Aspects of Pohlmeyer Reduction for superstrings in $AdS_5 \times S^5$

Arkady Tseytlin

- B. Hoare and AAT, arXiv:1104.2423
- Y. Iwashita, R. Roiban and AAT, to appear

"Pohlmeyer reduction":
reformulation of gauge-fixed $AdS_5 \times S^5$ superstring
in terms of current-type variables
preserving 2d Lorentz invariance:
a way towards exact solution of quantum $AdS_5 \times S^5$ superstring?

Aims:
solve string theory in $AdS_5 \times S^5$
using conformal invariance,
global supersymmetry and integrability

find S-matrix and justify Bethe Ansatz for the spectrum
from first principles;
then understand theory in finite volume: closed string theory

How to solve quantum string theory in $AdS_5 \times S^5$?

GS string on supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

not of known solvable type (cf. free oscillators; WZW)

analogy with exact solution of $O(n)$ model (Zamolodchikovs) or

principal chiral model (Polyakov-Wiegmann, ...) ?

2d CFT – no quantum mass generation

one problem of direct approaches:

lack of manifest 2d Lorentz symmetry

S-matrix depends on two rapidities, not on their difference,

symmetry constraints on it are not obviously clear...

An alternative approach?

Classically equivalent 2d Lorentz invariant action

describing same physical degrees of freedom

formulation in terms of currents rather than coordinate fields:

"Pohlmeyer reduction"

Integrable + 2d conformally invariant (UV finite) model –
fermionic generalization of non-abelian Toda theory

• intimately related (at least classically) to $AdS_5 \times S^5$ GS model
• contains fermions with standard kinetic terms
• has 2d Lorentz invariant S-matrix
for an equivalent set of 8+8 physical massive excitations
• interesting UV finite massive integrable model:
exact solution?
• deserves study regardless the issue of equivalence
to $AdS_5 \times S^5$ superstring at the quantum level

Some history

K. Pohlmeyer (1976):
Discovery of integrability (existence of ∞ of conservation laws) of *classical* O(3) sigma model via relation to sine-Gordon theory.
O(4) sigma model$\rightarrow$ complex sine-Gordon theory.
Integrability of O(n) model: Backlund transformations to generate solutions and higher conserved charges.

But why reduction relevant?
Assumed classical 2d conf. inv. which is broken at quantum level

Quantum O(3) and sin-Gordon theories are different
but integrability itself extends to quantum level
[Polyakov (1977); Zamolodchikov and Zamolodchikov (1979)]

Pohlmeyer reduction was not used much in the next 20 years...
but came to light again in the context of string theory:

Technical tool: to construct classical string solutions

• construction of *classical* string solutions in constant-curvature spaces like de Sitter and anti de Sitter [Barbashov, Nesterenko, 1981; de Vega, Sanchez, 1993]

• construction of *classical* string solutions in $AdS_5 \times S^5$ representing semiclassical string states in AdS/CFT context [Hofman, Maldacena, 2006; Dorey et al, 2006; Jevicki et al, 2007; Hoare, Iwashita, AT, 2009; Hollowood, Miramontes, 2009; ...]

• construction of euclidean open-string world-surfaces related to $N = 4$ SYM scattering amplitudes at strong coupling [Alday, Maldacena, 2009; Alday, Gaiotto, Maldacena, 2009; Dorn et al, 2009; Jevicki, Jin, 2009, ...]

Deeper role: reformulation/solution of quantum string theory

Quantum $AdS_5 \times S^5$ string is UV finite: Pohlmeyer reduction
– reformulation in terms of integrable massive theory –
may lead to an equivalent theory also at the quantum level
[Grigoriev and A.T, 2007; Mikhailov and Schafer-Nameki, 2007]

A way to exact solution of $AdS_5 \times S^5$ superstring?

• proof of UV finiteness of the reduced theory
[Roiban and A.T., 2009]

• equivalence of 1-loop quantum partition functions of string theory and reduced theory [Hoare, Iwashita and A.T., 2009]

• derivation of perturbative S-matrix of reduced theory
and its similarity to $AdS_5 \times S^5$ magnon S-matrix [Hoare and A.T.]
tree-level (2009) and one-loop (2010, 2011)

• comparison of soliton spectra and soliton S-matrices
[Hollowood and Miramontes, 2010, 2011; Hoare et al, 2011]

Pohlmeyer reduction: bosonic coset models

Prototypical example: S^2-sigma model $\to$ Sine-Gordon theory

$$L = \partial_+ X^m \partial_- X^m - \Lambda (X^m X^m - 1)\,, \qquad m = 1,2,3$$

Equations of motion:

$$\partial_+ \partial_- X^m + \Lambda X^m = 0\,, \quad \Lambda = \partial_+ X^m \partial_- X^m\,, \quad X^m X^m = 1$$

Stress tensor: $\mathrm{T}_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m$

$$\mathrm{T}_{+-} = 0\,, \quad \partial_+ \mathrm{T}_{--} = 0\,, \qquad \partial_- \mathrm{T}_{++} = 0$$

implies $\mathrm{T}_{++} = f(\sigma_+)$, $\mathrm{T}_{--} = h(\sigma_-)$

using the conformal transformations $\sigma_\pm \to F_\pm(\sigma_\pm)$ can set

$$\partial_+ X^m \partial_+ X^m = \mu^2\,, \qquad \partial_- X^m \partial_- X^m = \mu^2\,, \qquad \mu = \mathrm{const}$$

3 unit vectors in 3-dimensional Euclidean space:

$$X^m\,, \qquad X^m_+ = \mu^{-1} \partial_+ X^m\,, \qquad X^m_- = \mu^{-1} \partial_- X^m$$

X^m is orthogonal to X^m_+ and X^m_- $(X^m \partial_\pm X^m = 0)$
remaining $SO(3)$ invariant quantity is scalar product

$$\partial_+ X^m \partial_- X^m = \mu^2 \cos 2\varphi$$

then $\partial_+ \partial_- \varphi + \frac{\mu^2}{2} \sin 2\varphi = 0$
following from sine-Gordon action (Pohlmeyer, 1976)

$$\widetilde{L} = \partial_+ \varphi \partial_- \varphi + \frac{\mu^2}{2} \cos 2\varphi$$

2d Lorentz invariant despite explicit constraints
Classical solutions and integrable structures
(Lax pair, Backlund transformations, etc) are directly related
e.g., SG soliton mapped into rotating folded string on S^2:
"giant magnon" in the $J = \infty$ limit (Hofman, Maldacena 06)

Analogous construction for S^3 model gives
Complex sine-Gordon model (Pohlmeyer; Lund, Regge 76)

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\, \partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

φ, θ are $SO(4)$-invariants:
$\mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m$
$\mu^3 \sin^2\varphi\, \partial_\pm\theta = \mp\frac{1}{2}\epsilon_{mnkl} X^m \partial_+ X^n \partial_- X^k \partial_\pm^2 X^l$

In the case of AdS_2 or AdS_3:
replace $\sin\varphi \to \sinh\phi$, etc.

String-theory interpretation: string on $R_t \times S^n$

(i) conformal gauge and (ii) $t = \mu\tau$ to fix conformal diffeo's:

$\partial_\pm X^m \partial_\pm X^m = \mu^2$ are Virasoro constraints

e.g., reduced theory for string on $R_t \times S^3$

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

Similar construction for AdS_n case:

string on $AdS_n \times S^1_\psi$ with $\psi = \mu\tau$

e.g., reduced theory for string on $AdS_3 \times S^1$

$$\widetilde{L} = \partial_+\phi\partial_-\phi + \coth^2\varphi\,\partial_+\chi\partial_-\chi - \frac{\mu^2}{2}\cosh 2\phi$$

Comments:

- Virasoro constraints are solved by a special choice of variables related nonlocally to the original coordinates
- Reduced and string theories are equivalent as classical integrable systems: the respective Lax pairs are gauge-equivalent
- Although the reduction is not explicitly Lorentz invariant the resulting Lagrangian turns out to be 2d Lorentz invariant
- Reduced theory is formulated in terms of manifestly $SO(n)$ invariant variables: “blind” to original global symmetry
- PR may be thought of as a formulation in terms of physical d.o.f. – coset space analog of flat-space l.c. gauge (where 2d Lorentz is unbroken)

PR for bosonic string on F/G-coset

string on $F/G \times R_t$:
PR-theory: G/H gauged WZW model + integrable potential
F/G-coset sigma model: symmetric space

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}\,, \qquad [\mathfrak{g},\mathfrak{p}] \subset \mathfrak{p}\,, \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{g}$$

$$J = f^{-1}df = \mathcal{A} + P\,, \qquad \mathcal{A} \in \mathfrak{g}\,, \quad P \in \mathfrak{p}\,.$$

$$L = -\mathrm{Tr}(P_+P_-)\,, \qquad f \in F$$

G gauge transformations $f \to fg$;
global F-symmetry: $f \to f_0 f,\ f_0 \in F$;
classical conformal invariance
$J = \mathcal{A} + P$ as fundamental variables
$D_+P_- = 0\,, \quad D_-P_+ = 0\,, \qquad D = d + [\mathcal{A},\ ]$ – EOM
$D_-P_+ - D_+P_- + [P_+, P_-] + \mathcal{F}_{+-} = 0$ – Maurer-Cartan
$\mathrm{Tr}(P_+P_+) = -\mu^2\,, \quad \mathrm{Tr}(P_-P_-) = -\mu^2$ – Virasoro

Main idea: first solve EOM and Virasoro and then MC
special choice of G gauge condition and conformal diffs. $\rightarrow$
find reduced action giving eqs. resulting from MC
gauge fixing that solves the first Virasoro constraint

$$P_+ = \mu\, T = \text{const} \,, \qquad T \in \mathfrak{p} = \mathfrak{f} \ominus \mathfrak{g}, \qquad \text{Tr}(TT) = -1$$

choice of special element $T \rightarrow$ decomposition of algebra of F:

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g} \,, \qquad \mathfrak{p} = T \oplus \mathfrak{n} \,, \qquad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} \,, \qquad [T, \mathfrak{h}] = 0 \,,$$

$\mathfrak{h}$ is a centraliser of T in $\mathfrak{g}$
second Virasoro constraint is solved by

$$P_- = \mu\, g^{-1} T g \,, \qquad g \in G$$

EOM $D_- P_+ = 0$ is solved by $\mathcal{A}_- = (\mathcal{A}_-)_\mathfrak{h} \equiv A_-$
EOM $D_+ P_- = 0$ is solved by $\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g$
Thus new dynamical variables

$$G\text{-valued } g \,, \qquad \mathfrak{h}\text{-valued} \quad A_+,\ A_-, \quad [T, A_\pm] = 0$$

remaining Maurer-Cartan eq on $g, A_\pm$ follows from
G/H gauged WZW action with potential:

$$L = -\frac{1}{2}\,\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \text{WZ term}$$
$$-\mathrm{Tr}\big(A_+\,\partial_- g g^{-1} - A_-\,g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\big)$$
$$-\mu^2 \mathrm{Tr}(T g^{-1} T g)$$

Pohlmeyer-reduced theory for F/G coset sigma model
[Bakas,Park,Shin 95; Grigoriev, AT 07; Miramontes 08]

PR theory for string on $R_t \times F/G$ or $F/G \times S^1_\psi$:
equivalent eqs of motion; equivalent integrable structure (Lax pairs)
special case of non-abelian Toda theory:
"symmetric space Sine-Gordon model"
[Hollowood, Miramontes et al 96]

Reduced equation of motion in the "on-shell" gauge $A_\pm = 0$:

Non-abelian Toda equations:

$$\partial_-(g^{-1}\partial_+ g) - \mu^2 [T, g^{-1}Tg] = 0$$

$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = 0 \ , \qquad (\partial_- g g^{-1})_{\mathfrak{h}} = 0$$

parametrization of g in Euler angles (gauge fixing)

$g = e^{T_{n-2}\theta_{n-2}} ... e^{T_1\theta_1} e^{2T\varphi} e^{T_1\theta_1} ... e^{T_{n-2}\theta_{n-2}}$

integrating out $H = SO(n-1)$ gauge field $A_\pm$

leads to reduced theory that generalizes SG and CSG

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + G_{pq}(\varphi,\theta)\partial_+\theta^p\partial_-\theta^q + \frac{\mu^2}{2}\cos 2\varphi$$

gWZW for $G/H = SO(n)/SO(n-1)$:

$$ds^2_{n=2} = d\varphi^2 \ , \qquad ds^2_{n=3} = d\varphi^2 + \cot^2\varphi \, d\theta^2$$

$$ds^2_{n=4} = d\varphi^2 + \cot^2\varphi \, (d\theta_1 + \cot\theta_1 \tan\theta_2 d\theta_2)^2 + \tan^2\varphi \, \frac{d\theta_2^2}{\sin^2\theta_1}$$

String Theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to GS string: supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

$$S = T \int d^2\sigma \big[G_{mn}(x)\partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta\partial x + \bar{\theta}\theta\bar{\theta}\theta\partial x\partial x + ... \big],$$

tension $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset model

translates also to κ-symmetric $AdS_5 \times S^5$ superstring

Extends to quantum level: 1- and 2-loop computations and comparison to Bethe ansatz (work of last 8 years)

$AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

Killing vectors and Killing spinors of $AdS_5 \times S^5$:
$PSU(2,2|4)$ symmetry
replace $\frac{\widehat{F}}{G} = \frac{\text{SuperPoincare}}{\text{Lorentz}}$ in flat GS case by

$$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

$PSU(2,2|4)$ invariant action:
$I \sim \int \text{Tr}(f^{-1}df)^2_{F/G}$ + WZ-term
$J = f^{-1}df = J^m \mathcal{P}_m + J^I_\alpha \mathcal{Q}^\alpha_I + J^{mn} \mathcal{M}_{mn}$

$$I = \frac{\sqrt{\lambda}}{2\pi} \Big[\int d^2\sigma (J^m J^m + a \bar{J}^I J^I) + b \int \; J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ} \Big]$$

as in flat space $a = 0, \;\; b = \pm 1$ required by κ-symmetry
unique action with right symmetry and right flat-space limit

Equivalent form of the GS action:
$\frac{F}{G} = AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$
generalized to
$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$
basic superalgebra $\widehat{\mathfrak{f}} = psu(2,2|4)$
bosonic part $\mathfrak{f} = su(2,2) \oplus su(4) \cong so(2,4) \oplus so(6)$
admits Z_4-grading:

$$\widehat{\mathfrak{f}} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3 \,, \qquad\qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \,\mathrm{mod}\, 4}$$

$\mathfrak{f}_0 = \mathfrak{g} = sp(2,2) \oplus sp(4)$
$\mathfrak{f}_2 = AdS_5 \times S^5$
current $J = f^{-1}\partial_a f, \;\; f \in \widehat{F}$ (notation change: $J_0 \to \mathcal{A}$, etc)

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a}$$

$$\mathcal{A} \in \mathfrak{f}_0, \quad Q_1 \in \mathfrak{f}_1, \quad P \in \mathfrak{f}_2, \quad Q_2 \in \mathfrak{f}_3 \,.$$

GS Lagrangian:

$$L_{\rm GS} = \frac{1}{2}\,{\rm STr}(\sqrt{-g}g^{ab}P_aP_b + \varepsilon^{ab}Q_{1a}Q_{2b})\,,$$

fermionic currents in WZ term only

conformal gauge: $\sqrt{-g}g^{ab} = \eta^{ab}$

$$L_{\rm GS} = {\rm STr}[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})]$$

$${\rm STr}(P_+P_+) = 0\,, \qquad\qquad {\rm STr}(P_-P_-) = 0$$

Equations of motion in terms of currents: 1-st order form

$$\begin{aligned}
\text{EOM}: \quad & \partial_+P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0\,,\\
& \partial_-P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0\,,\\
& [P_+, Q_{1-}] = 0\,, \qquad [P_-, Q_{2+}] = 0\,.
\end{aligned}$$

$$\text{MC}: \quad \partial_-J_+ - \partial_+J_- + [J_-, J_+] = 0\,.$$

partial κ-symmetry gauge: $Q_{1-} = 0\,, \qquad Q_{2+} = 0$

remaining EOM:

$$\partial_+ P_- + [\mathcal{A}_+, P_-] = 0\,, \qquad \partial_- P_+ + [\mathcal{A}_-, P_+] = 0$$

Maurer-Cartan:

$$\begin{aligned} \partial_+ \mathcal{A}_- - \partial_- \mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-] + [P_+, P_-] + [Q_{1+}, Q_{2-}] &= 0\,, \\ \partial_- Q_{1+} + [\mathcal{A}_-, Q_{1+}] - [P_+, Q_{2-}] &= 0\,, \\ \partial_+ Q_{2-} + [\mathcal{A}_+, Q_{2-}] - [P_-, Q_{1+}] &= 0\,. \end{aligned}$$

apply Pohlmeyer reduction:

(i) start with GS equations in terms of currents

(ii) solve conformal gauge constraints algebraically introducing new set of field variables directly related to the currents

(iii) fix κ-symmetry gauge

(iv) reconstruct the action for new current variables

classical equivalence of original and "reduced" eqs:

both are integrable

Virasoro can be solved by fixing a special G-gauge
and residual conformal diffeomorpism gauge

$$P_+ = \mu T\,, \qquad P_- = \mu\, g^{-1} T g\,, \quad \mu = \text{const}$$

$$g \in G = Sp(2,2) \times Sp(4)$$

μ= an arbitary scale parameter – remnant of fixing
residual conformal diffeomorphisms, like p^+ in l.c. gauge
T is a fixed constant matrix, e.g., diag$(I, -I, I, -I)$, $\text{Str}\, T^2 = 0$
$H \in G$ that commutes with T, $[T, h] = 0$, $h \in H$:
$H = SU(2) \times SU(2) \times SU(2) \times SU(2)$
P_- is invariant under $g \to hg$ if $h \in H$
implies extra H gauge invariance of e.o.m. for g

$$A_+ \equiv g\mathcal{A}_+ g^{-1} + \partial_+ g g^{-1}\,, \qquad A_- \equiv (\mathcal{A}_-)_{\mathfrak{h}}$$

Thus $g \in G = Sp(2,2) \times Sp(4)$ and
A_+, A_- in $\mathfrak{h} = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$ of H
are new independent bosonic variables

impose partial κ-symmetry gauge

$$Q_{1-} = 0 \,, \qquad Q_{2+} = 0 \,,$$

define new fermionic variables

$$\Psi_1 = Q_{1+} \in \widehat{\mathfrak{f}}_1 \,, \qquad \Psi_2 = g Q_{2-} g^{-1} \in \widehat{\mathfrak{f}}_3$$

residual κ-symmetry fixed by $\Psi_{1,2} T = -T \Psi_{1,2}$
then define new fermionic variables

$$\Psi_{_R} = \frac{1}{\sqrt{\mu}} \Psi_1^{\|} \,, \qquad \Psi_{_L} = \frac{1}{\sqrt{\mu}} \Psi_2^{\|}$$

they are expressed in terms of real Grassmann
2×2 matrices $\xi_{R,L}$ and $\eta_{R,L}$: 8+8=16 components

Remarkably, exists local Lagrangian reproducing
resulting classical reduced equations:

