New Aspects of (Twisted) SCQM

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Based on:

O. Lechtenfeld, FT, arXiv:1105.4067 (N=8, 2D)
L. Baulieu, FT, (twisted) in preparation

The $1D$ $\mathcal{N}$-Extended Superalgebra, with $\mathcal{N}$ odd generators Q_I ($I = 1, 2, \ldots, \mathcal{N}$) and a single even generator H satisfying the (anti)-commutation relations

$$\begin{aligned} \{Q_I, Q_J\} &= \delta_{IJ} H, \\ [H, Q_I] &= 0, \end{aligned}$$

The *minimal* linear representations (also called *irreducible supermultiplets*) are given by the minimal number n_{min} of bosonic (fermionic) fields for a given value of $\mathcal{N}$.

$$\begin{aligned} \mathcal{N} &= 8l + m, \\ n_{min} &= 2^{4l} G(m), \end{aligned}$$

where $l = 0, 1, 2, \ldots$ and $m = 1, 2, 3, 4, 5, 6, 7, 8$.

$G(m)$ appearing in (1) is the Radon-Hurwitz function

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

The maximal *finite* number n_{max} of bosonic (fermionic) fields entering a non-minimal represesentation

$$n_{max} = 2^{\mathcal{N}-1}.$$

D-module reps of N-extended 1D susy:

minimal linear irreps: admissible field contents:
$(n_1, n_2, n_3, \ldots)$
Pashnev-FT, Kuznetsova-Rojas-FT.

graphs (adinkras) Gates-Faux

admissible connectivities, Kuznetsova-FT.

Nonminimal (indecomposable) linear reps. Gonzales-Khodaee-FT.

Nonlinear reps.

Inhomogeneous reps.

Minimal linear supermultiplets of extended supersymmetry in one dimension are usually formulated with homogeneous transformations for their component fields. In some selected cases it is possible to add an inhomogeneous term. This is admissible at

- $N=2$ for the supermultiplet $(0,2,2)$
- $N=4$ for the supermultiplets $(0,4,4)$ and $(1,4,3)$
- $N=8$ for the supermultiplets $(0,8,8)$, $(1,8,7)$, $(2,8,6)$ and $(3,8,5)$

The remaining $N = 2,4,8$ supermultiplets do not admit an inhomogeneous extension.

For $(1,4,3)$: Ivanov-Krivonos-Pashnev 1991.

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Examples:

(0,4,4). For the $N{=}4$ $(0,4,4)$ multiplet, we have ($i,j,k = 1,2,3$, $\epsilon_{123} = 1$)

$$\begin{aligned} &Q_0\psi = g \ , && Q_0\psi_j = g_j \ , \\ &Q_0 g = \dot\psi \ , && Q_0 g_j = \dot\psi_j \ , \\ &Q_i\psi = g_i \ , && Q_i\psi_j = -\delta_{ij} g + \epsilon_{ijk}\tilde g_k \ , \\ &Q_i g = -\dot\psi_i \ , && Q_i g_j = \delta_{ij}\dot\psi - \epsilon_{ijk}\dot\psi_k \ , \end{aligned}$$

and we may choose

$$\tilde g_1 = g_1 \ , \quad \tilde g_2 = g_2 \quad \text{but} \quad \tilde g_3 = g_3 + c \ .$$

(1,4,3). $N{=}4$ $(1,4,3)$:

$$\begin{aligned} &Q_0 x = \psi \ , && Q_0\psi = \dot x \ , \\ &Q_0\psi_j = g_j \ , && Q_0 g_j = \dot\psi_j \ , \\ &Q_i x = \psi_i \ , && Q_i\psi = -g_i \ , \\ &Q_i\psi_j = \delta_{ij}\dot x + \epsilon_{ijk}\tilde g_k \ , && Q_i g_j = -\delta_{ij}\dot\psi - \epsilon_{ijk}\dot\psi_k \ , \end{aligned}$$

with the same $\tilde g_k$ as in (0,4,4).

A special case of N=8 (2,8,6) sigma-model (Bellucci-Krivonos-Nersessian-Schcherbakov) is a 2D N=8 SCQM.

