String theory dualities and supergravity divergences

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based on work done with
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Guillaume Bossard, Paul Howe, Kelly Stelle

Motivations

$\mathcal{N} = 8$ maximal supergravity arises as the low-energy limit of type II string.

Analyzing its UV behaviour in various dimensions teaches about how supersymmetry acts, the role of the duality symmetries (in string and field theory) and the relation between string theory and its low-energy limit

In this talk we will discuss

- the role of supersymmetry in perturbative computation
- the role of non-perturbative duality symmetries in string theory

Constraints from supersymmetry

Supersymmetry implies various non-renormalisation theorems for higher dimension operators.

This constrains the candidate UV counter-terms.
But which fraction of supersymmetry is really needed?

Since we don't have an off-shell superspace approach this question is difficult to address

Constraints from supersymmetry: $\mathcal{N} = 4$ MSYM

- The case of $\mathcal{N} = 4$ super-Yang-Mills

$$\mathcal{S} = \frac{1}{g_{\rm YM}^2} \int d^4x \mathrm{tr}(F^2) + \cdots$$

- Coupling constant dimensionless in $D = 4$

Half of supersymmetries are enough for finiteness of $\mathcal{N} = 4$ SYM in $D = 4$
[Mandelstam; Howe, Stelle, West; Brink, Lindgren, Nilsson]

- Four points amplitude behave as

$$\mathfrak{A}_{4;L}^{(D)} \sim \Lambda^{(D-4)L-4}\, t_8 F^4$$

But this is not enough for understanding the correct critical ultraviolet behaviour in the single trace sector, and double trace sector of the 4-point amplitudes in dimensions $4 < D \leqslant 10$
[Berkovits, Green, Russo, Vanhove], [Bern et al.]

$\mathcal{N}=4$ SYM UV divergences [Berkovits, Green, Russo, Vanhove]

$$[\mathfrak{A}_{4;L}^{(D)}] = \Lambda^{(D-4)L-4-2\gamma_L}\,\partial^{2\gamma_L}\, t_8 \mathrm{tr} F^4 + \Lambda^{(D-4)L-4-2\beta_L}\,\partial^{2\beta_L}\, t_8(\mathrm{tr}F^2)^2$$

	L=1	L=2	L=3	L=4	L=5
$\partial^{2\gamma_L}\, t_8 \mathrm{tr}(F^4)$	$D_c = 8$ $\gamma_1 = 0$	$D_c = 7$ $\gamma_2 = 1$	$D_c = 6$ $\gamma_3 = 1$	$D_c = \frac{11}{2}$ $\gamma_4 = 1$	$D_c = \frac{26}{5}$ $\gamma_5 = 1$
$\partial^{2\beta_L}\, t_8(\mathrm{tr}F^2)^2$	$D_c = 8$ $\beta_1 = 0$	$D_c = 7$ $\beta_2 = 1$	$D_c = \frac{20}{3}$ $\beta_3 = 2$	$D_c = 6$ $\beta_4 = 2$	$D_c = \frac{28}{5}$ $\beta_5 = 2$

- Some F-term are in $D < 10$

$$\partial^2 t_8 \mathrm{tr}(F^4) \sim \int d^8\theta\, \mathrm{Tr}(W_\alpha^4)$$

$$\partial^4 t_8(\mathrm{tr}(F^2))^2 \sim \int d^{12}\theta\, (\mathrm{Tr}(W_\alpha^2))^2$$

- Gaugino superfield $D_{(\beta} W_{\alpha)} = (\gamma^{mn})_{\alpha\beta} F_{mn}$

$\mathcal{N} = 4$ SYM UV divergences [Berkovits, Green, Russo, Vanhove]

$$[\mathfrak{A}^{(D)}_{4;L}] = \Lambda^{(D-4)L-4-2\gamma_L}\,\partial^{2\gamma_L}\,t_8\mathrm{tr}F^4 + \Lambda^{(D-4)L-4-2\beta_L}\,\partial^{2\beta_L}\,t_8(\mathrm{tr}F^2)^2$$

