Alternative relativistic supersymmetries

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Abstract

Finite-dimensional alternative ($\mathbb{Z}_2\times\mathbb{Z}_2$-graded) Lie superalgebras are discussed parallel with the finite-dimensional standard ($\mathbb{Z}_2$-graded) Lie superalgebras. We also consider applications for some of these superalgebras as the relativistic supersymmetries (SUSY). We show that every standard relativistic SUSY (super-anti de Sitter, super-Poincarè, super-conformal, etc.) has an alternative variant. We also show that alternative ($\mathbb{Z}_2\times\mathbb{Z}_2$-graded) relativistic SUSY has different algebraic structure of the superspace and the representation theory with respect to the standard ($\mathbb{Z}_2$-graded) relativistic SUSY.

Contents

Standard relativistic SUSY (super-anti de Sitter, super-Poincarè, super-conformal, extended N-supersymmetry, etc) based on usual ($\mathbb{Z}_2$-graded) Lie superalgebras ($osp(4|1)$, $su(2,2|N)$, $osp(4|N)$ etc). It turns out that every standard relativistic SUSY has an alternative variant and these variants are based on alternative ($\mathbb{Z}_2\times\mathbb{Z}_2$-graded) Lie superalgebras ($osp(2;2|1)$, $su(2,1;1|N)$, $osp(2;2|N)$ etc).

Standard relativistic SUSY $\rightleftarrows$ Alternative relativistic SUSY

Distinctive features of the standard and alternative relativistic Poincarè SUSY are related with the relations between the four-momenta and the Q-charges and also between the space-time coordinates and the Grassmann variables. Namely, we have.
(I) *For the standard Poincarè SUSY:*

$$[P_\mu, Q_\alpha] \;=\; [P_\mu, \bar{Q}_{\dot\alpha}] = 0\,, \quad \{Q_\alpha, \bar{Q}_{\dot\alpha}\} \;=\; 2\sigma^\mu_{\alpha\beta}P_\mu, \tag{1}$$

$$[x_\mu, \theta_\alpha] \;=\; [x_\mu, \dot\theta_{\dot\alpha}] \;=\; \{\theta_\alpha, \bar\theta_{\dot\alpha}\} \;=\; 0, \tag{2}$$

(II) *For the altrenative Poincarè SUSY:*

$$\{P_\mu, Q_\alpha\} \;=\; \{P_\mu, \bar{Q}_{\dot\alpha}\} = 0\,, \quad [Q_\alpha, \bar{Q}_{\dot\alpha}] \;=\; 2\sigma^\mu_{\alpha\beta}P_\mu, \tag{3}$$

$$\{x_\mu, \theta_\alpha\} \;=\; \{x_\mu, \dot\theta_{\dot\alpha}\} \;=\; [\theta_\alpha, \bar\theta_{\dot\alpha}] \;=\; 0, \tag{4}$$

The standard ($\mathbb{Z}_2$-graded) superalgebra:

$$L \;=\; \sum_{a} L_a \;=\; L_0 + L_1 \tag{1}$$

$$< x_a, y_b > = \; (-1)^{ab} < y_b, x_a > \quad (x_a \in L_a,\; y_b \in L_b)\;, \tag{2}$$

$$<x_a,<y_b,z_c>>(-1)^{ac}+<z_c,<x_a,y_b>(-1)^{cb}+<y_b,<z_c,x_a>>(-1)^{ba}=0\;, \tag{3}$$

The alternative ($\mathbb{Z}_2\times\mathbb{Z}_2$-graded) superalgebra:

$$L \;=\; \sum_{\mathbf{a}=(a_1,a_2)} L_{\mathbf{a}} \;=\; L_{(0,0)} + L_{(1,1)} + L_{(1,0)} + L_{(0,1)} \tag{4}$$

$$< x_{\mathbf{a}}, y_{\mathbf{b}} > = \; (-1)^{(\mathbf{ab})} < y_{\mathbf{b}}, x_{\mathbf{a}} > \quad (x_{\mathbf{a}} \in L_{\mathbf{a}},\; y_{\mathbf{b}} \in L_{\mathbf{b}})\;, \tag{5}$$