Gonzales-Rojas-FT: for $N \geq 4$ and the presence of at least one physical boson one can set (manifest N=4)

$$\mathcal{S} = \int dt\mathcal{L} = \int dt\; Q_1Q_2Q_3Q_4\, F(x,y,\ldots)\;,$$

where $F(x,y,\ldots)$ is an unconstrained prepotential.

N=8 constraint:

$$Q_l\mathcal{L} = \partial_t W_l \qquad \text{for} \quad l = 5,6,7,8$$

imposes constraints on F.

In order to obtain scale invariance, the action should not contain any dimensionful coupling parameter, and therefore, due to $[Q_i] = \frac{1}{2}$, we demand that $[F] = -1$.

The inhomogeneous term gives rise to a Calogero-type potential.

The action may be complemented by the addition of a Fayet-Iliopoulos term

$$\mathcal{S}_{FI} = \int dt \sum_i (q_i g_i + r_i f_i) \qquad \text{with} \quad [q_i] = [r_i] = 1$$

FI $\Rightarrow$ the DFF oscillatorial damping of conformal mechanics.

Action $S \sim \frac{1}{x}(\dot{x})^2 + \ldots$.

Usual action recovered with the coordinate change $x = \frac{1}{2}w^2$.

Example: N=4 (1,4,3) multiplet (only x and g_i, no y or f_i), the proper choice for the prepotential is

$$
\begin{aligned}
F(x) &= \frac{1}{4} x \ln x \qquad \longrightarrow \\
\mathcal{L} + \mathcal{L}_{FI} &= F''(x)(\dot{x}^2 + g_i^2 \ + \ c\, g_3) + q_i g_i + \\
&\quad + fermions.
\end{aligned}
$$

After eliminating the auxiliary components g_i via their equations of motion and putting the fermions to zero, one gets

$$
\begin{aligned}
\mathcal{L}'_{\text{bos}} &= F''(x)(\dot{x}^2 - \frac{1}{4}c^2) \ - \frac{1}{4} q_i^2 / F''(x) \ - \ \frac{1}{2} c\, q_3 \\
&= \frac{1}{4}(\dot{x}^2 - \frac{1}{4}c^2)/x \ - \ g_i^2 x \ - \ \frac{1}{2} c\, q_3 \\
&= \frac{1}{2}\dot{w}^2 - \frac{1}{8}c^2 w^{-2} \ - \ \frac{1}{2} g_i^2 w^2 \ - \ \frac{1}{2} c\, q_3 \ ,
\end{aligned}
$$

and we have recovered the standard conformal action after the coordinate change $x = \frac{1}{2} w^2$.

For N=8 (2,8,6) the prepotential $F(x,y)$ must be harmonic,

$$\Delta F \equiv \ F_{xx} + F_{yy} = 0.$$

The general solution is encoded in a meromorphic function $H(z)$ via

$$F(x,y) \ = \ H(z) + \overline{H(z)}$$

The bosonic metric $g_{z\bar{z}} = H_{zz} + \bar{H}_{\bar{z}\bar{z}}$ is special Kähler of rigid type (Fre).

The harmonic prepotential with the correct scaling dimension $[H] = -1$ is

$$H(z) \ = \ \frac{1}{8} z \ln z$$

The Lagrangian is determined by

$$\Phi \ = \ \frac{1}{4}\frac{x}{x^2+y^2}$$
$$\widetilde{\Phi} \ = \ \frac{1}{4}\frac{y}{x^2+y^2}$$

In the bosonic limit, obtained by setting all fermions equal to zero, we obtain

$$\mathcal{L}_{\text{bos}} + \mathcal{L}_{\text{FI}} \ = \ \Phi\,(\dot{x}^2 + \dot{y}^2 + {g_i}^2 + {f_i}^2) + c\,(\Phi\,g_3 + \widetilde{\Phi} f_3) + q_i g_i + r_i f_i \ .$$