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- Some F-term are descendant of the Konishi operator $\mathrm{tr}(\Phi\cdot\Phi)$ in $D < 10$

$$\partial^2 t_8\mathrm{tr}(F^4) \sim \int d^{16}\theta\,\mathrm{tr}(\Phi\cdot\Phi)$$

$$\partial^4 t_8(\mathrm{tr}(F^2))^2 \sim \int d^{16}\theta\,(\mathrm{tr}(\Phi\cdot\Phi))^2$$

- These operators are not protected from quantum corrections

$\mathcal{N}=4$ SYM UV divergences [Berkovits, Green, Russo, Vanhove]

$$[\mathfrak{A}^{(D)}_{4;L}] = \Lambda^{(D-4)L-4-2\gamma_L}\,\partial^{2\gamma_L}\, t_8 \mathrm{tr} F^4 + \Lambda^{(D-4)L-4-2\beta_L}\,\partial^{2\beta_L}\, t_8 (\mathrm{tr} F^2)^2$$

	L=1	L=2	L=3	L=4	L=5
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For $L \geqslant 4$ the UV divergence is dominated by the single trace term

single trace	$\Lambda^{(D-4)L-6}\,\partial^2 t_8 \mathrm{tr}(F^4)$	$L \geqslant 2$
double trace	$\Lambda^{(D-4)L-8}\,\partial^4 t_8 (\mathrm{tr} F^2)^2$	$L \geqslant 3$

- $\mathcal{N}=3$ superspace explains the leading UV behaviour [Howe, Stelle]
- Confirmed by amplitude computation [Bern, Dixon, Carrasco, Johansson,Roiban]

The ultraviolet behaviour of $\mathcal{N} = 8$ supergravity

- Up to and including 4-loop order the critical UV behaviour is the same in for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA

$$[\mathfrak{M}_{4;L}^{(D)}] \sim \Lambda^{(D-4)L-6}\, \partial^{2L}\, \mathcal{R}^4 \qquad 2 \leqslant L \leqslant 4$$

[Bern et al.], [Green, Russo, Vanhove]

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[Bern et al.], [Green, Russo, Vanhove]

- After 4-loop it is expected a worse UV behaviour than for $\mathcal{N} = 4$ SYM

[Green, Russo, Vanhove], [Vanhove], [Green, Bjornsson]

$$[\mathfrak{M}_{4;L}^{(D)}] \sim \Lambda^{(D-2)L-14}\, \partial^{8}\, \mathcal{R}^4 \qquad L \geqslant 4$$

- At five-loop order the 4-point amplitude in
  - $\mathcal{N} = 4$ SYM divergences for $5 < 26/5 \leqslant D$
  - $\mathcal{N} = 8$ SUGRA divergences for $24/5 \leqslant D$

Would imply a *seven-loop* divergence in $D = 4$ with counter-term $\partial^8 \mathcal{R}^4$

$\mathcal{N} = 8$ SUGRA critical ultraviolet behaviour

Can we decide about the critical UV behaviour of the $\mathcal{N} = 8$ supergravity without doing a high loop computation?

The ultraviolet divergences in dimension D in the 4-graviton amplitudes are invariant under the duality symmetry groups of the theory.

In $3 \leqslant D \leqslant 11$ the vacuum of $\mathcal{N} = 8$ supergravity is invariant under Supersymmetry and the continuous duality symmetries $E_{11-D(11-D)}(\mathbb{R})$

The duality group is broken to a discrete subgroup $E_{11-D(11-D)}(\mathbb{Z})$ by non-perturbative effects and the massive string modes.

