-2-p)}}G_{A_1A_2A_3A_4}\wedge \\ & R_1^{A_1B_1\dots B_{s-2},A_2C_1\dots C_{s-2-p}D_1\dots D_p}\wedge R_1^{A_3}{}_{B_1\dots B_{s-2},}{}^{A_4C_{s-1-p}\dots C_{2(s-2-p)}}{}_{D_1\dots D_p}\,, \end{aligned}$$

$$a(s,p) = b(s)\lambda^{-2(p+1)}\frac{(d-5+2(s-p-2))!!\,(s-p-1)}{(s-p-2)!}\,,$$

The coefficients are chosen so that the variation of the action over all extra fields is identically zero: at the linearized level, only the frame-like and Lorentz-like fields contribute to the action

First On-Shell Theorem

$$R_1^{A_1\dots A_{s-1},B_1\dots B_{s-1}} \sim E_{0\,A_s}\wedge E_{0\,B_s} C^{A_1\dots A_s,B_1\dots B_s}\,.$$

Generalized Weyl tensors $C^{A_1\dots A_s,B_1\dots B_s}$ parametrize those components of the curvatures that may remain nonzero when the field equations and constraints on extra fields are imposed. $C^{A_1\dots A_s,B_1\dots B_s}$ is described by a traceless V^A–transversal two-row rectangular Young diagram of length s

$$C^{(A_1\dots A_s,A_{s+1})B_2\dots B_s}=0\,,\; C^{A_1\dots A_{s-2}CD,B_1\dots B_s}\eta_{CD}=0\,,\; C^{A_1\dots A_{s-1}C,B_1\dots B_s}V_C=0$$

Consequences:

$$R_{1\,A\dots}V^A\sim 0\,,\qquad R_{1\,A\dots}\wedge E^A\sim 0\,,\qquad G^{[A_1\dots A_q}\wedge R_1^{A_{q+1}]}{}_{\dots}\sim 0\,.$$

Dual curvature $(d-2)$–form

$$R'_{A_1\dots A_{s-1}\,,B_1\dots B_{s-1}\,,}=G_{A_{s-1}B_{s-1}}{}^{FG}R_{FA_1\dots A_{s-2}\,,GB_1\dots B_{s-2}}\,.$$

Important property: any d-form $F(R',R)$ bilinear in R and R' is symmetric

$$F(R',R)\sim F(R\,,R')\,.$$

Cubic interactions

Current Ω^{Φ} defined via

$$\delta S^3 = \int_{M^d} \delta\omega_\Phi \wedge \Omega^{\Phi}(\omega)\,, \qquad \Omega^{\Phi}(\omega) = \frac{\delta S^3}{\delta\omega_\Phi}\,,$$

respects gauge invariance under $\delta\omega_\Phi = D_0\omega_\Phi$ if it obeys the conservation condition

$$D_0\Omega^{\Phi} \sim 0\,.$$

in the case with several gauge fields the integrability condition should be respected

$$\frac{\delta\Omega^{\Lambda}}{\delta\omega_\Phi} = (-1)^{p_\Phi p_\Lambda}\frac{\delta\Omega^{\Phi}}{\delta\omega_\Lambda}\,.$$

It is this condition that makes difficult to introduce Noether current interactions for a system of gauge fields of different types

Different types of vertices

Abelian vertices

$$S^3 = \int_{M^d} V^{\Phi_1\Phi_2\Phi_3} \wedge R_{1\,\Phi_1} \wedge R_{2\,\Phi_2} \wedge R_{3\,\Phi_3}\,, \qquad R_{1\,\Phi} = D_0\omega_\Phi\,,$$

$$S^3 = \int_{M^d} U C_{s_1} C_{s_2} C_{s_3}\,,$$

C: **generalized Weyl tensors which parametrize on-shell nontrivial components of the linearized curvatures** $R_{1\,\Phi}$

Current vertices

$$S^3 = \int_{M^d} \omega_\Phi \wedge \tilde{\Omega}^\Phi(C)\,, \qquad D_0\tilde{\Omega}^\Phi(C) \sim 0$$

for a p**–form** ω_Φ **and a** D_0**on shell–closed** $(d-p)$**–form** $\tilde{\Omega}^\Phi(C)$ $(d-p)$ **bilinear in the HS Weyl zero-forms**

Non-Abelian curvatures

are typical for the actions constructed from bilinears of some non-Abelian curvatures $R = R_1 + \omega^2$ since the cubic part of the lagrangian $L = \frac{1}{2}RR$ has the structure $R_1\omega^2$.