Gauged WZW model for

$$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$$

with integrable potential and fermionic terms:

$$L_{tot} = L_B + L_F = L_{\rm gWZW}(g,A) + \mu^2\, {\rm Str}(g^{-1}TgT)$$

$$+ \,{\rm Str}\left(\Psi_{_L} T D_+ \Psi_{_L} + \Psi_{_R} T D_- \Psi_{_R} + \ \mu\, g^{-1}\Psi_{_L} g \Psi_{_R}\right)$$

fields are represented by 8×8 supermatrices, e.g.,

$g = {\rm diag}(a,b)$, $\quad a \in Sp(2,2), \quad b \in Sp(4)$

$D_\pm \Psi = \partial_\pm \Psi + [A_\pm, \Psi], \quad A_\pm \in \mathfrak{h} = su(2)\oplus ... \oplus su(2)$

$T = \frac{i}{2}{\rm diag}(1,1,-1,-1,1,1,-1,-1);$

$[T,h]=0, h\in H = [SU(2)]^4,$

invariant under H gauge transformations

$$g' = h^{-1} g h, \quad A'_\pm = h^{-1}A_\pm h + h^{-1}\partial_\pm h, \quad \Psi'_{_{L,R}} = h^{-1}\Psi_{_{L,R}} h$$

$$[T,h]=0, \quad h\in H = [SU(2)]^4$$

classically equivalent to GS model – integrable model:
Lax pair encoding equations of motion

$$\mathcal{L}_- = \partial_- + A_- + z^{-1}\sqrt{\mu}g^{-1}\Psi_{_L}g + z^{-2}\mu g^{-1}Tg\,,$$
$$\mathcal{L}_+ = \partial_+ + g^{-1}\partial_+ g + g^{-1}A_+ g + z\sqrt{\mu}\Psi_{_R} + z^2\mu T$$

- gWZW model coupled to fermions interacting minimally and through the "Yukawa" term
- 2d Lorentz invariant action with $\Psi_{_R}, \Psi_{_L}$ as 2d Majorana spinors with standard kinetic terms
- 8 real bosonic and 16 real fermionic independent variables; fermions link bosons from $Sp(2,2) \times Sp(4)$:
- 2d supersymmetry? yes, at least at quadratic level and in $AdS_2 \times S^2$ truncation: $n=2$ super sine-Gordon model
- μ-dependent interactions are equal to GS Lagrangian; gWZW produces MC eqs.: path integral derivation?
- action quadratic in fermions – in contrast to original GS action [quartic terms reflecting curvature will appear if we integrate out $A_\pm$ as in susy gauged WZW case]
- linearisation of e.o.m. in the gauge $A_\pm = 0$ around $g = \mathbf{1}$: gives 8+8 bosonic and fermionic d.o.f. with mass μ – same as in BMN limit

H gauge field $A_{\pm}$ can be gauged away on e.o.m. –
get fermionic generalization of non-abelian Toda equations:

$$\partial_-(g^{-1}\partial_+ g) + \mu^2[g^{-1}Tg, T] + \mu[g^{-1}\Psi_{_L} g, \Psi_{_R}] = 0,$$

$$T\partial_-\Psi_{_R} + \tfrac{1}{2}\mu(g^{-1}\Psi_{_L} g)^{\parallel} = 0\ ,$$

$$T\partial_+\Psi_{_L} + \tfrac{1}{2}\mu(g\Psi_{_R} g^{-1})^{\parallel} = 0\ ,$$

$$(g^{-1}\partial_+ g - \tfrac{1}{2}[[T, \Psi_{_R}], \Psi_{_R}])_{\mathfrak{h}} = 0\ ,$$

$$(g\partial_- g^{-1} - \tfrac{1}{2}[[T, \Psi_{_L}], \Psi_{_L}])_{\mathfrak{h}} = 0$$

fermions carry representations of both $Sp(2,2)$ and $Sp(4)$:
"intertwine" the two bosonic reduced sub-theories
Model resembles WZW models based on supergroups
rather than 2d supersymmetric WZW model
but fermions here have 1-st order kinetic term – a "hybrid"

Example: superstring on $AdS_2 \times S^2$

PR Lagrangian: same as $n=2$ supersymmetric sine-Gordon!

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$
$$+ \beta\partial_-\beta + \gamma\partial_-\gamma + \nu\partial_+\nu + \rho\partial_+\rho$$
$$- 2\mu\left[\cosh\phi\ \cos\varphi\,(\beta\nu + \gamma\rho) + \sinh\phi\ \sin\varphi\,(\beta\rho - \gamma\nu)\right] .$$

equivalent to

$$\widetilde{L} = \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2 + \psi^*_{L}\partial_+\psi_{L} + \psi^*_{R}\partial_-\psi_{R}$$
$$+ \left[W''(\Phi)\psi_{L}\psi_{R} + W^{*\prime\prime}(\Phi^*)\psi^*_{L}\psi^*_{R}\right] .$$

bosonic part is of $AdS_2 \times S^2$ bosonic reduced model if

$$W(\Phi) = \mu\cos\Phi\,, \qquad |W'(\Phi)|^2 = \frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)\ .$$

$$\psi_{L} = \nu + i\rho\,, \qquad \psi_{R} = -\beta + i\gamma\ ,$$

UV finiteness of reduced theory

[R. Roiban, A.T., 2009]

Reduction procedure may work at quantum level
only in conformally invariant case (like $AdS_5 \times S^5$ case)
Consistency requires that reduced theory is also UV finite
gWZW+ free fermions is finite;
μ is not renormalized, remains an arbitrary
conformal symmetry gauge fixing parameter at quantum level

Thus in contrast to l.c. gauge fixed GS superstring
the reduced model is 2d Lorentz invariant
and power counting renormalizable: in fact, finite.

Open questions

- Quantum equivalence of reduced theory and GS theory?
  Path integral argument of equivalence?
  Transformation may work only in quantum-conformal case like $AdS_5 \times S^5$
- Indication of equivalence: semiclassical expansion near counterparts of rigid strings in $AdS_5 \times S^5$ leads to same characteristic frequencies – same 1-loop partition function [Iwashita, Hoare, AAT 09]
- S-matrix for elementary excitations? [Hoare, AT, 09-11]
  Relation to magnon S-matrix in BA?
- Solve reduced theory $\to$ solve $AdS_5 \times S^5$ superstring

Recent work

Towards quantum S-matrix of the Pohlmeyer reduced form of $AdS_5 \times S^5$ superstring theory

Arkady Tseytlin

July 21, 2011

Based on Hoare, AAT, arXiv:1104.2423

Review of Pohlmeyer Reduction

Perturbative computation of S-matrix

q-deformed supersymmetry and exact S-matrix conjecture

Conclusions and open questions

Some history

- The Pohlmeyer reduction related the $O(n)$ sigma models to integrable Hamiltonian systems
- Led to the discovery of the integrability of the classical $O(3)$ sigma model via its relation to sine-Gordon.

Pohlmeyer, 1976
Luscher, Pohlmeyer, 1978
Pohlmeyer, Rehren, 1979
Eichenherr, Pohlmeyer, 1979

The Pohlmeyer reduction ...

- relates the currents of the original theory to the fields of the reduced theory.
- is carried out at the level of the equations of motion.
- gives rise to a 2-d Lorentz invariant integrable theory.

- Technical issue: equations of motion for higher dimensional models, e.g. $O(n)$, $n > 3$, apparently non-Lagrangian.
- Resolved by considering gauged WZW plus integrable potential.

Bakas, Park, Shin, 1996
Grigoriev, AAT, 2007
Miramontes, 2008

- Classical reduction - assumes conformal invariance, which is broken at quantum level– no equivalence at quantum level
- O(n) sigma model was shown to be integrable at quantum level.

Polyakov, 1977
Zamolodchikov and Zamolodchikov, 1979

Pohlmeyer reduction in string theory

- Used in the construction of classical string solutions representing semiclassical closed string states in AdS/CFT context.

Hofman, Maldacena, 2006
Dorey et al, 2006
Jevicki, Spradlin, Volovich et al, 2007
Hoare, Iwashita, AAT, 2009
Hollowood, Miramontes, 2009

- Used in the construction of Euclidean open-string world-surfaces related to $\mathcal{N} = 4$ super Yang-Mills scattering amplitudes at strong coupling.

Alday, Maldacena, 2009
Alday, Gaiotto, Maldacena, 2009
Dorn et. al, 2009
Jevicki, Jin, 2009 ...

Quantum equivalence?

- Quantum $AdS_5 \times S^5$ is UV finite so Pohlmeyer reduction may lead to an equivalent theory also at quantum level.

Grigoriev, AAT, 2007
Mikhailov, Schafer-Nameki, 2007

- Describes 8+8 physical degrees of freedom, solves Virasoro constraints and the resulting model is integrable – there exists a Lax connection
- Resulting reduced model is UV finite.

Roiban, AAT, 2009

- One-loop corrections to soliton energies match string ones

Hoare, Iwashita, AAT, 2009
Iwashita, 2010

- Two-loop corrections?
- S-matrix?

Pohlmeyer reduction - Aims

- Investigate this theory and its truncations in the hope that when fermions are included it will help us understand the quantum string theory.
- Consider the perturbative S-matrix and try to extend to exact S-matrix
- Construct solitons and conjecture exact S-matrix (cf. sine-Gordon)

Hollowood, Miramontes, 2010, 2011
Zamolodchikov and Zamolodchikov, 1979

- Earlier exact results for bosonic models with abelian H.