After eliminating the auxiliary fields via their algebraic equations of motion, we get

$$
\begin{aligned}
\mathcal{L}'_{\text{bos}} &= \frac{x}{x^2+y^2}\frac{\dot{x}^2+\dot{y}^2}{4} \;-\; \frac{(q_i^2+r_i^2)(x^2+y^2)}{x} \;- \\
&\quad c\,\frac{q_3 x+r_3 y}{2x} \;-\; \frac{c^2}{16x} \\
&=: \; K \;-\; V \;,
\end{aligned}
$$

Setting $x = \frac{1}{2}w^2$

$$
\begin{aligned}
\mathcal{L}'_{\text{bos}} &= \frac{1}{2}(1+\gamma^2)^{-1}(\dot{w}^2+\frac{\dot{y}^2}{w^2}) \;-\; \frac{1}{2}(1+\gamma^2)(q_i^2+r_i^2) \\
&\quad -\frac{1}{2}\,c\,(q_3+r_3\gamma) \;-\;-\frac{c^2}{8w^2}\;,
\end{aligned}
$$

where $\gamma = 2y/w^2$.

D-module reps of SCA's: example $sl(2|1)$ for $(1,2,1)$ multiplet:

$$
\begin{aligned}
H &= \begin{pmatrix} \partial_t & 0 & 0 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & \partial_t \end{pmatrix}, \\
W &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
D &= \begin{pmatrix} t\partial_t - \lambda & 0 & 0 & 0 \\ 0 & t\partial_t + 1 - \lambda & 0 & 0 \\ 0 & 0 & t\partial_t + \frac{1}{2} - \lambda & 0 \\ 0 & 0 & 0 & t\partial_t + \frac{1}{2} - \lambda \end{pmatrix}, \\
K &= \begin{pmatrix} -t^2\partial_t + 2\lambda t & 0 & 0 & 0 \\ 0 & -t^2\partial_t + (2\lambda - 2)t & 0 & 0 \\ 0 & 0 & -t^2\partial_t + (2\lambda - 1)t & 0 \\ 0 & 0 & 0 & -t^2\partial_t + (\end{pmatrix} \\
Q_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \partial_t \\ \partial_t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
Q_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\partial_t & 0 \\ 0 & -1 & 0 & 0 \\ \partial_t & 0 & 0 & 0 \end{pmatrix}, \\
\widetilde{Q}_1 &= \begin{pmatrix} 0 & 0 & t & 0 \\ 0 & 0 & 0 & t\partial_t - 2\lambda + 1 \\ t\partial_t - 2\lambda & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix}, \\
\widetilde{Q}_2 &= \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & -t\partial_t + 2\lambda - 1 & 0 \\ 0 & -t & 0 & 0 \\ t\partial_t - 2\lambda & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

The twisted N=2 SCA:

$$
\begin{aligned}
Q &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \bar{Q} &= \\
Q_V &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \partial_t \\ \partial_t & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \bar{Q}_V &= \\
Q_C &= \begin{pmatrix} 0 & 0 & t & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ -\lambda & t & 0 & 0 \end{pmatrix}, & \bar{Q}_C &= \\
H &= \begin{pmatrix} \partial_t & 0 & 0 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & \partial_t \end{pmatrix}, & c &= \\
S &= \begin{pmatrix} t\partial_t - \lambda & 0 & 0 & 0 \\ 0 & t\partial_t + 1 - \lambda & 0 & 0 \\ 0 & 0 & t\partial_t + 1 - \lambda & 0 \\ 0 & 0 & 0 & t\partial_t - \lambda \end{pmatrix}, & \bar{S} &= \\
Z &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \bar{Z} &=
\end{aligned}
$$

Common subalgebra: the $sl(2|1)$ Borel subalgebra:

$$\begin{aligned}\mathcal{B} &= \{D=\frac{1}{2}(S-\bar{S}), \quad H, \quad W=Z+\bar{Z}, \\ &\qquad Q_1=Q+Q_V-\bar{Q}, \quad Q_2=\bar{Q}-\bar{Q}_V-Q\}.\end{aligned}$$

Ordinary $N=2$ SCA:$(Y(t);\xi_1(t),\xi_2(t);g(t))$
Twisted $N=2$ SCA: $(X(t);\Psi(t),\bar{\Psi}(t);b(t))$.