For the curvature

$$R^\alpha = d\omega^\alpha + f^\alpha_{\beta\gamma}\omega^\beta\omega^\gamma \,,$$

$f^\alpha_{\beta\gamma}$ contribute linearly to the action while the on-shell analysis involves only Abelian (free) gauge transformation law.

For the cubic order analysis it does not matter whether or not $f^\alpha_{\beta\gamma}$ satisfy the Jacobi identities.

What does matter is the symmetry property of the coefficients: the existence of such an metric $g_{\alpha\beta}$ that the structure coefficients $f_{\alpha\beta\gamma} =$ $g_{\alpha\rho}f^\rho_{\beta\gamma}$ are totally (graded) antisymmetric

$$f_{\alpha\beta\gamma} = -f_{\alpha\gamma\beta} = -f_{\beta\alpha\gamma} \,.$$

Chern-Simons vertices

ω^3–type vertices we call Chern-Simons vertices.

Except for the case of true Chern-Simons vertices where ω^3 is a d–form, all other Chern-Simons vertices in AdS_d are equivalent up to total derivatives to some curvature-dependent vertices

Being true in AdS, this property may not be true in Minkowski geometry where nontrivial Chern-Simons vertices can exist.

Minkowski versus AdS

The situation in AdS_d geometry is different because of presence of the dimensionfull parameter $\rho = \lambda^{-1}$.

Let V_{AdS} be a deformation of a Minkowski vertex V_M. It may happen that it admits a representation

$$V_{AdS} \sim \lambda^{-2}(dU_{AdS} + \tilde{V}_{AdS})$$

with U and $\tilde{V}_{AdS}$ containing, respectively, $N+1$ and $N+2$ derivatives of the dynamical fields. This implies that V_{AdS} is equivalent to $\tilde{V}_{AdS}$ for all $\lambda \neq 0$.

AdS_d vertices that contain different numbers of derivatives can belong to the same equivalence class

Vertex tri-complex

Consider a differential form

$$F(\omega, R_1|V,E) = G^{A_1\dots A_q} V^{C_1}\dots V^{C_p} F_{[A_1\dots A_q],C_1\dots C_p}(\omega, R_1) = G^{A_1\dots A_q} F_{[A_1\dots A_q]}(\omega, R$$

Direct computation gives $dF = QF$

$$Q = Q^{top} + \lambda^2 Q^{sub} + Q^{cur}\,,$$

$$Q^{top}F = (-1)^{d-q}\frac{q}{d+1-q} G^{A_1\dots A_{q-1}} \frac{\partial}{\partial V_{A_q}} F_{A_1\dots A_q}(\omega, R_1|V)\,,$$

$$Q^{sub}F = (-1)^{d}\frac{q}{d+1-q} G^{A_2\dots A_q}\wedge V^{A_1}\Big(d+1-q+V^E\frac{\partial}{\partial V^E}\Big) F_{A_1\dots A_q}(\omega, R_1|V)$$

$$Q^{cur}F = (-1)^{d-q} R_1^{\alpha}\frac{\partial}{\partial\omega^{\alpha}} F(\omega, R_1|V)\,.$$

$$(Q^{top})^2 = 0\,,\qquad (Q^{sub})^2 = 0\,,\qquad (Q^{cur})^2 = 0\,,$$

$$\{Q^{top}, Q^{sub}\} = 0\,,\qquad \{Q^{top}, Q^{cur}\} = 0\,,\qquad \{Q^{cur}, Q^{sub}\} = 0\,.$$

Q^{top}, Q^{sub} **and** Q^{cur} **form tri-complex.**

Vertex cohomology

Let $F(\omega\,,R_1)$ be a d–form. Consider the action

$$S=\int_{M^d}F(\omega\,,R_1)\,.$$

Using convention $\varepsilon^\alpha\frac{\partial}{\partial\omega^\alpha}(R_1)=0$ gauge variation is

$$\delta\int_{M^d}F(\omega\,,R_1)=\int_{M^d}\varepsilon^\alpha\frac{\partial}{\partial\omega^\alpha}(QF(\omega\,,R_1))\,,$$

Gauge invariance requires

$$QF(\omega\,,R_1)\sim G(R_1)\,.$$

Gauge invariant vertices: $H^d(Q)$.