Duality symmetries

[Hull,Townsend]

D	$E_{11-D(11-D)}(\mathbb{R})$	K_D	$E_{11-D(11-D)}(\mathbb{Z})$
10A	$\mathbb{R}^+$	1	1
10B	$Sl(2,\mathbb{R})$	$SO(2)$	$Sl(2,\mathbb{Z})$
9	$Sl(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$Sl(2,\mathbb{Z})$
8	$Sl(3,\mathbb{R}) \times Sl(2,\mathbb{R})$	$SO(3) \times SO(2)$	$Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$
7	$Sl(5,\mathbb{R})$	$SO(5)$	$Sl(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{R})$	$USp(8)$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{R})$	$SO(16)$	$E_{8(8)}(\mathbb{Z})$

- $E_{11-D(11-D)}$ real split forms, K_D maximal compact subgroup.
- The vacuum of the theory is the scalar manifold $\vec{\varphi} \in \mathcal{M}_D = G_D/K_D$

Constraint from Supersymmetry

- Higher derivative 1/2, 1/4 and 1/8-BPS couplings corrections to the effective action

$$\mathcal{S} = \frac{1}{\ell_D^{D-2}} \int d^D x \sqrt{-g} \left(\mathcal{R} + \ell_D^6 \mathcal{E}^{(D)}_{(0,0)} \mathcal{R}^4 + \ell_D^{10} \mathcal{E}^{(D)}_{(1,0)} \nabla^4 \mathcal{R}^4 + \ell_D^{12} \mathcal{E}^{(D)}_{(0,1)} \nabla^6 \mathcal{R}^4 + \cdots \right)$$

- Invariance under the duality symmetries

$$\mathcal{E}^{(D)}_{(p,q)}(\gamma \cdot \vec{\varphi}) = \mathcal{E}^{(D)}_{(p,q)}(\vec{\varphi})$$

- If no anomalies $\gamma \in G_D(\mathbb{R})$ in perturbative $\mathcal{N} = 8$ supergravity: eg $D = 4$ E_7 not anomalous [Marcus; Bossard, Hillmann, Nicolai]

- Broken to $\gamma \in G_D(\mathbb{Z})$ by non-perturbative effects and massive string modes

Constraint from Supersymmetry

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$$\mathcal{S}=\frac{1}{\ell_D^{D-2}}\int d^Dx\,\sqrt{-g}\left(\mathcal{R}+\ell_D^6\,\mathcal{E}^{(D)}_{(0,0)}\,\mathcal{R}^4+\ell_D^{10}\,\mathcal{E}^{(D)}_{(1,0)}\,\nabla^4\mathcal{R}^4+\ell_D^{12}\,\mathcal{E}^{(D)}_{(0,1)}\,\nabla^6\mathcal{R}^4+\cdots\right)$$

- On-shell supersymmetry invariance

$$\delta_\epsilon\mathcal{S}=0;\qquad \delta_\epsilon=\delta^0_\epsilon+\ell_D^6\,\delta^3_\epsilon+\cdots$$

- Leads to a set of Noether deformations at increasing derivative order

$$\delta^0_\epsilon\mathcal{S}^n+\sum_{r_1+r_2=n}\delta^{r_1}_\epsilon\mathcal{S}^{r_2}=\delta^n_\epsilon\mathcal{S}^0$$

Constraint from Supersymmetry

In $D = 10$ dimensions the theory is invariant under $SL(2,\mathbb{R})/SO(2)$

On-shell maximal supersymmetry in $D = 10$ dimensions implies the differential equations [Green, Sethi] [Sinha] [Green, Vanhove]

$$
\begin{aligned}
\delta^0_\epsilon \mathcal{S}^3 \simeq 0 \qquad & (\Delta^{(10)} - \frac{3}{4})\,\mathcal{E}^{(10)}_{(0,0)} &=& \quad 0;\\
\delta^0_\epsilon \mathcal{S}^5 \simeq 0 \qquad & (\Delta^{(10)} - \frac{5}{4})\,\mathcal{E}^{(10)}_{(1,0)} &=& \quad 0;\\
\delta^0_\epsilon \mathcal{S}^6 + \delta^3_\epsilon \mathcal{S}^3 \simeq 0 \qquad & (\Delta^{(10)} - 12)\,\mathcal{E}^{(10)}_{(0,1)} &=& \quad -(\mathcal{E}^{(10)}_{(0,0)})^2
\end{aligned}
$$