With a convenient normalization we can present the free Lagrangians as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\dot{Y}^2+g^2-\xi_1\dot{\xi}_1-\xi_2\dot{\xi}_2\right), \\ \mathcal{L}^\sharp &= b^2-b\dot{X}+\bar{\Psi}\dot{\Psi}.\end{aligned}$$

In order to identify them ($\mathcal{L}^\sharp=\mathcal{L}$), we have to provide the invertible "twist transformation" T linking the two sets of fields. We have

$$\begin{aligned}X &= \sqrt{2}iY, \\ b &= \frac{1}{\sqrt{2}}(g+i\dot{Y}), \\ \Psi &= \frac{1}{\sqrt{2}}(\xi_1-\xi_2), \\ \bar{\Psi} &= \frac{i}{\sqrt{2}}(\xi_1+\xi_2).\end{aligned}$$

BRST cohomologies:

The $N = 2$ free Lagrangian is invariant under the 6 nilpotent fermionic generators, provided that $\lambda = \frac{1}{2}$.

$$
\begin{aligned}
\int dt\mathcal{L} &= \int dt\left(b^\mu\eta_{\mu\nu}b^\nu - b^\mu\eta_{\mu\nu}\dot{X}^\nu + \bar{\Psi}^\mu\dot{\Psi}^\nu\right)\\
&= \int dtQ\left(\bar{\Psi}^\mu\eta_{\mu\nu}(b^\nu - \dot{X}^\nu)\right) = -\int dt\bar{Q}\left(\Psi^\mu\eta_{\mu\nu}(b^\nu - \dot{X}^\nu)\right)\\
&= \int dtQQ_V\left(\bar{\Psi}^\mu\eta_{\mu\nu}\Psi^\nu\right) = -\int dt\bar{Q}\bar{Q}_V\left(\Psi^\mu\eta_{\mu\nu}\bar{\Psi}^\nu\right)\\
&= \int dtQ_C\bar{Q}\left(\bar{\Psi}^\mu\eta_{\mu\nu}\dot{\Psi}^\nu\right) = -\int dt\bar{Q}_CQ\left(\Psi^\mu\eta_{\mu\nu}\dot{\bar{\Psi}}^\nu\right)\\
&= \int dtQ_CQ_V\left(\frac{1}{2}b^\mu\eta_{\mu\nu}b^\nu\right) \quad = -\int dt\bar{Q}_C\bar{Q}_V\left(\frac{1}{2}b^\mu\eta_{\mu\nu}b^\nu\right).
\end{aligned}
$$

Interactions:

the supersymmetric interaction is introduced in term of the "prepotential" $W[X^\mu]$.
A manifest Q-invariant term can be added to the action by setting

$$\mathcal{L}_{int} = Q\left(\bar{\Psi}^\mu \frac{\delta W}{\delta X^\mu}\right) = b^\mu \frac{\delta W}{\delta X^\mu} - \bar{\Psi}^\mu \frac{\delta^2 W}{\delta X^\mu \delta X^\nu} \Psi^\nu .$$

The Q, Q_V, $\bar{Q}$ and $\bar{Q}_V$ invariances, modulo a time derivative, of $\mathcal{L}_{int}$ are warranted because

$$\mathcal{L}_{int} = Q\bar{Q}\,(W) = QQ_V(W) = -\bar{Q}\bar{Q}_V\,(W)\,.$$

The Q_C, $\bar{Q}_C$ invariances, modulo a time derivative, of $\mathcal{L}_{int}$ imply the following condition on the prepotential:

$$\Psi^\mu \frac{\partial W}{\partial X^\nu} - X^\nu \frac{\partial^2 W}{\partial X^\mu \partial X^\nu} \Psi^\nu = 0 \;\Rightarrow\; \frac{\partial}{\partial X^\rho}(X^\mu \frac{\partial W}{\partial X^\mu}) = 0.$$

Therefore, the condition for having a Q_C-invariance is

$$X^\mu \frac{\partial W}{\partial X^\mu} \;=\; C,$$

whose general solution is

$$W \;=\; C \ln R + f\left(\frac{X^\mu}{R}\right).$$

C is an arbitrary constant and f is an arbitrary function of the non-dimensional quantities $\frac{X^\mu}{R}$, where $R^2 \equiv X^\mu \eta_{\mu\nu} X^\nu$.