Flat limit

$$Q = F^{fl} + \lambda^2 Q^{sub}\,, \qquad Q^{fl} = Q^{top} + Q^{cur}\,.$$

λ^2 iin front of Q^{sub} signals that $Q^{sub}F$ contains at least two less space-time derivatives than $Q^{top}F$ and $Q^{cur}F$. Since the term with Q^{sub} disappears in the flat limit, vertices in Minkowski space are controlled by Q^{fl}.

Gauge invariant vertices in Minkowski space: $H^d(Q^{fl})$.

Vertex F is pure iff

$$F \in H^d(Q^{fl})\,, \qquad Q^{sub}F = 0 \qquad \Rightarrow QF = 0$$

In terms of HS connections, pure vertices have the same form in Minkows and AdS_d.

Spin two

Two Q-closed vertices

$$B_1 = G^{A_1\dots A_5} V^C R_{A_1\,,A_2} R_{A_3\,,A_4} \omega_{A_5\,,C}$$

$$B_2 = G^{A_1\dots A_4} R_{A_1\,,A_2} \omega_{A_3}{}^{B} \omega_{A_4\,,B}\,.$$

B_1 **can be represented in the form**

$$B_1 = Q^{sub} U\,, \qquad U = -\frac{1}{2}(-1)^d G^{A_1\dots A_6} R_{A_1\,,A_2} R_{A_3\,,A_4} \omega_{A_5\,,A_6}\,.$$

$$Q^{top} U = 0\,, \qquad Q^{cur} U = -\frac{1}{2} G^{A_1\dots A_6} R_{A_1\,,A_2} R_{A_3\,,A_4} R_{A_5\,,A_6}\,.$$

B_1 **is** Q^{fl}**–closed. Since** B_1 **is** Q^{sub}**–exact, it is pure hence being gauge invariant both in Minkowski space and** AdS_d.

B_2 **is also pure. Both** B_1 **and** B_2 **contain up to four space-time derivatives of the vielbein: some combination of** B_1 **and** B_2 **should be** Q^{fl} **exact.**

$$(d-4)B_1 - 3B_2 \sim (-1)^d Q^{fl}(E_2 - (d-4)E_1))\,,$$

$$E_1 = G^{A_1\dots A_5} V^C R_{A_1,A_2}\omega_{A_3,A_4}\omega_{A_5,C}\,, \qquad E_2 = G^{A_1\dots A_4}\omega_{A_1}{}^B R_{A_2,A_3}\omega_{A_4,B}\,.$$

However, being equivalent in Minkowski space, B_1 and B_2 are not equivalent in AdS_d. Indeed,

$$(d-4)B_1 - 3B_2 \sim (-1)^d (Q - \lambda^2 Q^{sub})(E_2 - (d-4)E_1)\,,$$

AdS deformation of $(d-4)B_1 - 3B_2$, that was trivial in Minkowski case gives rise to the vertex

$$V_3 = \frac{1}{2}(-1)^d Q^{sub}(E_2 - (d-4)E_1)\,.$$

V_3 is Q-closed since B_1 and B_2 are. Being Q^{sub}–closed, V_3 is pure. Its explicit form is

$$V_3 = (d-3)G^{A_1\dots A_4}V^C V^C R_{A_1A_2}\omega_{A_3C}\omega_{A_4C} + G^{A_1A_2A_3}V^C\Big(\omega_C{}^D\omega_{A_1A_2}\omega_{A_3D} + \omega_{A_1}{}^D\omega$$

V_3 contains two derivatives, reproducing cubic part of the Einstein action with the cosmological term.