Dorey, Hollowood, 1994
Miramontes, Hollowood et. al, 1995 - present

Pohlmeyer reduction example: $\mathbb{R}_t \times S^2$

- classical sigma model on $\mathbb{R}_t \times S^2$ – S^2 embedded in $\mathbb{R}^3$

$$\mathcal{L} = \frac{R^2}{4\pi\alpha'} \int d^2x \; [-\partial t \partial t + \partial \mathbf{X} \cdot \partial \mathbf{X}] + \Lambda(\mathbf{X} \cdot \mathbf{X} - 1) \tag{1}$$

- Coordinate on $\mathbb{R}^3$– $\mathbf{X} = (X_1, X_2, X_3)$

Conventions

- Worldsheet coordinates – (τ, σ)
- Lightcone coordinates – $x_\pm = \tau \pm \sigma\,,\;\; \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

- Fix conformal gauge and static gauge – $t = \mu\tau$

Equations of motion

- with respect to $\mathbf{X}$

$$\partial_+\partial_-\mathbf{X} + (\partial_+\mathbf{X} \cdot \partial_-\mathbf{X})\mathbf{X} = 0 \tag{2}$$

- with respect to 2d metric – Virasoro constraints

$$\partial_\pm\mathbf{X} \cdot \partial_\pm\mathbf{X} = \mu^2 \tag{3}$$

- with respect to Λ – sphere constraints

$$\mathbf{X} \cdot \mathbf{X} = 1 \tag{4}$$

- “Solve” the Virasoro constraints: replace $\mathbf{X}$ by single field φ

$$\partial_+\mathbf{X} \cdot \partial_-\mathbf{X} = \mu^2 \cos 2\varphi \tag{5}$$

three vectors $\mathbf{X}$, $\partial_+\mathbf{X}$, $\partial_-\mathbf{X}$ span $\mathbb{R}^3$.

- Therefore we can write $\partial_+\partial_+\mathbf{X}$ and $\partial_-\partial_-\mathbf{X}$ as linear combinations.
- The equation of motion for φ is then

$$\partial_+\partial_-\varphi + \frac{\mu^2}{2} \sin 2\varphi = 0 \tag{6}$$

Pohlmeyer, 1976

- Sine-gordon equation of motion – single degree of freedom.
- Resulting equations of motion are Lorentz invariant, though the reduction is not.
- Blind to original $SO(3)$ global symmetry.
- implies classical integrability
- Method generalises to larger target spaces, e.g. $\mathbb{R}_t \times S^3$ is related to complex sine-Gordon.

$AdS_5 \times S^5$ superstring

$AdS_5 \times S^5$ superstring worldsheet sigma model

Metsaev, AAT, 1998

- Based on the coset

$$\frac{\hat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)} \tag{7}$$

- Bosonic part of the coset is $\frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)} \cong AdS_5 \times S^5$.
- $\mathbb{Z}_4$ decomposition of algebra

$$\mathfrak{psu}(2,2|4) = \hat{\mathfrak{f}} = \oplus_{i=1}^{4} \hat{\mathfrak{f}}_i \,, \qquad [\hat{\mathfrak{f}}_i, \hat{\mathfrak{f}}_j] \subset \hat{\mathfrak{f}}_{i+j \mod 4} \tag{8}$$

$$\hat{\mathfrak{g}} = \hat{\mathfrak{f}}_0 = \mathfrak{sp}(2,2) \oplus \mathfrak{sp}(4) \qquad \hat{\mathfrak{f}}_{1,3} \text{ fermionic} \tag{9}$$

$$\hat{\mathfrak{f}}_2 \text{ bosonic part of coset} \tag{10}$$

- Action is constructed by taking a group valued field

$$f \in PSU(2,2|4) \tag{11}$$

and considering the Maurer-Cartan one-form

$$\mathcal{J} = f^{-1}df \in \hat{\mathfrak{f}} \tag{12}$$

- Under the $\mathbb{Z}_4$ decomposition $\mathcal{J} = \mathcal{A} + \mathcal{Q}_1 + \mathcal{P} + \mathcal{Q}_3$
- Under the G-gauge symmetry – $f \to fg$
  – $\mathcal{A}$ transforms as a connection,
  – $\mathcal{P}$ and $\mathcal{Q}_{1,3}$ transform covariantly.
- Action is constructed from $\mathcal{P}$ and $\mathcal{Q}_{1,3}$ – with the bosonic part given by usual coset sigma model

$$\mathcal{L} = \mathrm{STr}(\mathcal{P}_+\mathcal{P}_-) + \text{fermionic} \tag{13}$$

- In addition to G-gauge symmetry there is a global $\hat{F}$ symmetry – $f \to f_0 f$

Pohlmeyer reduction – $AdS_5 \times S^5$ superstring

- Solve the equations of motion and the Virasoro constraints using G-gauge symmetry and κ-symmetry
- In solving the Virasoro constraints we introduce a mass μ and a constant matrix $T \in \hat{\mathfrak{f}}_2$.
- Constant matrix T induces a further $\mathbb{Z}_2$ decomposition of the algebra

$$\hat{\mathfrak{f}} = \hat{\mathfrak{f}}^{\parallel} \oplus \hat{\mathfrak{f}}^{\perp} \tag{14}$$

$$[\hat{\mathfrak{f}}^{\parallel}, \hat{\mathfrak{f}}^{\parallel}] \subset \hat{\mathfrak{f}}^{\perp} \qquad [\hat{\mathfrak{f}}^{\parallel}, \hat{\mathfrak{f}}^{\perp}] \subset \hat{\mathfrak{f}}^{\parallel} \qquad [\hat{\mathfrak{f}}^{\perp}, \hat{\mathfrak{f}}^{\perp}] \subset \hat{\mathfrak{f}}^{\perp} \tag{15}$$

- $\hat{\mathfrak{f}}_0^{\perp} = [\mathfrak{su}(2)]^4$ is an algebra – denote $\mathfrak{h}$ and the corresponding group H
- The equations of motion for the reduced theory are given by the flatness condition for $\mathcal{J}$.

Action of Pohlmeyer reduced $AdS_5 \times S^5$ superstring

- Reduced equations of motion have $H \times H$-gauge symmetry
- If we gauge-fix to leave a H-gauge symmetry then the resulting equations come from the following action

$$\mathcal{S} = -\frac{k}{4\pi}\mathrm{STr}\Big[\frac{1}{2}\int d^2x\; g^{-1}\partial_+ g g^{-1}\partial_- g - \frac{1}{3}\int d^3x\; \epsilon^{mnl} g^{-1}\partial_m g g^{-1}\partial_n g g^{-1}\partial_l g \tag{16}$$

$$+\int d^2x\,(A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+A_-) \tag{17}$$

$$+\int d^2x\,(\Psi_{{}_L} T D_+ \Psi_{{}_L} + \Psi_{{}_R} T D_- \Psi_{{}_R} + \mu g^{-1}\Psi_{{}_L} g \Psi_{{}_R} + \mu^2\, g^{-1} T g T)\Big] \tag{18}$$

- $g \in G = Sp(2,2) \times Sp(4)$
- $A_\pm \in \mathfrak{h} = [\mathfrak{su}(2)]^4$
- $\Psi_{{}_L} \in \hat{\mathfrak{f}}_1^{\|}$
- $\Psi_{{}_R} \in \hat{\mathfrak{f}}_3^{\|}$

Grigoriev, AAT, 2007, 2008

Comments

- Fermionic extension of gauged WZW theory plus integrable potential (generalised sine-Gordon model)
- Lorentz invariant
- H-gauge symmetry
- If H is non-abelian – no global symmetry
- Classically integrable – Lax connection
- Blind to original global $\hat{F}$ symmetry of string theory.
- No apparent supersymmetry – target-space or spacetime

Truncated models

For $AdS_3 \times S^3$

- $\hat{F} = SU(1,1|2) \times SU(1,1|2)$
- $G = U(1,1) \times U(2)$
- $H = [U(1)]^4$
- Complex sine-Gordon + complex sinh-Gordon coupled to fermions

For $AdS_2 \times S^2$

- $\hat{F} = PSU(1,1|2)$
- $G = SO(1,1) \times SO(2)$
- H is trivial
- $\mathcal{N} = 2$ supersymmetric sine-Gordon theory

Tree-level S-matrix of the PR theory

- Lorentz invariant two-particle tree-level S-matrix for Pohlmeyer reduced theory constructed using a particular gauge choice.

  Hoare, AAT, 2009

- Similar structure to the $AdS_5 \times S^5$ light-cone gauge tree-level S-matrix.

  Klose, McLoughlin, Roiban, Zarembo, 2006
  Arutyunov, Frolov, Zamaklar, 2006

- same group factorisation properties: arising from supersymmetry in $AdS_5 \times S^5$ case, not manifest in reduced theory.
- Suggests hidden fermionic symmetry.

Gauge choice

Under the H-gauge symmetry

- $g \to h^{-1} g h$
- $A_\pm \to h^{-1} A_\pm h + h^{-1} \partial_\pm h$
- Gauge fix $A_+ = 0$
- Path-integral over A_- gives the constraint equation appearing as a delta-function

$$(g^{-1}\partial_+ g - [\Psi_{_R} T,\, \Psi_{_R}])|_{\mathfrak{h}} = 0 \qquad (19)$$

- Can use this equation perturbatively to eliminate the unphysical $\mathfrak{h}$ part of g.

Explicitly

- write $g = X + \xi, \quad X \in \mathfrak{g} \ominus \mathfrak{h}, \quad \xi \in \mathfrak{h}$
- solve perturbatively for ξ as function of X and $\Psi_{_R}$.