$\Delta^{(10)} = \tau_2^2\,(\partial^2_{\tau_1} + \partial^2_{\tau_2})$ is the invariant Laplacian for $SL(2,\mathbb{R})/SO(2)$

Dimensional reduction

- Let's $\Delta^{(D)}$ be the Laplace operator on the scalar manifold $\mathcal{M}_D = E_{11-D(11-D)}/K_{11-D}$

$$\Delta^{(D)} \to \Delta^{(D+1)} + \frac{D-2}{2(D-1)}(\partial_{\log r^2})^2 + \frac{D^2-3D-58}{2(D-1)}\partial_{\log r^2}$$

- r is the radius of compactification between dimension $D+1$ and D
- Eigenvalues $(\Delta^{(D)} - \lambda^{(D)}_{(p,q)})\mathcal{E}^{(D)}_{(p,q)} = S_{(p,q)}$ are related

$$\lambda^{(D)}_{(p,q)} - \lambda^{(D+1)}_{(p,q)} = \frac{2p+3(q+1)}{(D-1)(D-2)}(D^2-3D-53+4p+6q)$$

Differential equations and critical dimensions

We deduce that for $D \geqslant 3$

$$\begin{aligned}
\left(\Delta^{(D)} - \frac{3(11-D)(D-8)}{D-2}\right) \mathcal{E}^{(D)}_{(0,0)} &= 6\pi\delta_{D-8,0} \\
\left(\Delta^{(D)} - \frac{5(12-D)(D-7)}{D-2}\right) \mathcal{E}^{(D)}_{(1,0)} &= 80\zeta(2)\delta_{D-7,0} \\
\left(\Delta^{(D)} - \frac{6(14-D)(D-6)}{D-2}\right) \mathcal{E}^{(D)}_{(0,1)} &= -(\mathcal{E}^{(D)}_{(0,0)})^2 + \zeta(3)\delta_{D-6,0}
\end{aligned}$$

- The eigenvalues vanish where $D^{2L}\mathcal{R}^4$ appears as UV counter-term [Green, Russo, Vanhove]

$$D_c = \begin{cases} 8 & \text{for } L = 1 \\ 4 + 6/L & \text{for } 2 \leqslant L \leqslant 4 \end{cases}$$

- The eigenvalues are defined for $D \geqslant 3$. For $D = 2$ string theory becomes strongly coupled and the group becomes an infinite Kac-Moody algebra

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\end{aligned}$$

Field derivation of the eigenvalue equations

- dualities in field theory and superspace methods in higher dimensions [Bossard, Stelle, Howe]
- Soft-scalar limit analysis in $D = 4$ [Elvang, Kiermaier; Beisert et al.]

Perturbative contributions

- The $\mathcal{N}=8$ supergravity limit is

$$\ell_s = \ell_D\, g_D^{-\frac{1}{D-2}} \to 0 \text{ with } \ell_D \text{ fixed } g_D \to \infty$$

The perturbative expansion for the $\partial^{2k}\mathcal{R}^4$ interactions with $k = 0, 2, 3$ have

$$\mathcal{E}^{(D)}_{(0,0)}\Big|_{\text{pert}} = g_D^{-2\frac{8-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^2} + I_{1-\text{loop}} \right)$$

$$\mathcal{E}^{(D)}_{(1,0)}\Big|_{\text{pert}} = g_D^{-4\frac{7-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^4} + \frac{1}{g_D^2} I_{1-\text{loop}} + I_{2-\text{loop}} \right)$$

$$\mathcal{E}^{(D)}_{(0,1)}\Big|_{\text{pert}} = g_D^{-6\frac{6-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^6} + \frac{1}{g_D^4} I_{1-\text{loop}} + \frac{1}{g_D^2} I_{2-\text{loop}} + I_{3-\text{loop}} + O(e^{-\frac{1}{g_D}}) \right)$$

- In $D = 4$ these couplings have moduli dependence on g_4, they therefore violate E_7 invariance. No allowed counter-terms.