Spin two example illustrates the general phenomenon that AdS deformation of a higher-derivative vertex trivial in flat space may become nontrivial in AdS_d.

Spin three

Two pure vertices

$$
\begin{aligned}
F^3 \;&=\; G^{A_1A_2A_3}V^{C(3)}tr\Big(\omega_{A_1B,\,A_2E}\Big(-2\{\omega_{A_3}{}^{B}{}_{,CD},\,\omega^{ED}{}_{,C(2)}\} - \{\omega_{A_3D,\,C}{}^{B},\omega^{ED}{}_{,}
\\
&+\; \omega_{A_1B,\,FC}\Big(2\{\omega_{A_2}{}^{B}{}_{,CG},\omega_{A_3}{}^{F}{}_{,}{}^{G}{}_{C}\} + \frac{4}{3}\{\omega_{A_2G,}{}^{B}{}_{C},\omega_{A_3}{}^{F}{}_{,}{}^{G}{}_{C}\}\Big)\Big)\,.
\end{aligned}
$$

F^3 **contains three spin three fields that carry at most three derivatives.**

$$
\begin{aligned}
H^5 \;=&\; G^{A_1A_2A_3A_4}V^{C}V^{C}tr\Big(R_{A_1}{}^{G}{}_{,A_2}{}^{D}\Big(2\{\omega_{A_3C,\,A_4E},\omega_{CD,\,G}{}^{E}\} - 3\{\omega_{A_3DD,G}{}^{E},\omega_{A_4}
\\
&+6\{\omega_{A_3G,C}{}^{E},\omega_{A_4D,CE}\} + 8\{\omega_{A_3}{}^{E}{}_{,CG},\omega_{A_4D,CE}\} + \{\omega_{A_3}{}^{E}{}_{,CG},\omega_{A_4E,C}{}^{D}\}\Big)\Big)
\end{aligned}
$$

H^5 **contains up to five derivatives. Using the on shell condition it is straightforward, although somewhat lengthy, to check that**

$$
Q^{top}H^5 \sim 0\,, \qquad Q^{sub}H^5 \sim 0\,, \qquad Q^{cur}H^5 \sim 0\,.
$$

That H^5 **is not** Q^{top}**–exact is easy to see.**

F^3 **is of Chern-Simons type. Hence it is equivalent to a higher-derivative vertex.**

Higher-spin algebra

Generic element

$$A(\hat{Y}) = \sum_n A_{A_1\dots A_n\,,B_1\dots B_n} T^{A_1\dots A_n\,,B_1\dots B_n}\,,$$

where the coefficients are projected to two-row traceless Young diagrams

$$A^{\{A_1\dots A_n,A_{n+1}\}B_2\dots B_n} = 0\,, \qquad A^{A_1\dots A_{n-2}C}{}_{C,}{}^{B_1\dots B_n} = 0\,.$$

$$(A\circ B)^{A(n),B(n)} = \sum_{kl} f^{A(n),B(n)}_{C(k),D(k)\,;F(l),G(l)}(h)A^{C(k),D(k)}B^{F(l),G(l)}$$

HS algebra possesses the invariant trace operation

$$tr(A\circ B) = tr(B\circ A)\,, \qquad tr(A) = A_0\,, \qquad A_0 = A_{A(0),B(0)}\,.$$

$$R_{A(n),B(n)} = d\omega_{A(n),B(n)}(x) + (\omega(x)\circ\wedge\,\omega(x))_{A(n),B(n)}\,,$$

Cubic Action

$$S = \frac{1}{2}\int_{M^d}\sum_s\sum_{p=0}^{s-2} a(s,p) V_{C_1}\dots V_{C_{2(s-2-p)}}\wedge$$
$$tr\Big(R^{B_1\dots B_{s-1},}{}^{C_1\dots C_{s-2-p}D_1\dots D_{p+1}}\wedge R'_{B_1\dots B_{s-1},}{}^{A_4C_{s-1-p}\dots C_{2(s-2-p)}}{}_{D_1\dots D_{p+1}}\Big)\,.$$

tr is the trace over matrix indices in the case of HS algebras with non-Abelian Yang-Mills symmetries.