The target-space reparametrization covariant action with worldline $N=2$ supersymmetry is expressed by the Lagrangian

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}g_{\mu\nu}\dot{X}^{\nu}\dot{X}^{\nu}+\bar{\Psi}^{\mu}(g_{\mu\nu}\dot{\Psi}^{\nu}+ \\ &\quad \Gamma_{\mu,\rho\sigma}\dot{X}^{\rho}\Psi^{\nu})+\frac{1}{4}R_{\mu\nu\rho\sigma}\bar{\Psi}^{\rho}\Psi^{\sigma}\bar{\Psi}^{\nu}\Psi^{\nu}.\end{aligned}$$

The t-dependent coordinates $X^{\mu}(t)$ are bosons, while $\Psi^{\mu}(t)$ and $\bar{\Psi}^{\mu}(t)$ are fermions.

Using an auxiliary field $b^{\mu}(t)$, one can express $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L} &= g_{\mu\nu}b^{\nu}b^{\mu}+b^{\mu}(-g_{\mu\nu}\dot{X}^{\nu}+\Gamma_{[\mu,\rho]\sigma}\bar{\Psi}^{\rho}\Psi^{\sigma})+ \\ &\quad \partial_{\rho}g_{\mu\nu}\bar{\Psi}^{\mu}\Psi^{\rho}X^{\nu}+\bar{\Psi}^{\mu}(g_{\mu\nu}\dot{\Psi}^{\nu}+\Gamma_{\mu,\rho\sigma}\dot{X}^{\rho}\Psi^{\nu}).\end{aligned}$$

The general covariance in the curved target-space with coordinates X^{μ} is explicit for the action . However, such an important invariance is only enforced after the elimination from the action of the auxiliary fields b^{μ} via their algebraic equations of motion $b_{\mu}=g_{\mu\nu}\dot{X}^{\nu}-\Gamma_{[\mu,\rho]\sigma}\bar{\Psi}^{\rho}\Psi^{\sigma}$.

$$\begin{aligned}\int dtL &= QQ_V\int dt(\bar{\Psi}^{\mu}g_{\mu\nu}\Psi^{\nu}) \\ &= -\bar{Q}\bar{Q}_V\int dt(\bar{\Psi}^{\mu}g_{\mu\nu}\Psi^{\nu}).\end{aligned}$$

Very economical way to implement the $N = 2$ SCA: invariance under

$$\mathcal{G}^{\sharp}_{min} \;=\; \{Q, Q_C, c, N_{gh}\}.$$

$$N_{gh} \;:=\; S + \bar{S},$$

while

$$D \;:=\; \frac{1}{2}(S - \bar{S}) = t\frac{d}{dt} + d_s$$

contains the diagonal matrix d_s with the engineering or scaling dimension of the component fields. The ghost number and the scale dimensions are given by

	N_{gh}	d_s
X	0	$-\frac{1}{2}$
b	0	$\frac{1}{2}$
Ψ	1	0
$\bar{\Psi}$	-1	0

In particular the $sl(2)$ conformal invariance of the one-dimensional conformal quantum mechanics is obtained as a bonus:

on an arbitrary s-dimensional field $\Phi_s(t)$ (in our case $s = -\frac{1}{2}$ for X, $s = 0$ for Ψ and $\bar{\Psi}$, $s = \frac{1}{2}$ for b):

$$
\begin{aligned}
L_{-1} &= \frac{d}{dt}, \\
L_0 &= t\frac{d}{dt} + s, \\
L_1 &= -t^2\frac{d}{dt} - 2st.
\end{aligned}
$$

The non-vanishing commutators are

$$
\begin{aligned}
[L_0, L_{\pm 1}] &= \pm L_{\pm 1}, \\
[L_1, L_{-1}] &= 2L_0.
\end{aligned}
$$