Dilatation operator: general structures

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Outline

1. Introduction
2. $\mathcal{N} = 4$ SYM
3. General case
   - $4D$ theory
   - $3D$ theory
4. Conclusion & Outlook

☞ based on (ArXiv):
1003.5272, 0811.2669, 0804.1835, 0707.3517, hep-th/0701089, hep-th/0611274, hep-th/0608028, hep-th/0411261, hep-th/0410010, hep-th/0409086, hep-th/0408102, hep-th/0404066

General Motivation

- Known examples of solvable gauge models bring to geometric description

  Examples:
  - Matrix models
  - Seiberg–Witten model
  - Chiral model for strongly coupled QCD
  - etc...

$$\begin{aligned} &\text{gauge theory} && \longrightarrow \text{microscopic} \\ &\text{geometric model} && \longrightarrow \text{macroscopic} \end{aligned}$$

General Motivation

- **Macroscopic description:** introduce collective variables taking the values in the phase/moduli space of the model.
- Non-trivial symmetries of the microscopic model (apart from gauge invariance) translate to the symmetries of the macroscopic one
- The description can not depend on the parametrization of the space of collective modes $\Rightarrow$ The effective theory should be geometric, i.e. phase/moduli space reparametrization invariant
- The scale appears as a (thermo)dynamical parameter

(non)AdS/(non)CFT

- Most striking example of such a description is provided by AdS/CFT correspondence.
- Originally it was formulated as a property of the string theory, but in the present it extended outside the string theory framework.
- It is a two-way weak/strong coupling correspondence
- Most studied case: correspondence between $4D$ $\mathcal{N} = 4$ super Yang–Mills theory and string/gravity on $\mathrm{AdS}_5 \times S^5$
- Intensively studied: correspondence between $3D$ Chern–Simons–matter conformal theory and $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ and relative theories...

AdS/CFT correspondence ingredients ($\mathcal{N} = 4$ SYM)

In the limit of large gauge group rank N, we have the correspondence [Maldacena]

$$(\mathcal{N} = 4 \text{ SYM})_{\mathcal{M}_{1,3}} \Leftrightarrow (\text{string theory})_{\mathrm{AdS}_5 \times S^5}$$

Identification of symmetry groups PSU(2,2|4)⊃SO(2,4)×SO(6); The correspondence between operators of SYM and states of ST. Dilatations correspond to time shifts

Non-planar extension

For $N < \infty$ the string interactions should be included with the rate $\sim N^{-1}$.

$$g_s \sim J^2/N, \qquad J - \text{classical dimension/length}$$

SYM: $N \to \infty$ — invariance of single trace operators. Single trace operators do not mix with multi-trace ones under renormalization.
Integrability [Minahan-Zarembo,Beisert-Staudacher, etc]
$\to$ "AdS/CFT dictionary"

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$AdS_5 \times S^5$ strings	$\mathcal{N} = 4$ SYM
quantum states	Local gauge invariant composite operators (LGICO)
AdS isometry	Conformal symmetry
Sphere isometry	R-symmetry
Time shift	Dilatation, RG-flow
Hamiltonian, H	Dilatation operator, Mixing matrix, Δ
...	...

This *dictionary* was checked in various regimes, but there is (will be?) *no* mathematical proof.

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Explicit construction of the dynamical system

$\mathcal{N} = 4$ SYM field content: A_μ, ψ, ϕ^i, $i = 1, \ldots, 6$
"Alphabet": $\{W_A\} = \{F_{\mu\nu}, \phi, \psi, \nabla F, \nabla\phi, \nabla\psi \ldots\}$
"Language": gauge invariant combinations of letters
"Words": simplest gauge invariants, one-trace composite operators,

$$\mathcal{O}_{A_1 A_2 \ldots A_L} = \mathrm{tr}\, W_{A_1} W_{A_2} \ldots W_{A_L}$$

"Phrases" (LGICO):

$$\mathcal{O}_{A_1 A_2 \ldots A_{L_1}} \mathcal{O}_{B_1 B_2 \ldots B_{L_2}} \ldots \mathcal{O}_{C_1 C_2 \ldots C_{L_r}}$$

Operator mixing: as $N \to \infty$ the trace structure becomes invariant: linear combinations of *words* form invariant spaces