Choosing the coefficients in such a way that the on-shell (V^A-independent takes the form

$$S \sim \frac{1}{2}\int_{M^d} Tr(R\wedge\circ R')$$

implies its gauge invariance under the HS gauge transformations

$$\delta R = [R\,,\epsilon]_\circ$$

due to the cyclic property of trace and on-shell property

$$F(R',R)\sim F(R\,,R')\,.$$

$sp(2)$ invariance

In terms of generating functions

$$A(Y) = \sum_n A_{A_1 \dots A_n\,, B_1 \dots B_n} Y_1^{A_1} \dots Y_1^{A_n} Y_2^{B_1} \dots Y_2^{B_n}\,.$$

That $A_{A_1 \dots A_n\,, B_1 \dots B_n}$ obeys the properties of two-row traceless Young diagrams is encoded by the constraints

$$\tau_{ij} A(Y) = 0\,, \qquad \Delta^{ij} A(Y) = 0\,,$$

$$\tau_i{}^j = Y_i^A \frac{\partial}{\partial Y_j^A} - \frac{1}{2}\delta_i^j Y_k^A \frac{\partial}{\partial Y_k^A}\,, \qquad \Delta^{ij} = \frac{\partial^2}{\partial Y_i^A \partial Y_{Aj}}$$

$$i, j, \dots = 1, 2\,, \qquad a^i = \epsilon^{ij} a_j\,, \qquad a_i = a^j \epsilon_{ji}\,, \qquad \epsilon_{ij}\epsilon_{ij} = -\epsilon_{ji}\,, \qquad \epsilon_{12} = 1$$

$\tau_{ij} = \tau_{ji}$ **generate $sp(2)$ with the invariant symplectic form ϵ_{ij}.**

Non-zero vertices are represented by $sp_\mu(2)$ singlets for all μ

Vertex generating functions

It is convenient to replace $G^{A_1 \dots A_q}$ by a product of anticommuting variables $\psi^{A_1} \dots \psi^{A_q}$. All $o(d-1,2)$ invariant contractions are represented by operators

$$\Delta^{i^\mu j^\nu} = \Delta^{j^\nu i^\mu} = \frac{\partial^2}{\partial Y^A_{i^\mu} \partial Y_{A j^\nu}}\,, \qquad p_{i^\mu} = V^A \frac{\partial}{\partial Y^{A i^\mu}}\,, \qquad \sigma_{i^\mu} = \psi^A \frac{\partial}{\partial Y^{A i^\mu}}\,.$$

General vertex

$$F(A) = F(\Delta, p, \sigma) \prod_{\rho=1}^{N} A_\rho(Y_\rho)\Big|_{Y_\sigma=0}$$

That $A_\nu(Y_\nu)$ describes traceless tensors implies $\Delta^{i^\mu j^\mu} A_\mu(Y_\mu) = 0\,.$

$F(\Delta, p, \sigma)$ should be $sp_\mu(2)$ invariant for any μ.

Using labels $\bar\nu$ for $A(Y_\nu) = R_1(Y_\nu)$, the on-shell conditions are

$$p_{i^{\bar\nu}} F \sim 0\,, \qquad \frac{\partial}{\partial Y^{A_{i^\nu}}} \frac{\partial}{\partial \psi_A} F \sim 0$$

Non-Abelian and current vertices

In terms of generating functions pure non-Abelian vertex

$$L(V) = V_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta)R'_{\alpha^1}(Y_1)\omega_{\alpha^2}(Y_2)\omega_{\alpha^3}(Y_3)\Big|_{Y_i=0}, \qquad V_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta) = -V_{132}^{\alpha^1\alpha^3\alpha^2}($$

$R'(Y)$ is the dual curvature $(d-2)-$form.