See [Elvang et al.; Beisert et al.; Bossard et al.] for field theory arguments

Extracting the UV divergence [Green, Russo, Vanhove]

- The $\mathcal{N}=8$ supergravity limit is

$$\ell_s = \ell_D\, g_D^{-\frac{1}{D-2}} \to 0 \text{ with } \ell_D \text{ fixed } g_D \to \infty$$

In the critical dimension the L-loop counter-term

$$(D=8, L=1)\, \mathcal{R}^4: \quad \mathcal{E}^{(8)}_{(0,0)}\Big|_{\text{pert}} \sim \frac{2\pi}{3} \log g_8^2 + o(\log g_8^2)$$

$$(D=7, L=2)\, \partial^4\mathcal{R}^4: \quad \mathcal{E}^{(7)}_{(1,0)}\Big|_{\text{pert}} \sim \frac{8\pi^2}{15} \log g_7^2 + o(\log g_7^2)$$

$$(D=6, L=3)\, \partial^6\mathcal{R}^4: \quad \mathcal{E}^{(6)}_{(0,1)}\Big|_{\text{pert}} \sim 15\zeta(3) \log g_6^2 + o(\log g_6^2)$$

The coefficient are matching the coefficients of the $1/\epsilon$ pole the field theory UV divergences evaluated by [Bern, Dixon, et al.] for the $L=1$ loop in $D=8$, $L=2$ in $D=7$ and $L=3$ in $D=6$ (up to a factor of 6)

What about non-protected operators?

- So far we have discussed protected $1/2$, $1/4$ and $1/8$-BPS operators
- What about non-protected operators?

For $\mathcal{N} = 8$ supergravity that with 32 supercharges dimension analysis indicates that the dimensions 16 operator $\nabla^8\mathcal{R}^4$ could be an D-term given by the volume of superspace

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Доверяй, но проверяй!

Linearized $D = 4$ $\mathcal{N} = 8$ supergravity

- At the linearized level one can construct the invariants

$$\int d^4x \int d^{8+2L}\theta d^{8+2L}\bar{\theta}\, (W\bar{W})^2 \sim \partial^{2L}\,\mathcal{R}^4, \qquad L = 0, 2, 3, 4$$

where W_{ijkl} and $\bar{W}^{ijkl} = \frac{1}{24}\,\epsilon^{ijklmnpq}W_{mnpq}$ are the **70** scalar fields parametrizing the coset space $E_7/(SU(8)/\mathbb{Z}_2)$

- The structure of these F-terms is determined by the $SU(2,2|8)$ superconformal representations [Petkova, Dobrev], [Drummond, Heslop, Howe, Kerstan]
- Classification can be obtained as well from soft limit properties of scattering amplitudes [Elvang et al.; Beisert et al.]
- expression $SU(8)$ invariant but not E_7 invariant

Harmonic superspace

Harmonic superspace is an extension of the usual superspace $\mathbb{R}^{4|4\mathcal{N}}$ with the addition of extra bosonic coordinates in the flag manifold [Rosly; Galperin, Ivanov, Ogievetsky, Sokatchev]

$$\mathbb{F}_{q,p} = S(U(p) \times U(\mathcal{N}-p-q) \times U(q))\backslash SU(\mathcal{N})$$

- For $\mathcal{N}=8$ at the linearized level we can consider the 1/2, 1/4, and 1/8 BPS measure [Drummond, Heslop, Howe, Kerstan]

$$\int d^4x d\tilde{\mu}_{(8,4-L,4-L)} \, (W\bar{W})^4 \sim \int d^4x \, \partial^{2L} \, \mathcal{R}^4 \qquad L = 0,2,3,4$$

$$d\tilde{\mu}_{(8,p,p)} = d^{16-2p}\theta d^{16-2p}\bar{\theta} \, du$$

Harmonic superspace

Only the harmonic measures with $d\mu_{(\mathcal{N},1,1)}$ can be extended to full superspace

[Bossard, Howe, Stelle, Vanhove]