Scale dependence: Renormalization & Operator mixing

Scale dependence is induced by the renormalization
Consider a set of composite operators $\{\mathcal{O}_J\}$ closed under renormalization (mixing)

$$\mathcal{O}_J^{Ren} = Z(\Lambda)_J{}^I \mathcal{O}_I$$

Dilatation Operator (Generator of RG-flows, now our Hamiltonian)

$$\Delta = Z^{-1} \cdot \frac{\partial Z}{\partial \log \Lambda}$$

Anomalous dimensions

$$\Delta \mathcal{O}_\lambda = \lambda \mathcal{O}_\lambda$$

General case

- So far we considered the case of $\mathcal{N} = 4$ in $4D$. Generalizations to other cases are possible.
- Various deformations of $\mathcal{N} = 4$ SYM in 4d; Chern-Simons-matter theories (ABJM & friends) in 3d were considered since that...
- How about the general case?
  - Can we construct a corresponding model in the general case of a renormalizable theory?
  - What are the necessary ingredients?
  - And which are the universal structures?
  - What are the model dependent features?

Generalization to renormalizable theories (à la Connes)

One can construct a quantum theory model from the original field theory. The new quantum theory is defined by,

- Hilbert space of States: space of LGICO (Local gauge invariant composite operators)
- Hamiltonian: Dilatation Operator, (RG-flow generator)
- Observables: ... Automorphisms of the algebra of LGICOs.

What else?

- The states should form a Hilbert space!
- Hermitian Hamiltonian...
- etc...

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Hermitian product and Hamiltonian for a CFT

Consider first a Conformal Field Theory. The primary operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$ of dimensions Δ_1 and Δ_2 have the following correlator,

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\rangle = \frac{C\delta_{\Delta_1\Delta_2}}{x^{\Delta_1+\Delta_2}}$$

The identification $\mathcal{O} \mapsto |\mathcal{O}\rangle$ with $\langle \mathcal{O}_1 \mid \mathcal{O}_2\rangle = C\delta_{\Delta_1\Delta_2}$ and

$$H\,|\mathcal{O}_i\rangle = \Delta_i\,|\mathcal{O}_i\rangle$$

solves the problem. . .

Hermitian product and Hamiltonian for a generic QFT

☞ We can extend this for a generic renormalizable theory: Hermitian product and Hamiltonian can be introduced through the correlators

$$\langle \mathcal{O}^\dagger(x)\mathcal{O}'(0)\rangle = \langle \mathcal{O}| (\mu^2 x^2)^{-\mathbf{D}} |\mathcal{O}'\rangle \equiv \langle \mathcal{O}| \mathrm{e}^{-\tau \mathbf{D}} |\mathcal{O}'\rangle$$

$\tau = \log(\mu^2 x^2)$

$$\langle \mathcal{O} \mid \mathcal{O}'\rangle = \langle \mathcal{O}^\dagger(x)\mathcal{O}'(0)\rangle|_{\mu^2 x^2 = 1}$$

$$\langle \mathcal{O}| \mathbf{D} |\mathcal{O}'\rangle = -\frac{1}{2}\mu\frac{\partial}{\partial\mu}\langle \mathcal{O}^\dagger(x)\mathcal{O}'(0)\rangle|_{\mu^2 x^2 = 1}$$

. . . as soon as we identify local operators with quantum states

General Construction for Hermitian product and Hamiltonian

We have to analyze the RG-transformation of a composite operator $\mathcal{O}$ in perturbation theory. Mixing matrix $Z(\Lambda)$ can be found considering divergent terms in correlators of two probe operators $\mathcal{O}$ and $\mathcal{O}'$,

$$\langle : \mathcal{O}'_y(\phi) :: \mathcal{O}_0 : \rangle = \langle : \mathcal{O}'_y : \mathrm{e}^{-\int :V(\phi):} : \mathcal{O}_0 : \rangle_0$$

The source of relevant divergences is the Wick expansion of products

$$e^{-\int :V(\phi):} : \mathcal{O}_0 : = \left(1 - \int : V(\phi) : + \frac{1}{2!} \iint : V(\phi) :: V(\phi) : + \ldots \right) : \mathcal{O}_0 :$$

So, we should modify $\mathcal{O}_0$ in such a way to cancel divergences and find the scale dependence after the cancelation.