$L(V)$ is Q^{top} and Q^{sub}-closed because it does not contain the compensator

$$Q^{cur}L = (-1)^d\Big(V_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta) - V_{231}^{\alpha^2\alpha^3\alpha^1}(\Delta)\Big)R'_{\alpha^1}(Y_1)R_{\alpha^2}(Y_2)\omega_{\alpha^3}(Y_3)\Big|_{Y_i=0},$$

$Q^{cur}L(V) = 0$ with totally antisymmetric $V_{123}^{\alpha^1\alpha^2\alpha^3}$ in which case $L(V)$ is

pure: $Q^{fl}L(V) = 0$, $Q^{sub}L(V) = 0$

Pure current vertices

$$F(\tilde{U}) = -Q^{sub}T(\tilde{U}) = \sigma_{i^1}\sigma^{i^i}\sigma_{j^2}\sigma^{j^2}\sigma_{k^3}p^{k^3}\tilde{U}_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta)R_{\alpha^1}(Y_1)R_{\alpha^2}(Y_2)\omega_{\alpha^3}(Y_3)\Big|_{Y_i=($$

Derivative reduction

All non-Abelian vertices $L(V)$ and current vertices $F(\tilde{U})$ are pure vertices with $s_1+s_2+s_3-2$ derivatives. Since there exists just one nontrivial Minkowski vertex of this order of derivatives, most of vertices $L(V)$ and $F(\tilde{U})$ should be quasi exact.

Consider

$$I=-w_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta)\sigma_{i^1}\sigma^{i^1}\sigma_{j^2}\sigma^{j^2}\sigma_{k^3}p^{k^3}R_{\alpha^1}(Y_1)R_{\alpha^2}(Y_2)\omega_{\alpha^3}(Y_3)\Big)\Big|_{Y=0},$$

$$J=\frac{3}{d-4}w_{(123)}^{(\alpha^1\alpha^2\alpha^3)}(\Delta)\sigma_{i^1}\sigma^{i^1}\sigma_{j^2}\sigma_{k^3}\Delta^{j^2k^3}R_{\alpha^1}(Y_1)\omega_{\alpha^2}(Y_2)\omega_{\alpha^3}(Y_3)$$

with arbitrary $w_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta)$ and $w_{(123)}^{(\alpha^1\alpha^2\alpha^3)}(\Delta)=\frac{1}{6}\sum_{\mu\neq\nu\neq\rho}w_{\mu\nu\rho}^{\alpha^\mu\alpha^\nu\alpha^\rho}(\Delta)$.

One can see that

$$I+J=(-1)^dQ^{fl}(H+W),$$

$$H=w_{(123)}^{(\alpha^1\alpha^2\alpha^3)}(\Delta)(\Delta)\sigma_{i^1}\sigma^{i^1}\sigma_{j^2}\sigma_{k^3}\Delta^{j^2k^3}\omega_{\alpha^1}(Y_1)\omega_{\alpha^2}(Y_2)\omega_{\alpha^3}(Y_3)\Big|_{Y=0}$$

$$W=w_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta)\sigma_{1i}\sigma_1^i\sigma_{2j}\sigma_2^j\sigma_{k^3}p^{k^3}R_{\alpha^1}(Y_1)\omega_{\alpha^2}(Y_2)\omega_{\alpha^3}(Y_3)\Big|_{Y=0}$$

This implies that the sum of the current vertex I and non-Abelian vertex J is quasi exact. (cf the case of spin two).

To perform further reduction more tricky consequences of the on-shell conditions should be used

$$Q^{top} N \sim \frac{(-1)^d}{d-5} \sigma_{k^1} \sigma^{k^1} \sigma_{j^2} \sigma^{j^2} \sigma_{l^3} p^{l^3} (2\Phi_{12}\Phi_{13}\Phi_{23} + \Phi^2) N_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta) R_{\alpha^1}(Y_1) R_{\alpha^2}(Y_2$$

$$N = \Phi_{12} N_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta) \sigma_{i^1}\sigma^{i^1} \sigma_{j^2}\sigma^{j^2} \sigma_{k^3}\sigma^{k^3} p_{n^3} p_{m^3} \Delta^{n^3 u^1} \Delta_{u^1 v^2} \Delta^{v^2 m^3} R_{\alpha^1}(Y_1) R_{\alpha^2}(Y_2$$

$$\Phi_{\mu\nu} = \Delta_{i^\mu}{}^{j^\nu} \Delta_{j^\nu}{}^{i^\mu} \qquad \mu \neq \nu\,, \qquad \Phi \equiv \Phi_{123} = \Delta_{i^1}{}^{j^2} \Delta_{j^2}{}^{k^3} \Delta_{k^3}{}^{i^1}\,.$$

As a result any vertex I with

$$w_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta) = \left(2\Phi_{12}\Phi_{13}\Phi_{23} + \Phi^2\right) v_{123}^{\alpha^1\alpha^2\alpha^3}(\Delta)$$

is quasi-exact and, hence, is equivalent some lower derivative vertex. Similarly one proceeds with further reductions of derivatives.