$$<x_{\mathbf{a}},<y_{\mathbf{b}},z_{\mathbf{c}}>>(-1)^{(\mathbf{ac})}+<z_{\mathbf{c}},<x_{\mathbf{a}},y_{\mathbf{b}}>>(-1)^{(\mathbf{cb})}+<y_{\mathbf{b}},<z_{\mathbf{c}},x_{\mathbf{a}}>>(-1)^{(\mathbf{ba})}=0\;, \tag{6}$$

where

$$(\mathbf{ab}) = a_1 b_1 + a_2 b_2. \tag{7}$$

The standard ($\mathbb{Z}_2$-graded) matrix superalgebras

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$$M = \begin{pmatrix} A_{(0)} & A_{(1)} \\ D_{(1)} & D_{(0)} \end{pmatrix} = M_{(0)} + M_{(1)} = \tag{8}$$

$$= \begin{pmatrix} A_{(0)} & 0 \\ 0 & D_{(0)} \end{pmatrix} + \begin{pmatrix} 0 & A_{(1)} \\ D_{(1)} & 0 \end{pmatrix}. \tag{9}$$

The matrix M is $(m+n)\times(m+n)$-dimensional. Here $A_{(00)}$ is $m\times m$-, $A_{(1)}$ is $m\times n$-, $D_{(1)}$ is $m\times n$-, $D_{(0)}$ is $n\times n$-dimensional.

$$< M_{(a)}, M_{(b)} > := M_{(a)}M_{(b)} - (-1)^{ab}M_{(b)}M_{(a)} = M_{(a+b)} \tag{10}$$

$$(a+b) \mod 2, \; i.e.: \; (1)+(1)=(0), \; (1)+(0)=(1), \; (0)+(0)=(0), \tag{11}$$

$$ab = 0,1, \quad \Rightarrow \quad [M_{(a)}, M_{(b)}] \text{ for } ab=0, \; \{M_{(a)}, M_{(b)}\} \text{ for } ab=1, \tag{12}$$

The alternative ($\mathbb{Z}_2 \times \mathbb{Z}_2$-graded) matrix superalgebras

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$$M = \begin{pmatrix} A_{(00)} & A_{(11)} & A_{(10)} & A_{(01)} \\ B_{(11)} & B_{(00)} & B_{(01)} & B_{(10)} \\ C_{(10)} & C_{(01)} & C_{(00)} & C_{(11)} \\ D_{(01)} & D_{(10)} & D_{(11)} & D_{(00)} \end{pmatrix} = M_{(00)} + M_{(11)} + M_{(10)} + M_{(01)} = \tag{13}$$

$$= \begin{pmatrix} A_{(00)} & 0 & 0 & 0 \\ 0 & B_{(00)} & 0 & 0 \\ 0 & 0 & C_{(00)} & 0 \\ 0 & 0 & 0 & D_{(00)} \end{pmatrix} + \begin{pmatrix} 0 & A_{(11)} & 0 & 0 \\ B_{(11)} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{(11)} \\ 0 & 0 & D_{(11)} & 0 \end{pmatrix} + \tag{14}$$

$$+ \begin{pmatrix} 0 & 0 & A_{(10)} & 0 \\ 0 & 0 & 0 & B_{(10)} \\ C_{(10)} & 0 & 0 & 0 \\ 0 & D_{(10)} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & A_{(01)} \\ 0 & 0 & B_{(01)} & 0 \\ 0 & C_{(01)} & 0 & 0 \\ D_{(01)} & 0 & 0 & 0 \end{pmatrix}.$$

The matrix M is $(m+s+n+r)\times(m+s+n+r)$-dimensional. Here $A_{(0)}$ is $m\times m$-, $A_{(11)}$ is $m\times s$-, $A_{(10)}$ is $m\times n$-, $A_{(01)}$ is $m\times r$-dimensional.