Using integration by parts and the linearised equations of motion one can write the Lagrangian in the following local form

$$\mathcal{L} = \frac{k}{4\pi}\ \mathrm{STr}\Big(\frac{1}{2}\partial_+ X \partial_- X - \frac{\mu^2}{2} X^2 + \psi_L T \partial_+ \psi_L + \psi_R T \partial_- \psi_R + \mu \psi_L \psi_R \tag{20}$$

$$+ \frac{1}{12}[X, \partial_+ X][X, \partial_- X] + \frac{\mu^2}{24}[X, [X, T]]^2 \tag{21}$$

$$- \frac{1}{4}[\psi_L T, \psi_L][X, \partial_+ X] - \frac{1}{4}[\psi_R, T\psi_R][X, \partial_- X] \tag{22}$$

$$- \frac{\mu}{2}[X, \psi_R][X, \psi_L] + \frac{1}{2}[\psi_L T, \psi_L][\psi_R, T\psi_R] + \dots\Big) \tag{23}$$

Residual gauge symmetry - Lagrangian is invariant under the global part of the gauge group H

$$(X,\, \Psi_{R},\, \Psi_{L}) \to h^{-1}(X,\, \Psi_{R},\, \Psi_{L})h \tag{24}$$

Lagrangian can be written in terms of fields transforming in representations of H

$$X = Y + Z \qquad \Psi = \zeta + \chi \tag{25}$$

$$\begin{pmatrix} SU(2)_1 & Y & 0 & \zeta \\ Y & SU(2)_{\dot{1}} & \chi & 0 \\ 0 & \chi & SU(2)_2 & Z \\ \zeta & 0 & Z & SU(2)_{\dot{2}} \end{pmatrix} \tag{26}$$

Fundamental indices of $SU(2)_1$ and $SU(2)_2$ – a and α

Fundamental indices of $SU(2)_{\dot{1}}$ and $SU(2)_{\dot{2}}$ – $\dot{a}$ and $\dot{\alpha}$

Treat the indices $a, \dot{a}$ as bosonic, i.e. $[a] = [\dot{a}] = 0$

Treat the indices $\alpha, \dot{\alpha}$ as fermionic, i.e. $[\alpha] = [\dot{\alpha}] = 1$

The fields transform as follows under the $[SU(2)]^4$ symmetry

$$Y_{a\dot{a}} \qquad Z_{\alpha\dot{\alpha}} \qquad \zeta_{a\dot{\alpha}} \qquad \chi_{\alpha\dot{a}} \tag{27}$$

Can also expand out the Lagrangian to give

$$\mathcal{L}_5 = \frac{1}{2}\partial_+ Y_{a\dot{a}}\partial_- Y^{\dot{a}a} - \frac{\mu^2}{2} Y_{a\dot{a}} Y^{\dot{a}a} + \frac{1}{2}\partial_+ Z_{\alpha\dot{\alpha}}\partial_- Z^{\dot{\alpha}\alpha} - \frac{\mu^2}{2} Z_{\alpha\dot{\alpha}} Z^{\dot{\alpha}\alpha} \tag{28}$$

$$+ \frac{i}{2}\zeta_{L\,a\dot{\alpha}}\partial_+\zeta_L{}^{\dot{\alpha}a} + \frac{i}{2}\zeta_{R\,a\dot{\alpha}}\partial_-\zeta_R{}^{\dot{\alpha}a} - i\mu\zeta_{L\,a\dot{\alpha}}\zeta_R{}^{\dot{\alpha}a} \tag{29}$$

$$+ \frac{i}{2}\chi_{L\,\alpha\dot{a}}\partial_+\chi_L{}^{\dot{a}\alpha} + \frac{i}{2}\chi_{R\,\alpha\dot{a}}\partial_-\chi_R{}^{\dot{a}\alpha} - i\mu\chi_{L\,\alpha\dot{a}}\chi_R{}^{\dot{a}\alpha} \tag{30}$$

$$+ \frac{\pi}{2k}\Big[-\frac{2}{3}\big(Y_{a\dot{a}}Y^{\dot{a}a}\partial_+ Y_{b\dot{b}}\partial_- Y^{\dot{b}b} - Y_{a\dot{a}}\partial_+ Y^{\dot{a}a} Y_{b\dot{b}}\partial_- Y^{\dot{b}b} + \frac{\mu^2}{2} Y_{a\dot{a}}Y^{\dot{a}a}Y_{b\dot{b}}Y^{\dot{b}b}\big) \tag{31}$$

$$+ \frac{2}{3}\big(Z_{\alpha\dot{\alpha}}Z^{\dot{\alpha}\alpha}\partial_+ Z_{\beta\dot{\beta}}\partial_- Z^{\dot{\beta}\beta} - Z_{\alpha\dot{\alpha}}\partial_+ Z^{\dot{\alpha}\alpha} Z_{\beta\dot{\beta}}\partial_- Z^{\dot{\beta}\beta} + \frac{\mu^2}{2} Z_{\alpha\dot{\alpha}}Z^{\dot{\alpha}\alpha}Z_{\beta\dot{\beta}}Z^{\dot{\beta}\beta}\big) \tag{32}$$

$$+ i\big(\zeta_{L\,a\dot{\alpha}}\zeta_L{}^{\dot{\alpha}b}Y^{\dot{b}a}\partial_+ Y_{b\dot{b}} + \zeta_{R\,a\dot{\alpha}}\zeta_R{}^{\dot{\alpha}b}Y^{\dot{b}a}\partial_- Y_{b\dot{b}} + \mu\,\zeta_{R\,a\dot{\alpha}}\zeta_L{}^{\dot{\alpha}a}Y_{b\dot{b}}Y^{\dot{b}b}\big) \tag{33}$$

$$- i\big(\zeta_{L\,a\dot{\alpha}}\zeta_L{}^{\dot{\beta}a}Z^{\dot{\alpha}\beta}\partial_+ Z_{\beta\dot{\beta}} + \zeta_{R\,a\dot{\alpha}}\zeta_R{}^{\dot{\beta}a}Z^{\dot{\alpha}\beta}\partial_- Z_{\beta\dot{\beta}} + \mu\,\zeta_{R\,a\dot{\alpha}}\zeta_L{}^{\dot{\alpha}a}Z_{\beta\dot{\beta}}Z^{\dot{\beta}\beta}\big) \tag{34}$$

$$+ i\big(\chi_{L\,\alpha\dot{a}}\chi_L{}^{\dot{b}\alpha}Y^{\dot{a}b}\partial_+ Y_{b\dot{b}} + \chi_{R\,\alpha\dot{a}}\chi_R{}^{\dot{b}\alpha}Y^{\dot{a}b}\partial_- Y_{b\dot{b}} + \mu\,\chi_{R\,\alpha\dot{a}}\chi_L{}^{\dot{a}\alpha}Y_{b\dot{b}}Y^{\dot{b}b}\big) \tag{35}$$

$$- i\big(\chi_{L\,\alpha\dot{a}}\chi_L{}^{\dot{a}\beta}Z^{\dot{\beta}\alpha}\partial_+ Z_{\beta\dot{\beta}} + \chi_{R\,\alpha\dot{a}}\chi_R{}^{\dot{a}\beta}Z^{\dot{\beta}\alpha}\partial_- Z_{\beta\dot{\beta}} + \mu\,\chi_{R\,\alpha\dot{a}}\chi_L{}^{\dot{a}\alpha}Z_{\beta\dot{\beta}}Z^{\dot{\beta}\beta}\big) \tag{36}$$

$$+ 4i\mu\big(\zeta_{R\,a\dot{\alpha}}\chi_{L\,\beta\dot{b}}Y^{\dot{b}a}Z^{\dot{\alpha}\beta} - \chi_{R\,\alpha\dot{a}}\zeta_{L\,b\dot{\beta}}Y^{\dot{a}b}Z^{\dot{\beta}\alpha}\big) \tag{37}$$

$$+ 2\big(\zeta_{L\,a\dot{\alpha}}\zeta_{L\,b\dot{\beta}}\zeta_R{}^{\dot{\alpha}b}\zeta_R{}^{\dot{\beta}a} - \chi_{L\,\alpha\dot{a}}\chi_{L\,\beta\dot{\beta}}\chi_R{}^{\dot{a}\beta}\chi_R{}^{\dot{b}\alpha}\big)\Big] + \mathcal{O}(k^{-2})\,. \tag{38}$$

Computation of tree-level S-matrix

- p_1 and p_2 are two on-shell momenta of the particles
- Convenient to use rapidities $p_i = \mu \sinh \vartheta_i$
- Lorentz symmetry: S-matrix only depends on $\theta = \vartheta_1 - \vartheta_2$

- combine four fields $Y_{a\dot{a}}$, $Z_{\alpha\dot{\alpha}}$, $\zeta_{a\dot{\alpha}}$, $\chi_{\alpha\dot{a}}$ in a single field

$$\Phi_{A\dot{A}}\,, \qquad A = (a, \alpha) \tag{39}$$

- two-particle S-matrix takes the following form

$$\mathbb{S}\,|\Phi_{A\dot{A}}(\vartheta_1)\Phi_{B\dot{B}}(\vartheta_2)\rangle = S_{A\dot{A},B\dot{B}}^{C\dot{C},D\dot{D}}(\theta)\,|\Phi_{C\dot{C}}(\vartheta_1)\Phi_{D\dot{D}}(\vartheta_2)\rangle \tag{40}$$

- The tree-level S-matrix factorises as

$$S_{A\dot{A},B\dot{B}}^{C\dot{C},D\dot{D}}(\theta) = (-1)^{[B][\dot{A}]+[D][\dot{C}]} S_{AB}^{CD}(\theta) S_{\dot{A}\dot{B}}^{\dot{C}\dot{D}}(\theta) \tag{41}$$

- Same factorisation as in the light-cone gauge string theory

Klose, McLoughlin, Roiban, Zarembo, 2006

- There it relied on integrability and $PSU(2|2) \times PSU(2|2)$ global symmetry
- We have integrability but no manifest supersymmetry