This mechanism works for $d \geq 6$. In $d = 4$ quasi exact current vertices do not exist. Hence $4d$ triangle vertices result from non-Abelian vertices. Indeed, the list of independent $4d$ vertices is shorter than for any d.

Comparison with flat space results

Minkowski vertices Metsaev (2005); Manvelyan, Mkrtchyan, Ruhl (2010)

$$s_1 + s_2 - s_3 \le 2N \le s_1 + s_2 + s_3$$

The vertices obtained in AdS_d require $s_i - 1$ satisfy triangle inequalities

$$s_i + s_j - s_k - 1 \ge 0\,, \qquad i \ne j \ne k\,.$$

Vertices, that can be constructed in terms of connection one-forms and curvature two-forms, should respect the triangle inequalities otherwise otherwise contraction of indices between two-row Young diagrams gives zero. The reason why some of vertices were missed is that we did not consider vertices that contain Weyl 0-forms directly.

Particular examples:

The vertex with maximal number of derivatives $s_1 + s_2 + s_3$ is not on the list since all vertices considered considered so far contain at most $s_1 + s_2 + s_3 - 2$ derivatives since all R^3 vertices are quasi exact.

interactions of a spin-s gauge field with two spin zero scalar fields. Scalar is described by the zero-form $C(x)$ and its derivatives = elements of the Weyl module for the spin zero field. In this case of current interactions between a spin-s gauge field and HS currents built from (derivatives) of the scalar field the triangle inequalities are not respected.

To incorporate vertices of general type into the scheme it is necessary extend First-On-Shell Theorem to the Central on-shell theorem that contains the equations on zero-forms in the Weyl module $C(x)$

$$\tilde{D}C = 0\,.$$

Towards full nonlinear action

Extension to the full system of fields that enter the free unfolded formulation of massless HS fields, including Weyl zero-forms, reduces analysis of cubic HS interactions to the analysis of vertices that are on-shell-closed by virtue of unfolded field equations. The idea is to look for a nonlinear action

$$S = \int L \,,$$

where d-form L is on-shell closed

$$dL \sim 0$$

by virtue of the nonlinear unfolded equations and such that the quadratic part of the action coincides with the standard free action of massless fields.

Lagrangians of this type will describe HS dynamics modulo local field redefinitions.

Conclusions

Vertices with different numbers of derivatives are related in AdS

Vertex tri-complex classifies nontrivial vertices in AdS_d

AdS–tri-complex contains a Minkowski sub-bicomplex

Vertex tri-complex applies to mixed symmetry type of general type and higher-order vertices

HS vertices for symmetric fields are uniformally formulated as non-Abelian and current interactions that carry $s_1 + s_2 + s_3 - 2$ derivatives

Missed vertices do not respect triangle inequalities for spins, requiring explicit appearance of the Weyl tensor and its derivatives

Full nonlinear Lagrangian as a on-shell closed form $dL \sim 0$ by virtue of full nonlinear unfolded equations

The novel feature of our proposal is that the analysis of the nonlinear action is (and, in fact, should be) on-shell beyond its free field part. This suggestion changes the strategy of the action construction: instead of looking for an action, that gives rise to the unfolded equations, having unfolded equations one should find a lagrangian that is on-shell closed by virtue of unfolded equations.

Such an approach fits very well the analysis of HS theory performed by Giombi and X.Yin 2009, in the context of AdS/CFT interpretation of HS theory, which is solely based on the unfolded dynamics approach.

Remarkably, this construction gets very similar to the effective action construction of QFT.