Wick expansion

Wick expansion in functional form can be cast into [see one of the Kleinert's books]

$$:\mathcal{O}'_y :: \mathcal{O}_x := \mathrm{e}^{\check{\phi}_{Ay} D_{AB}(y-x) \check{\phi}_{Bx}} \mathcal{O}'_y \mathcal{O}_x \equiv \mathcal{O}' * \mathcal{O}(x,y)$$

$*$ — star product resembles one in noncommutative theories, but is different

$$\check{\phi}_{Ax} = \frac{\partial}{\partial \phi_A(x)} \qquad \text{Not a functional derivative!}$$

e.g. Euclidean massless propagator
Functional Wick expansion can be generalized to the product of 3, 4,. . . factors

A note on Notations

We have to deal with complicate expressions ☞ notations are important.

- Condensed (multi-index)

$$\mu_1\mu_2\ldots\mu_n \to \mathbf{n}, \qquad \phi_{\mathbf{n}} = \partial_{\mathbf{n}}\phi(0)$$

- A traceless set of indices,

$$(\mu_1\mu_2\ldots\mu_n) \to (\mathbf{n}): \qquad \phi_{(\mathbf{n})}$$

- In general, treat space-time indices as sets,

$$\mathbf{n}+\mathbf{m} \to \mu_1\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_m$$

$$\mathbf{n}\backslash\mathbf{r} \to \mu_1\ldots\check{\mu}_{i_1}\ldots\check{\mu}_{i_r}\ldots\mu_n, \quad \mathbf{r} \to \mu_{i_1}\ldots\mu_{i_r}, \quad \mathbf{r}\subset\mathbf{n}$$

- summation over intersecting sets...

The $4D$ theory

The scale dependence of the two-point function is dimension (and model) dependent

$$S = \int \mathrm{d}x \left(-\tfrac{1}{2}\phi \cdot D^{-1} \cdot \phi + V(\phi)\right)$$

The basic propagators in four-dimensions are

$$D(x) = \begin{cases} \sim \frac{1}{4\pi^2}\frac{1}{x^2} & \text{scalars, gauge bosons, etc,} \\ \sim \gamma^\mu \partial_\mu \frac{1}{4\pi^2}\frac{1}{x^2}, & \text{fermions,} \end{cases}$$

- LGICOs are polynomials in fundamental letter and their derivatives.
- LGICOs are defined modulo EoM ☞ can eliminate the traces of derivatives

Tools: differential renormalization

Differential regularization/renormalization scheme in real space allows to regularize singular expressions like [Freedman-Johnson-Latorre],

$$\frac{1}{x^{2k}} = -\frac{1}{4^{k-1}(k-1)!(k-2)!}\Box^{k-1}\frac{\ln \mu^2 x^2}{x^2}, \qquad k \geq 2$$

introduces a scale dependence:

$$\mu\frac{\partial}{\partial\mu}\left[\frac{1}{x^{2k}}\right]_{\mathrm{reg}} \equiv \left[\frac{1}{x^{2k}}\right] = \frac{8\pi^2}{4^{k-1}(k-1)!(k-2)!}\Box^{k-2}\delta(x)$$

where we used the property

$$\Box\frac{1}{x^2} = -4\pi^2\delta(x)$$

One vertex level

In the leading one-loop approximation the contribution comes only from one- and two-vertex Feynman diagrams. Consider first the one-vertex level Regularizing the terms in the Wick Expansion we get for the first order in interaction potential

$$
\begin{aligned}
-\int \mathrm{d}y\, [V_{\mathrm{int}}(y)*] &= -\int \mathrm{d}y\ \left[\mathrm{e}^{\check{\phi}_y \cdot D_y \cdot \check{\phi}} \right] V_y \\
&= -\int \mathrm{d}y \left(\check{\phi}_y \cdot [D_y] \cdot \check{\phi} + \frac{1}{2} (\check{\phi}_y \otimes \check{\phi}_y) \cdot [D_y \otimes D_y] \cdot (\check{\phi} \otimes \check{\phi}) \right. \\
&\quad \left. + \frac{1}{3!} (\check{\phi}^{\otimes 3}) \cdot [D_y^{\otimes 3}] \cdot (\check{\phi}^{\otimes 3}) + \frac{1}{4!} (\check{\phi}^{\otimes 4}) \cdot [D_y^{\otimes 4}] \cdot (\check{\phi}^{\otimes 4}) + \dots \right) V_y,
\end{aligned}
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&= -\int \mathrm{d}y\,\Bigg(\frac{1}{2}(\check{\phi}_y\otimes\check{\phi}_y)\cdot[D_y\otimes D_y]\cdot(\check{\phi}\otimes\check{\phi}) \\
&\quad + \frac{1}{3!}(\check{\phi}^{\otimes 3})\cdot[D_y^{\otimes 3}]\cdot(\check{\phi}^{\otimes 3}) + \frac{1}{4!}(\check{\phi}^{\otimes 4})\cdot[D_y^{\otimes 4}]\cdot(\check{\phi}^{\otimes 4}) \Bigg) V_y,
\end{aligned}
$$