$$< M_{(a_1 a_2)}, M_{(b_1 b_2)} > := M_{(a_1 a_2)} M_{(b_1 b_2)} - (-1)^{a_1 b_1 + a_2 b_2} M_{(b_1 b_2)} M_{(a_1 a_2)} = M_{(a_1+b_1\; a_2+b_2)} \tag{15}$$

$$\mathbf{a}+\mathbf{b} := (a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2) \mod 2, \tag{16}$$

$$(11)+(10) = (01), \quad (11)+(01) = (10), \quad \text{etc.} \tag{17}$$

$$(\mathbf{ab}) = 0,1,2 \;\Rightarrow\; [M_{(\mathbf{a})}, M_{(\mathbf{b})}] \text{ for } (\mathbf{ab}) = 0,2; \quad \{M_{(\mathbf{a})}, M_{(\mathbf{b})}\} \text{ for } (\mathbf{ab}) = 1. \tag{18}$$

Classification of the $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-graded simple Lie superalgebras (LSA).

The standard ($\mathbb{Z}_2$-graded) simple Lie superalgebras. The first papers about classification: *V. Rittenberg, M.Sheunert. J. Math. Phys.* **19** *(1978), 709–??*; *V. Kac. Ad. Math.* **26***(1977), 8–??*.

Later a complete list of simple Lie superalgebras was obtained. The following scheme resumes the classification:

Simple SLSA
→ Classical SLSA
  → Basis SLSA: $sl(m|n)$, $osp(m|2n)$, $F(4)$, $G(3)$, $D(2,1;\alpha)$
  → Strange SLSA: $P(n)$, $Q(n)$
→ Cartan type SLSA: $W(n)$, $S(n)$, $\tilde{S}(n)$, $H(n)$

The basis SLSA were well studied, in particular, their the structure of root systems, Cartan-Weyl and Chevalley bases, Weyl groups, Dynkin diagrams, etc. These superalgebras are largely applied in theoretical physics (integrable systems, quantum field theory, relativistic physics, etc.)

The classical supergroups $SL(m|n)$, $OSP(2m|n)$ and their different real form were firstly studied by F.A. Berezin (1930–1980). As a good example of a detailed description of $UOSP(1|2)$ was presented in:*F.A. Berezin, V.N. Tolstoy. The group with Grassmann structure* $\mathrm{UOSP}(1|2)$. *Commun. Math. Phys.* **78** *(1981), No. 3, 409–428.*

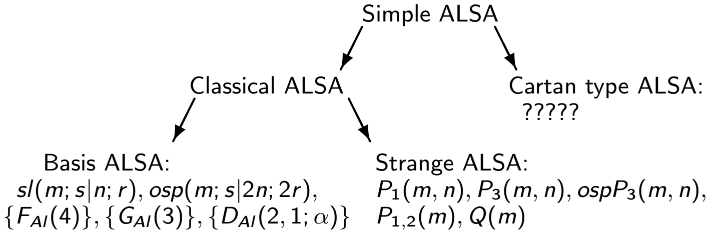
The basis SLSA are largely applied in theoretical physics (integrable systems, quantum field theory, relativistic physics, etc.) and they were well studied, in particular, their the structure of root systems, Cartan-Weyl and Chevalley bases, Weyl groups, Dynkin diagrams, etc. Alternative analogs of the basis LSA almost were not studied and they were not applied in theoretical physics. In my opinion, there are only two papers are devoted to structure of ALSA and their representation theory, namely,

V. Rittenberg, D. Wyler. Sequences of $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie algebras and superalgebras. J. Math. Phys. **19** *(1978), No. 10, 2193–2200.*

*V. Rittenberg, D. Wyler. Nucl.Phys. B***139** *(1978), No. 10, 189–???.*

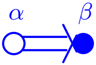
In these papers the authors explicit constructed the following theories of the alternative ($\mathbb{Z}_2\times\mathbb{Z}_2$-graded) LSA: $\mathfrak{sl}(m,s|n,r)$, $\mathfrak{osp}(2m,2s|n,n)$, $P_1(m,n)$, $P_3(m,n)$, $ospP_3(m,n)$, $P_{1,2}(m)$, and $Q(m)$.