- We can make special the coordinates $\zeta^\alpha := \theta_1^\alpha$ and $\bar\zeta^{\dot\alpha} := \bar\theta^{\mathcal{N}\,\dot\alpha}$ because of the obstruction from the dimension 1/2 torsion $T_{\alpha\beta}^{ij\dot\gamma\,k} = \epsilon_{\alpha\beta}\bar\chi^{\dot\gamma\,ijk}$

$$\begin{aligned} \hat E_{\hat A} &:= \{\tilde E^1_\alpha, \tilde E_{\dot\alpha\,\mathcal{N}}, d^1{}_r, d^r{}_{\mathcal{N}}, d^1{}_{\mathcal{N}}\}\,, \qquad 2\leqslant r\leqslant \mathcal{N}-1 \\ \{\hat E_{\hat A}, \hat E_{\hat B}\} &= C_{\hat A\hat B}{}^{\hat C}\,\hat E_{\hat C}\,, \end{aligned}$$

preserved by the structure group $SL(2,\mathbb{C})\times U(1)\times U(\mathcal{N}-2)\times U(1)$

- Normal coordinates

$$\zeta^{\hat A} := \{\zeta^\alpha, \bar\zeta^{\dot\alpha}, z^r{}_1, z^{\mathcal{N}}{}_r, z^{\mathcal{N}}{}_1\},$$

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- G-analytic field $D_{\alpha\,1}B_{\alpha\dot\beta} = \bar D_{\dot\alpha\,\mathcal{N}}B_{\alpha\dot\beta} = 0$

$$B_{\alpha\dot\beta} = \begin{cases} \bar\chi^{1ij}_{\dot\beta}\chi_{\alpha\,\mathcal{N}ij} & \text{for } \mathcal{N} = 4,5,8 \\ \bar\chi^{1ij}_{\dot\beta}\chi_{\alpha\,6ij} + \frac{1}{3}\chi^{1ijkl}_\alpha\bar\chi_{\dot\beta\,6ijkkl} & \text{for } \mathcal{N} = 6\,, \end{cases}$$

- Extending to $\mathcal{N} \geqslant 4$ the flot equation in [Kuzenko et al.] one shows that

$$\zeta^{\hat\alpha}\partial_{\hat\alpha}\ln E = -\frac{1}{3}B_{\alpha\dot\beta}\zeta^\alpha\bar\zeta^{\dot\beta} + \frac{1}{18}B_{\alpha\dot\beta}B_{\alpha\dot\alpha}\zeta^\alpha\zeta^\beta\bar\zeta^{\dot\alpha}\bar\zeta^{\dot\beta}\,.$$

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- G-analytic field $D_{\alpha\,1}B_{\alpha\dot\beta} = \bar D_{\dot\alpha\,\mathcal{N}}B_{\alpha\dot\beta} = 0$

$$B_{\alpha\dot\beta} = \begin{cases} \bar\chi_{\dot\beta}^{1ij}\chi_{\alpha\,\mathcal{N}ij} & \text{for } \mathcal{N} = 4,5,8 \\ \bar\chi_{\dot\beta}^{1ij}\chi_{\alpha\,6ij} + \frac{1}{3}\chi_\alpha^{1ijkl}\bar\chi_{\dot\beta\,6ijkkl} & \text{for } \mathcal{N} = 6\,, \end{cases}$$

- The supervielbein takes the very simple form without a quadratic term in $\zeta^2\bar\zeta^2$

$$E(\hat x,\zeta,\bar\zeta) = E|_{\zeta=0}\left(1 - \frac{1}{6}B_{\alpha\dot\beta}\zeta^\alpha\zeta^{\dot\beta} + 0\zeta^2\bar\zeta^2\right)\,,$$

Harmonic superspace

Only the harmonic measures with $d\mu_{(\mathcal{N},1,1)}$ can be extended to full superspace

[Bossard, Howe, Stelle, Vanhove]

- One then defines the $1/\mathcal{N}$ harmonic measure (over $4(\mathcal{N}-1)$ θs)

$$\int d^4x d^{4\mathcal{N}}\theta E(x,\theta)\,\Phi(x,\theta) =: \int d^4x\, d\mu_{(\mathcal{N},1,1)}\,(D^1)^2(\bar{D}_{\mathcal{N}})^2\Phi|_{\zeta=0}$$