Tree-level result

$$S_{AB}^{CD}(\theta,k) = \begin{cases} K_1(\theta,k)\delta_a^c\delta_b^d + K_2(\theta,k)\delta_a^d\delta_b^c\,, & \\ K_3(\theta,k)\delta_\alpha^\gamma\delta_\beta^\delta + K_4(\theta,k)\delta_\alpha^\delta\delta_\beta^\gamma\,, & \\ K_5(\theta,k)\epsilon_{ab}\epsilon^{\gamma\delta}\,, & K_6(\theta,k)\epsilon_{\alpha\beta}\epsilon^{cd}\,, \\ K_7(\theta,k)\delta_a^d\delta_\beta^\gamma\,, & K_8(\theta,k)\delta_\alpha^\delta\delta_b^c\,, \\ K_9(\theta,k)\delta_a^c\delta_\beta^\delta\,, & K_{10}(\theta,k)\delta_\alpha^\gamma\delta_b^d\,, \end{cases} \tag{42}$$

$$K_1(\theta,k) = K_3(\theta,-k) = 1 + \frac{i\pi}{2k}\tanh\frac{\theta}{2} + \mathcal{O}(k^{-2}) \tag{43}$$

$$K_2(\theta,k) = K_4(\theta,-k) = -\frac{i\pi}{k}\coth\theta + \mathcal{O}(k^{-2}) \tag{44}$$

$$K_5(\theta,k) = -K_6(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2} + \mathcal{O}(k^{-2}) \tag{45}$$

$$K_7(\theta,k) = -K_8(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2} + \mathcal{O}(k^{-2}) \tag{46}$$

$$K_9(\theta,k) = K_{10}(\theta,-k) = 1 + \mathcal{O}(k^{-2}) \tag{47}$$

Comments

- Have group factorisation, but not satisfaction of Yang-Baxter – common to all theories with non-abelian H
- Light-cone gauge superstring theory result is not Lorentz invariant but does satisfy Yang-Baxter.
- Unitarity and crossing.
- Corresponding results for $AdS_3 \times S^3$ and $AdS_2 \times S^2$ do satisfy Yang-Baxter – H is abelian (or trivial).
- Coefficients are exactly those of a quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ R-matrix.

Beisert, Koroteev, 2008
Beisert, 2010

- However these coefficients parametrise the R-matrix in a deformed way that breaks the manifest $SU(2)\times SU(2)$ symmetry.

One-loop S-matrix of the PR theory

- Computation extended to one-loop – can be carried out with just the quartic Lagrangian – standard perturbation theory

  Hoare, AAT, 2011

- Relevant Feynman diagrams are bubble and tadpole
- tadpole has a vanishing finite contribution in 2d: not relevant

One-loop result - $AdS_5 \times S^5$

$$K_i = \; p_{0_5}(\theta,k) \; \hat{K}_i \tag{48}$$

$$\hat{K}_1(\theta,k) = \hat{K}_3(\theta,-k) = 1 + \frac{i\pi}{2k}\tanh\frac{\theta}{2} - \frac{5\pi^2}{8k^2} - \frac{i\pi\theta}{2k^2} + \mathcal{O}(k^{-3}) \tag{49}$$

$$\hat{K}_2(\theta,k) = \hat{K}_4(\theta,-k) = -\frac{i\pi}{k}\coth\theta + \frac{\pi^2}{2k^2} + \frac{i\pi\theta}{k^2} + \mathcal{O}(k^{-3}) \tag{50}$$

$$\hat{K}_5(\theta,k) = -\hat{K}_6(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3}) \tag{51}$$

$$\hat{K}_7(\theta,k) = -\hat{K}_8(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3}) \tag{52}$$

$$\hat{K}_9(\theta,k) = \hat{K}_{10}(\theta,-k) = 1 + \mathcal{O}(k^{-3}) \tag{53}$$

$$p_{0_5}(\theta,k) = 1 + \; \frac{\pi\,\mathrm{cosech}\,\theta}{4k^2}\Big(i\big[2 + (i\pi - 2\theta)\coth\theta\big] - \pi\,\mathrm{cosech}\,\theta\Big) \tag{54}$$

Comments

- Coefficients are still similar to those of the q-deformed S-matrix though no longer exactly the same – differ by extra θ terms.
- Coefficients of q-deformed S-matrix satisfy q-deformed crossing relations whereas those of the perturbative S-matrix satisfy standard relations.
- Phase factor is the same as the expansion of the $\mathcal{N} = 2$ supersymmetric sine-Gordon phase factor.

Zamolodchikov and Zamolodchikov, 1979
Shankar, Witten, 1978
Ahn, 1991
Kobayashi, Uematsu, 1991

Pohlmeyer reduced $AdS_2 \times S^2$:

- The one-loop computation agrees with the exact results for the $\mathcal{N} = 2$ supersymmetric sine-Gordon S-matrix.

Kobayashi, Uematsu, 1991

Supersymmetry of the reduced theories - $AdS_2 \times S^2$

The Pohlmeyer reduced $AdS_2 \times S^2$ theory has a $\mathcal{N} = 2$ susy

$$\mathfrak{so}(1,1) \Subset ([\mathfrak{psu}(1|1)]^2 \ltimes \mathbb{R}^2) \tag{55}$$

- The algebra in the brackets is precisely $\hat{\mathfrak{f}}^{\perp}$.
- The projection to subalgebra of $\hat{\mathfrak{f}} = \mathfrak{psu}(1,1|2)$ defined by the constant matrix T.
- The reduction procedure: different grades of the algebra under the Lorentz group

$$\mathbb{R}^2 \;:\qquad [\mathfrak{P}_+] = 1 \qquad [\mathfrak{P}_-] = -1 \tag{56}$$

$$\mathfrak{psu}(1|1) \;:\qquad [\mathfrak{Q}_R] = \frac{1}{2} \qquad [\mathfrak{Q}_L] = -\frac{1}{2} \tag{57}$$

Digression: complex sine-Gordon model

- First perturbative study of S-matrix – local counterterms were required at one-loop to restore satisfaction of the Yang-Baxter equation.

  de Vega and Maillet, 1981

- Semiclassical corrections to soliton masses, and conjecture of exact quantum spectrum.

  de Vega and Maillet, 1983

- Formulation of complex sine-Gordon as a $SU(2)/U(1)$ gauged WZW model plus integrable potential.

  Bakas, Park, Shin, 1994

- Conjecture of full quantum S-matrix for soliton scattering.

Dorey and Hollowood, 1994

- Special points when $k \in \mathbb{N}$ – evidence that gauged WZW may play an important role.
- Consider functional determinant that arises from solving the delta-function in the path integral.
- This functional determinant gives rise to local counterterms that precisely restore the satisfaction of the Yang-Baxter equation and match the expansion of the Dorey/Hollowood S-matrix.

Hoare, AAT, 2010

Pohlmeyer reduction of superstring theory on $AdS_3 \times S^3$

- Like in complex sine-Gordon one-loop S-matrix does not satisfy Yang-Baxter.
- Can be restored by the addition of local counterterms.
- Group factorisation is also restored.
- Counterterms can be derived from a functional determinant, but a more complicated one than that arising from solving the delta-function.
- Suggests there may be an alternative formulation of the action that is more symmetric with bosons and fermions.

Pohlmeyer reduction of superstring theory on $AdS_5 \times S^5$ –

- No counterterms required to restore group-factorisation.
- The functional determinant identified in the $AdS_3 \times S^3$ case gives vanishing correction when extended to the $AdS_5 \times S^5$ case.

Reduced theory for $AdS_3 \times S^3$

Drop down to $AdS_3 \times S^3$ – easier to identify supersymmetry

Analogous to the $AdS_5 \times S^5$ case –

- $a, \dot{a}, \alpha, \dot{\alpha}$ are vector $SO(2)$ indices
- The fields $Y_{a\dot{a}}$, $Z_{\alpha\dot{\alpha}}$, $\zeta_{a\dot{\alpha}}$, $\chi_{\alpha\dot{a}}$ satisfy a constraint to reduce the number of degrees of freedom to $4 + 4$

$$Y_{a\dot{a}} = \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}Y_{b\dot{b}} \qquad \text{etc.} \tag{58}$$

- The fields can be again packaged into a single field $\Phi_{A\dot{A}}$
- With the addition of contribution of from local counterterms restoring for Yang-Baxter the one-loop perturbative S-matrix factorises

$$S_{A\dot{A},B\dot{B}}^{C\dot{C},D\dot{D}}(\theta) = (-1)^{[B][\dot{A}]+[D][\dot{C}]} S_{AB}^{CD}(\theta) S_{\dot{A}\dot{B}}^{\dot{C}\dot{D}}(\theta) \tag{59}$$

S-matrix of reduced $AdS_3 \times S^3$ theory

$$S_{AB}^{CD} = \left\{ \begin{array}{l} L_1(\theta,k)\delta_{ac}\delta_{bd} + L_2(\theta,k)\epsilon_{ac}\epsilon_{bd}\,, \\ L_3(\theta,k)\delta_{\alpha\gamma}\delta_{\beta\delta} + L_4(\theta,k)\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\,, \\ L_5(\theta,k)\delta_{ac}\delta_{\beta\delta} + L_6(\theta,k)\epsilon_{ac}\epsilon_{\beta\delta}\,, \\ L_7(\theta,k)\delta_{\alpha\gamma}\delta_{bd} + L_8(\theta,k)\epsilon_{\alpha\gamma}\epsilon_{bd}\,, \\ L_9(\theta,k)\big(\delta_{ab}\delta_{\gamma\delta} + \epsilon_{ab}\epsilon_{\gamma\delta}\big)\,, \\ L_{10}(\theta,k)\big(\delta_{\alpha\beta}\delta_{cd} + \epsilon_{\alpha\beta}\epsilon_{cd}\big)\,, \\ L_{11}(\theta,k)\big(\delta_{ad}\delta_{\gamma\beta} + \epsilon_{ad}\epsilon_{\gamma\beta}\big)\,, \\ L_{12}(\theta,k)\big(\delta_{\alpha\delta}\delta_{cb} + \epsilon_{\alpha\delta}\epsilon_{cb}\big)\,, \end{array} \right. \tag{60}$$