The following classification scheme of the simple ALSA is assumed to be:



It turns aut the classical ALSA as the classical SLSA have Cartan-Weyl and Chevalley bases, Weyl groups, Dynkin diagrams, etc. Let us to consider the Dynkin diagrams. In the case of SLSA the nodes of the Dynkin diagram and corresponding simple roots occur three types white, gray and dark while in the case ALSA we have six types of nodes: white and white "up-down"; grey "up"and gray "down"; dark "up"and dark "down".

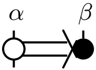
The standard ($\mathbb{Z}_2$-graded) superalgebra $\mathfrak{osp}(4|1)$.

The Dynkin diagram: 

The Serre relations: $[e_{\pm\alpha},[e_{\pm\alpha},e_{\pm\beta}]]=0$, $[\{[e_{\pm\alpha},e_{\pm\beta}],e_{\pm\beta}\},e_{\pm\beta}]=0$.

The root system Δ_+: $\underbrace{2\beta,2\alpha+2\beta,\alpha,\alpha+2\beta}_{deg(\cdot)=0},\ \underbrace{\beta,\alpha+\beta}_{deg(\cdot)=1}$.

The alternative ($\mathbb{Z}_2\times\mathbb{Z}_2$-graded) superalgebra $\mathfrak{osp}(4|1)$.

The Dynkin diagram: 

The Serre relations: $\{e_{\pm\alpha},\{e_{\pm\alpha},e_{\pm\beta}\}\}=0$, $\{[\{e_{\pm\alpha},e_{\pm\beta}\},e_{\pm\beta}],e_{\pm\beta}\}=0$.

The root system Δ_+: $\underbrace{2\beta,2\alpha+2\beta}_{deg(\cdot)=(00)},\underbrace{\alpha,\alpha+2\beta}_{deg(\cdot)=(11)},\ \underbrace{\beta}_{deg(\cdot)=(10)},\ \underbrace{\alpha+\beta}_{deg(\cdot)=(01)}$.

The involution $(*)$ $((x^*)^* = x)$:

$$\begin{array}{rclrcl} e^*_{\pm\alpha} & = & -e_{\mp\alpha}\ , & e^*_{\pm\beta} & = & -e_{\pm(\alpha+\beta)}\ , \\ e^*_{\pm2\beta} & = & -e_{\pm(2\alpha+2\beta)}\ , & e^*_{\pm(\alpha+2\beta)} & = & -e_{\pm(\alpha+2\beta)}\ , \\ h^*_{\alpha} & = & h_{\alpha}\ , & h^*_{\beta} & = & -h_{\alpha} - h_{\beta}\ . \end{array} \qquad (1)$$

The Lorentz algebra $\mathfrak{osp}(3,1)$:

$$\mathcal{M}_{13} = \frac{1}{\sqrt{2}}\Big(e_{2\beta} + e_{2\alpha+2\beta} + e_{-2\beta} + e_{-2\beta-2\alpha}\Big), \quad \mathcal{M}_{12} = 2(h_{\alpha} + 2h_{\beta})\ , \qquad (2)$$

$$\mathcal{M}_{23} = \frac{1}{\sqrt{2}}\Big(e_{2\beta} + e_{2\alpha+2\beta} - e_{-2\beta} - e_{-2\beta-2\alpha}\Big), \quad \mathcal{M}_{03} = 2ih_{\alpha} \qquad (3)$$

$$\mathcal{M}_{01} = \frac{i}{\sqrt{2}}\Big(e_{2\beta} - e_{2\alpha+2\beta} + e_{-2\beta} - e_{-2\beta-2\alpha}\Big), \qquad (4)$$

$$\mathcal{M}_{02} = \frac{i}{\sqrt{2}}\Big(e_{2\beta} - e_{2\alpha+2\beta} - e_{-2\beta} + e_{-2\beta-2\alpha}\Big), \qquad (5)$$

The anti-de-Sitter $\mathfrak{osp}(3,2)$ is generated by $\mathfrak{osp}(3,1)$ and the elements:

$$\mathcal{M}_{14} = e_{\alpha+2\beta} + e_{-\alpha-2\beta}, \quad \mathcal{M}_{24} = i(e_{\alpha+2\beta} - e_{-\alpha-2\beta}), \qquad (6)$$

$$\mathcal{M}_{34} = e_{\alpha} + e_{-\alpha}, \quad \mathcal{M}_{04} = e_{\alpha} - e_{-\alpha}\ . \qquad (7)$$