- For $\mathcal{N}=8$ the 1/8 BPS coupling $\nabla^6\mathcal{R}^4$

$$\int d^4x d\mu_{(8,1,1)}\,E(x,\theta,u)\,F(\mathcal{V}) = \int d^4x\, e\,(f_{(0,1)}(\varphi)\,\nabla^6\mathcal{R}^4 + \text{susy completion})$$

- Fully supersymmetric, $SU(8)$ invariant but not E_7 invariant expression $\mathcal{V}\in E_7/(SU(8)/\mathbb{Z}_2)$

Harmonic superspace

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[Bossard, Howe, Stelle, Vanhove]

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$$\int d^4x d^{4\mathcal{N}}\theta E(x,\theta)\,\Phi(x,\theta) =: \int d^4x\, d\mu_{(\mathcal{N},1,1)}\,(D^1)^2(\bar{D}_{\mathcal{N}})^2\Phi|_{\zeta=0}$$

- With $\Phi = 1$ one shows that the duality invariant volume is vanishing

$$\int d^4x d^{4\mathcal{N}}\theta E(x,\theta) = 0, \qquad 4 \leqslant \mathcal{N} \leqslant 8$$

- The $\nabla^8\mathcal{R}^4$ term is explicitly E_7-invariant (dim 1/2 torsion $T \sim \chi$)

$$\int d\mu_{(8,1,1)}\,\bar{\chi}^{1mn}\chi_{8mn}\bar{\chi}^{1pq}\chi_{8pq} \sim \int d^4xe\,(\nabla^8\,\mathcal{R}^4 + \cdots)$$

- Protected against quantum correction?

Full superspace integrals for $\mathcal{N} = 8$

- In case there is no 7-loop divergence in $D = 4$, we can construct a host of E_7 invariant full superspace integral, like the 8-loop candidate counter-term [Kallosh; Howe, Lindstrom]

$$\int d^4x d^{32}\theta E(x,\theta)\,(\chi\bar{\chi})^4 \sim \int d^4x e(x)\,\nabla^{10}R^4 + \cdots$$

- Since E_7 is not anomalous in perturbation [Marcus; Bossard, Hillman, Nicolai] nothing seem to rule them out as candidate counter-terms to UV divergences

$\mathcal{N} = 8$ supergravity UV divergences road map

- Green: explicitly checked by field theory or string theory computation
- Blue: ruled out by duality E_7 arguments
- Black 'allowed': Allowed by not explicitly checked
- Red: First possible ultraviolet divergence. Coefficient has not been evaluated

	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^8 R^4$	$\partial^{10} R^4$	$\partial^{12} R^4$
D=11	– –	– –	– –	– –	– –	L=2 yes
D=10	– –	– –	– –	– –	L=2 yes	– –
D=9	– –	– –	– –	L=2 yes	– –	– –
D=8	L=1 yes	– –	L=2 yes	– –	– –	L=3 yes
D=7	– –	L=2 yes	– –	– –	– –	– –
D=6	– –	– –	L=3 yes	– –	L=4 yes	– –
D=5	L=2 no	– –	L=4 no	– –	– –	L=6
D=4	L=3 no	L=5 no	L=6 no	L=7 !	L=8 ?	L=9

Conclusion and Outlook

We have shown how to incorporate the counter-term to $\mathcal{N} = 8$ supergravity UV divergences in a duality invariant function of the moduli

- The duality invariant coupling have *zero* eigenvalue when there is a UV divergence
- At 7-loop order
  - We shown that the candidate counter-term $\partial^8\mathcal{R}^4$ has a E_7-invariant supersymmetric form.
  - Need to understand if this operator is protected (F-term) or not (D-term) from quantum corrections.
  - Need to address this question from string theory point of view
- If no 7-loop divergence
  - Need to examine the situation at 8-loop and higher where susy E_7 invariant counter-terms exist
  - Search for unsuspected symmetries