$$L_i(\theta,k) = p_{0_3}(\theta,k)\ \hat{L}_i(\theta,k) \tag{61}$$

$$p_{0_3} = 1 + \frac{\pi\,\mathrm{cosech}\,\theta}{2k^2}\Big(i\big[2 + (i\pi - 2\theta)\coth\theta\big] - \pi\,\mathrm{cosech}\,\theta\Big) \tag{62}$$

$$\hat{L}_1(\theta,k)=\hat{L}_3(\theta,-k)=1-\frac{i\pi}{k}\operatorname{cosech}\theta-\frac{\pi^2}{2k^2}+\mathcal{O}(k^{-3}) \tag{63}$$

$$\hat{L}_2(\theta,k)=\hat{L}_4(\theta,-k)=\frac{i\pi}{k}\coth\theta-\frac{i\pi}{k^2}\coth\theta-\frac{i\pi}{2k^2}(i\pi-2\theta)(\operatorname{cosech}\theta)^2 \tag{64}$$

$$+\frac{\pi^2}{2k^2}\coth\theta\operatorname{cosech}\theta+\mathcal{O}(k^{-3}) \tag{65}$$

$$\hat{L}_5(\theta,k)=\hat{L}_7(\theta,-k)=1+\mathcal{O}(k^{-3}) \tag{66}$$

$$\hat{L}_6(\theta,k)=\hat{L}_8(\theta,-k)=-\frac{i\pi}{k^2}\coth\theta-\frac{i\pi}{2k^2}(i\pi-2\theta)(\operatorname{cosech}\theta)^2 \tag{67}$$

$$+\frac{\pi^2}{2k^2}\coth\theta\operatorname{cosech}\theta+\mathcal{O}(k^{-3}) \tag{68}$$

$$\hat{L}_9(\theta,k)=-\hat{L}_{10}(\theta,-k)=\frac{i\pi}{2k}\operatorname{sech}\frac{\theta}{2}+\mathcal{O}(k^{-3})\,, \tag{69}$$

$$\hat{L}_{11}(\theta,k)=-\hat{L}_{12}(\theta,-k)=-\frac{i\pi}{2k}\operatorname{cosech}\frac{\theta}{2}+\mathcal{O}(k^{-3}) \tag{70}$$

Supersymmetry of the reduced theories - $AdS_3 \times S^3$

bosonic symmetry of the Pohlmeyer reduced $AdS_3 \times S^3$ theory

$$\mathfrak{so}(1,1) \in (\mathfrak{u}(1)^3 \oplus \mathbb{R}^2) \qquad (71)$$

This is the bosonic subgroup of the full perpendicular algebra

$$\hat{\mathfrak{f}}^{\perp} = \mathfrak{so}(1,1) \in ([\mathfrak{u}(1) \in \mathfrak{psu}(1|1)]^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^2) \qquad (72)$$

By analogy with $AdS_2 \times S^2$ conjecture this to be the symmetry of the PR $AdS_3 \times S^3$ theory.
For invariance of one-loop perturbative S-matrix require quantum deformation.

$AdS_3 \times S^3$ supersymmetry – classical algebra

- Consider "half" the algebra

$$[\mathfrak{u}(1) \Subset \mathfrak{psu}(1|1)] \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^2 \tag{73}$$

- This is the symmetry that should act on the factorised S-matrix
- Central extensions act in the same way on both factors
- Classical algebra

$$[\mathfrak{R}, \mathfrak{R}] = 0, \qquad [\mathfrak{L}, \mathfrak{L}] = 0, \tag{74}$$
$$[\mathfrak{R}, \mathfrak{Q}_{\pm\mp}] = \pm i\mathfrak{Q}_{\pm\mp}, \qquad [\mathfrak{L}, \mathfrak{Q}_{\pm\mp}] = \mp i\mathfrak{Q}_{\pm\mp}, \tag{75}$$
$$[\mathfrak{R}, \mathfrak{S}_{\pm\mp}] = \pm i\mathfrak{S}_{\pm\mp}, \qquad [\mathfrak{L}, \mathfrak{S}_{\pm\mp}] = \mp i\mathfrak{S}_{\pm\mp}, \tag{76}$$
$$\{\mathfrak{S}_{\pm\mp}, \mathfrak{Q}_{\pm\mp}\} = 0, \qquad \{\mathfrak{S}_{\pm\mp}, \mathfrak{Q}_{\mp\pm}\} = \pm\frac{i}{2}(\mathfrak{R} + \mathfrak{L}) \equiv \pm\mathfrak{A}, \tag{77}$$
$$\{\mathfrak{Q}_{\pm\mp}, \mathfrak{Q}_{\pm\mp}\} = 0, \qquad \{\mathfrak{Q}_{\pm\mp}, \mathfrak{Q}_{\mp\pm}\} = -\mathfrak{P}_+, \tag{78}$$
$$\{\mathfrak{S}_{\pm\mp}, \mathfrak{S}_{\pm\mp}\} = 0, \qquad \{\mathfrak{S}_{\pm\mp}, \mathfrak{S}_{\mp\pm}\} = \mathfrak{P}_- . \tag{79}$$

$AdS_3 \times S^3$ q-deformed supersymmetry

- Bosonic subalgebra is abelian
- $U(1) \times U(1)$ invariant factorised S-matrix satisfies the Yang-Baxter equation
- Therefore the quantum deformation of the supersymmetry should affect only

$$\{\mathfrak{S}_{\pm\mp}, \mathfrak{Q}_{\mp\pm}\} = \pm\mathfrak{A} \tag{80}$$

- This is deformed to

$$\{\mathfrak{S}_{\pm\mp}, \mathfrak{Q}_{\mp\pm}\} = \pm[\mathfrak{A}]_q \tag{81}$$

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \qquad [0]_q = 0, \; [1]_q = 1 \tag{82}$$

A quantum deformed $\mathcal{N} = 4$ 2-d spacetime supersymmetry, with a $U(1)^3$ bosonic R-symmetry.

Hoare, AAT, 2010

$AdS_3 \times S^3$ q-deformed supersymmetry – coproduct

Usual Leibniz coproduct tells us the action of the symmetry on the two-particle states

$$\Delta(\mathfrak{J}) = \mathbb{I} \otimes \mathfrak{J} + \mathfrak{J} \otimes \mathbb{I} \tag{83}$$

Coproduct should respect the commutation relations – if we deform the algebra we need to deform the coproduct

- Bosonic generators (including momenta) have usual coproduct
- Fermionic generators

$$\Delta(\mathfrak{Q}_{\pm\mp}) = \mathfrak{Q}_{\pm\mp} \otimes q^{-\mathfrak{A}} + \mathbb{I} \otimes \mathfrak{Q}_{\pm\mp}\,, \tag{84}$$

$$\Delta(\mathfrak{S}_{\pm\mp}) = \mathfrak{S}_{\pm\mp} \otimes \mathbb{I} + q^{\mathfrak{A}} \otimes \mathfrak{S}_{\pm\mp}\,, \tag{85}$$

$AdS_3 \times S^3$ – invariance of perturbative S-matrix

The perturbative PR $AdS_3 \times S^3$ S-matrix is invariant under the q-deformed supersymmetry for

$$q = 1 - \frac{2i\pi}{k} - \frac{2\pi^2}{k^2} + \dots \quad \left(= e^{-\frac{2i\pi}{k}}\right) \tag{86}$$

Assuming the quantum deformed supersymmetry is exact can conjecture an exact S-matrix for the Pohlmeyer reduced $AdS_3 \times S^3$

- Phase factor is fixed by unitarity, crossing and matching with the perturbative computation.
- S-matrix satisfies the Yang-Baxter equation.
- The perturbative expansion of the S-matrix agrees with the one-loop computation

$AdS_3 \times S^3$ exact S-matrix conjecture

$$L_{1,3}(\theta,k) = \frac{1}{2}\Big[P_1(\theta,k)\,\mathsf{cosh}\left(\frac{\theta}{2} \pm \frac{i\pi}{k}\right)\mathrm{sech}\,\frac{\theta}{2} + P_2(\theta,k)\,\mathsf{sinh}\left(\frac{\theta}{2} \mp \frac{i\pi}{k}\right)\mathrm{cosech}\,\frac{\theta}{2}\Big] \tag{87}$$

$$L_{2,4}(\theta,k) = \frac{1}{2}\Big[P_1(\theta,k)\,\mathsf{cosh}\left(\frac{\theta}{2} \pm \frac{i\pi}{k}\right)\mathrm{sech}\,\frac{\theta}{2} - P_2(\theta,k)\,\mathsf{sinh}\left(\frac{\theta}{2} \mp \frac{i\pi}{k}\right)\mathrm{cosech}\,\frac{\theta}{2}\Big] \tag{88}$$

$$L_{5,7}(\theta,k) = \frac{1}{2}\Big[P_1(\theta,k) + P_2(\theta,k)\Big], \qquad L_{6,8}(\theta,k) = \frac{1}{2}\Big[P_1(\theta,k) - P_2(\theta,k)\Big] \tag{89}$$

$$L_{9,10}(\theta,k) = \frac{i}{2}P_1(\theta,k)\,\mathsf{sin}\,\frac{\pi}{k}\,\mathrm{sech}\,\frac{\theta}{2}, \qquad L_{11,12}(\theta,k) = -\frac{i}{2}P_2(\theta,k)\,\mathsf{sin}\,\frac{\pi}{k}\,\mathrm{cosech}\,\frac{\theta}{2} \tag{90}$$

$$P_1(\theta,k) = \sqrt{\frac{\mathsf{cosh}\left(\frac{\theta}{2} + \frac{i\pi}{k}\right)}{\mathsf{cosh}\left(\frac{\theta}{2} - \frac{i\pi}{k}\right)}} \prod_{l=1}^{\infty} \frac{\Gamma(\frac{i\theta}{2\pi} - \frac{1}{k} + l - \frac{1}{2})\Gamma(\frac{i\theta}{2\pi} + \frac{1}{k} + l + \frac{1}{2})}{\Gamma(-\frac{i\theta}{2\pi} - \frac{1}{k} + l - \frac{1}{2})\Gamma(-\frac{i\theta}{2\pi} + \frac{1}{k} + l + \frac{1}{2})} \tag{91}$$

$$\frac{\Gamma(-\frac{i\theta}{2\pi} + l - \frac{1}{2})\Gamma(-\frac{i\theta}{2\pi} + l + \frac{1}{2})}{\Gamma(\frac{i\theta}{2\pi} + l - \frac{1}{2})\Gamma(\frac{i\theta}{2\pi} + l + \frac{1}{2})} \tag{92}$$

$$P_2(\theta,k) = P_1(i\pi - \theta)) \tag{93}$$

Hoare, AAT, 2011

Summary of supersymmetry

PR $AdS_2 \times S^2$:

$$\hat{\mathfrak{f}}^{\perp} = \mathfrak{so}(1,1) \ni ([\mathfrak{psu}(1|1)]^2 \ltimes \mathbb{R}^2) \qquad (94)$$

- There is nothing to quantum deform!