The anti-de-Sitter SA $\mathfrak{osp}(3,2|1)$ is generated by $\mathfrak{osp}(3,2)$ and the "supercharges":

$$e_{\pm1} = e_{\pm\beta}\ , \quad e_{\pm2} = e_{\pm(\alpha+\beta)}\ . \qquad (8)$$

The elements $\mathcal{M}_{AB}$ $(A,B=0,1,2,3,4)$ satisfy the standard relations

$$[\mathcal{M}_{AB},\mathcal{M}_{CD}] = i\big(g_{BC}\,\mathcal{M}_{AD}-g_{BD}\,\mathcal{M}_{AC}+g_{AD}\,\mathcal{M}_{BC}-g_{AC}\,\mathcal{M}_{BD}\big)\;, \qquad (9)$$

$$\mathcal{M}_{AB} = -\mathcal{M}_{BA}\;, \qquad \mathcal{M}^{\star}_{AB}=-\mathcal{M}_{AB} \qquad (10)$$

where $g_{AB}=\mathrm{diag}\,(1,-1,-1,-1,\ 1)$.

Using the standard contraction procedure for the super-anti de Sitter algebra: $\mathcal{M}_{\mu 4} = R\, P_\mu$ $(\mu = 0,1,2,3)$, $e_{\pm k} = \sqrt{iR}\, Q_{\pm k}$ $(k = \pm 1, \pm 2)$ for $R \to \infty$ we obtain the super-Poincarè algebra (standard and alternative) which is generated by $\mathcal{M}_{\mu\nu}$, P_μ, Q_α, $Q_{\dot\alpha}$ where $\mu,\nu = 0,1,2,3$; $\alpha,\dot\alpha = 1,2$ with the relations (we write down only those which are changed in the standard and alternative Poincarè SUSY).

(I) *for the standard Poincarè SUSY:*

$$\{P_\mu, Q_\alpha\} \;=\; \{P_\mu, \bar{Q}_{\dot\alpha}\} = 0\,, \quad \{Q_\alpha, \bar{Q}_{\dot\alpha}\} \;=\; 2\sigma^\mu_{\alpha\beta} P_\mu, \tag{1}$$

(II) *for the altrenative Poincarè SUSY:*

$$\{P_\mu, Q_\alpha\} \;=\; \{P_\mu, \bar{Q}_{\dot\alpha}\} = 0\,, \quad [Q_\alpha, \bar{Q}_{\dot\alpha}] \;=\; 2\sigma^\mu_{\alpha\beta} P_\mu, \tag{2}$$

Let us consider the supergroup associated to the alternative Poincarè superalgebra. A group element g is given by the exponential of the alternative super-Poincarè generators, namely

$$g(x^\mu, \omega^{\mu\nu}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \;=\; \exp(x^\mu P_\mu + \omega^{\mu\nu} M_{\mu\nu} + \theta^\alpha Q_\alpha + \bar{Q}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}). \qquad (1)$$

Because the grading of the exponent is (00) and the grading of P_μ is (11) therefore the grading x_μ is (11). Analogously the grading θ^α is (10) and $\bar{\theta}^{\dot{\alpha}}$ is (01). This means that

$$\{x_\mu, \theta_\alpha\} \;=\; \{x_\mu, \dot{\theta}_{\dot{\alpha}}\} \;=\; [\theta_\alpha, \bar{\theta}_{\dot{\alpha}}] \;=\; 0. \qquad (2)$$

These relations are different from the standard SUSY.
One defines the superspaces as the coset space of the alternative super-Poincarè group by the Lorentz subgroup, parameterized the coordinates x^μ, θ^α, $\bar{\theta}^{\dot{\alpha}}$, subject to the condition $\bar{\theta}^{\dot{\alpha}} = (\theta^\alpha)^*$.
We can define a superfield $\mathcal{F}$ as a function of superspace.