PR $AdS_3 \times S^3$:

$$\hat{\mathfrak{f}}^{\perp} = \mathfrak{so}(1,1) \ni ([\mathfrak{u}(1) \ni \mathfrak{psu}(1|1)]^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^2) \qquad (95)$$

- For Lorentz invariant S-matrix matching perturbative result need to quantum-deform fermionic part of this symmetry

PR $AdS_5 \times S^5$:

$$\hat{\mathfrak{f}}^{\perp} = \mathfrak{so}(1,1) \ni ([\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}^2) \qquad (96)$$

- There a relativistic S-matrix with a q-deformed symmetry that satisfies Yang-Baxter!

Given by $g \to \infty$ limit of q-deformed $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}^3$ R-matrix constructed by Beisert and Koroteev.

Beisert, Koroteev, 2008
Beisert, 2010
Hoare, AAT, 2011

Similarities to the perturbative S-matrix.

However, the bosonic subgroup is non-abelian

– the quantum deformation affects it non-trivially

– lack of manifest H symmetry in S-matrix.

- Same story for non-abelian bosonic theories – conjectured that the physical S-matrix for the solitons is the q-deformed one

Hollowood, Miramontes, 2009

How to match with perturbative computation?

Open question

- What excitations are we scattering? Are they physical? Solitons?
- Maybe S-matrix for physical excitations is related by non-unitarity rotation arising from subtlety in solving for the unphysical field ξ.

Further evidence that $\hat{\mathfrak{f}}^{\perp}$ is the symmetry algebra for these theories comes from study of solitons in the PR $AdS_5 \times S^5$ theory.

Hollowood, Miramontes, 2010, 2011

- recent work of Hoare, Hollowood, Miramontes: – fusion procedure with the q-deformed S-matrix supports the mass spectrum of solitons computed by Hollowood and Miramontes.

Conclusions and open questions

- Important to understand perturbation theory in order to test conjectures for exact results.

Bosonic models with abelian H, e.g. complex sine-Gordon –

- The perturbative S-matrix plus the contribution from the functional determinant satisfies Yang-Baxter and agrees with the conjectured exact result.

de Vega, Maillet, 1981, 1983
Dorey, Hollowood, 1994
Hoare, AAT, 2010

- gauged WZW plus potential correct way of defining the integrable theory (extensions beyond one-loop?)

- What is the origin of the quantum-deformation from Lagrangian viewpoint?
- Relation of q-deformed supersymmetry of S-matrix to (non-local) supersymmetry of the reduced theory action?

Goyhman, Ivanov, 2011
Hollowood, Miramontes, 2011

- Study of the solitons supports q-deformation.

Hollowood, Miramontes, 2009-2011

- Relation between quantum $AdS_5 \times S^5$ string theory and quantum reduced theory?
- k vs. $\sqrt{\lambda}$?
- classical equivalence; one-loop partition functions match
- 2-loop partition functions for infinite spin limit of folded string are closely related: Catalan's constant term matches, but there are extra $(\ln 2)^2$ terms

suggests $\ln Z_2 = \ln Z_2^{(string)} + a(\ln Z_1^{(string)})^2$

Iwashita, Roiban, AT, to appear

$$\mathcal{S}\,|\phi_1\phi_1\rangle = \big(J_1 + J_2\big)\,|\phi_1\phi_1\rangle \tag{97}$$

$$\mathcal{S}\,|\phi_1\phi_2\rangle = J_1 \sec\frac{\pi}{k}\,|\phi_1\phi_2\rangle + \big(J_2 - iJ_1 \tan\frac{\pi}{k}\big)\,|\phi_2\phi_1\rangle \tag{98}$$

$$- J_5 \sec\frac{\pi}{k}\,|\psi_3\psi_4\rangle + J_5(1 + i\tan\frac{\pi}{k})\,|\psi_4\psi_3\rangle \tag{99}$$

$$\mathcal{S}\,|\phi_2\phi_1\rangle = J_1 \sec\frac{\pi}{k}\,|\phi_2\phi_1\rangle + \big(J_2 + iJ_1 \tan\frac{\pi}{k}\big)\,|\phi_1\phi_2\rangle \tag{100}$$

$$- J_5 \sec\frac{\pi}{k}\,|\psi_4\psi_3\rangle + J_5(1 - i\tan\frac{\pi}{k})\,|\psi_3\psi_4\rangle \tag{101}$$

$$\mathcal{S}\,|\phi_2\phi_2\rangle = \big(J_1 + J_2\big)\,|\phi_2\phi_2\rangle \tag{102}$$

$$\mathcal{S}\,|\psi_3\psi_3\rangle = \big(J_3 + J_4\big)\,|\psi_3\psi_3\rangle \tag{103}$$

$$\mathcal{S}\,|\psi_3\psi_4\rangle = J_3 \sec\frac{\pi}{k}\,|\psi_3\psi_4\rangle + \big(J_4 - iJ_3 \tan\frac{\pi}{k}\big)\,|\psi_4\psi_3\rangle \tag{104}$$

$$- J_6 \sec\frac{\pi}{k}\,|\phi_1\phi_2\rangle + J_6(1 + i\tan\frac{\pi}{k})\,|\phi_2\phi_1\rangle \tag{105}$$

$$\mathcal{S}\,|\psi_4\psi_3\rangle = J_3 \sec\frac{\pi}{k}\,|\psi_4\psi_3\rangle + \big(J_4 + iJ_3 \tan\frac{\pi}{k}\big)\,|\psi_3\psi_4\rangle \tag{106}$$

$$- J_6 \sec\frac{\pi}{k}\,|\phi_2\phi_1\rangle + J_6(1 - i\tan\frac{\pi}{k})\,|\phi_1\phi_2\rangle \tag{107}$$

$$\mathcal{S}\,|\psi_4\psi_4\rangle = \big(J_3 + J_4\big)\,|\psi_4\psi_4\rangle \tag{108}$$

$$\mathcal{S}\,\big|\phi_a\psi_\beta\big\rangle = J_7\ \delta_a^d\delta_\beta^\gamma\,|\psi_\gamma\phi_d\rangle + J_9\ \delta_a^c\delta_\beta^\delta\,|\phi_c\psi_\delta\rangle \tag{109}$$

$$\mathcal{S}\,|\psi_\alpha\phi_b\rangle = J_8\ \delta_\alpha^\delta\delta_b^c\,|\phi_c\psi_\delta\rangle + J_{10}\ \delta_\alpha^\gamma\delta_b^d\,|\psi_\gamma\phi_d\rangle \tag{110}$$

$$J_{1,3}(\theta,k) = P_0(\theta,k)\cos\frac{\pi}{k}\,\mathrm{sech}\,\frac{\theta}{2}\cosh\left(\frac{\theta}{2}\pm\frac{i\pi}{2k}\right) \tag{111}$$

$$J_{2,4}(\theta,k) = \mp iP_0(\theta,k)\left[1-\cos\frac{\pi}{k}+\cosh\theta+\cosh\left(\theta\pm\frac{i\pi}{k}\right)\right]\sin\frac{\pi}{2k}\,\mathrm{cosech}\,\theta \tag{112}$$

$$J_{5,6}(\theta,k) = -iP_0(\theta,k)\cos\frac{\pi}{k}\sin\frac{\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2} \tag{113}$$

$$J_{7,8}(\theta,k) = -iP_0(\theta,k)\sin\frac{\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2} \tag{114}$$

$$J_{9,10}(\theta,k) = P_0(\theta,k) \tag{115}$$

$$P_0(\theta,k) = \sqrt{\frac{\sinh\theta - i\sin\frac{\pi}{k}}{\sinh\theta + i\sin\frac{\pi}{k}}}\;Y(\theta,k)\;Y(i\pi-\theta,k) \tag{116}$$

$$Y(\theta,k) = \prod_{l=1}^{\infty}\frac{\Gamma\left(\frac{1}{2k}-\frac{i\theta}{2\pi}+l\right)\Gamma\left(-\frac{1}{2k}-\frac{i\theta}{2\pi}+l-1\right)}{\Gamma\left(\frac{1}{2k}-\frac{i\theta}{2\pi}+l+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2k}-\frac{i\theta}{2\pi}+l-\frac{1}{2}\right)} \tag{117}$$

$$\frac{\Gamma\left(-\frac{i\theta}{2\pi}+l-\frac{1}{2}\right)\Gamma\left(-\frac{i\theta}{2\pi}+l+\frac{1}{2}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}+l-1\right)\Gamma\left(-\frac{i\theta}{2\pi}+l\right)} \tag{118}$$

Hoare, AAT, 2011
Beisert, Koroteev, 2008
Beisert, 2010