Two vertex level

Second level yields

$$\frac{1}{2!}\int \mathrm{d}y_1 \int \mathrm{d}y_2 [V_{\mathrm{int}}(y_1) * V_{\mathrm{int}}(y_2) *]$$

$$= \frac{1}{2}\int \mathrm{d}y_1 \int \mathrm{d}y_2 \times$$

$$\Big\{ (\check{\phi}_{y_1} \otimes \check{\phi}_{y_1} \otimes \check{\phi}_{y_2}) \cdot [D_{y_1} \otimes D_{y_1 - y_2} \otimes D_{y_2}] \cdot (\check{\phi} \otimes \check{\phi}_{y_2} \otimes \check{\phi}) +$$

$$(\check{\phi}_{y_1}^{\otimes 3} \otimes \check{\phi}_{y_2}) \cdot [D_{y_1}^{\otimes 2} \otimes D_{y_1 - y_2} \otimes D_{y_2}] \cdot (\check{\phi}^{\otimes 2} \otimes \check{\phi}_{y_2} \otimes \check{\phi}) + \dots \Big\} V_{y_1} V_{y_2} .$$

The The $3D$ theory

The leading contribution in $3D$ comes from the two loop level
Basic propagators

$$D(x) = \begin{cases} \frac{1}{4\pi x}, & \text{for scalars, gauge bosons, etc,} \\ \gamma^\mu \partial_\mu \frac{1}{4\pi x}, & \text{for fermions,} \\ \epsilon_{\mu\nu\lambda} \partial_\lambda \frac{1}{4\pi x}, & \text{for Chern-Simons fields.} \end{cases}$$

Basic objects are still LGICO $\mathcal{O}(\Phi)$, depending on fundamental fields $\phi(x)$ and their derivatives

$$\phi_{\mu_1 \mu_2 \dots \mu_n} \equiv \partial_{\mu_1} \partial_{\mu_2} \dots \partial_{\mu_n} \phi(0)$$

EoMs can be used to eliminate the dependence on CS gauge fields traces of derivatives

"Functional Wick Expansion"

Functional form of Wick expansion can be introduced through the following equations

$$:\mathcal{O}_1::\mathcal{O}_2:=\mathrm{e}^{\check{D}_{12}}:\mathcal{O}_1\mathcal{O}_2:$$

$$:\mathcal{O}_1::\mathcal{O}_2::\mathcal{O}_3:=\mathrm{e}^{\check{D}_{12}+\check{D}_{13}+\check{D}_{23}}:\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3:$$

$$:\mathcal{O}_1::\mathcal{O}_2::\mathcal{O}_3::\mathcal{O}_4:=\mathrm{e}^{\check{D}_{12}+\check{D}_{13}+\check{D}_{23}+\check{D}_{14}+\check{D}_{24}+\check{D}_{34}}:\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4:$$

$$:\mathcal{O}_1::\mathcal{O}_2:\cdots:\mathcal{O}_k:=\mathrm{e}^{\left(\sum_{l<m}\check{D}_{lm}\right)}:\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_k:$$

where $\check{D}_{xy}$ is a two-point differential operator

$$\check{D}_{xy}=\check{\Phi}_x\cdot\mathbf{D}_{xy}\cdot\check{\Phi}_y=\sum_{(\mathbf{n})}D_{xy}^{(\mathbf{n})}\check{s}_{xy}^{(\mathbf{n})},$$

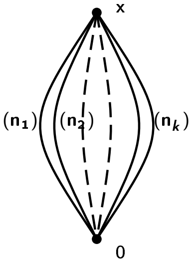
$$\check{s}_{xy}^{(\mathbf{n})}=\sum_{\substack{\mathbf{r},\mathbf{s}\\ \mathbf{r}+\mathbf{s}=\mathbf{n}}}(-1)^s\check{\phi}_x^{(\mathbf{r})}\cdot\check{\phi}_y^{(\mathbf{s})}+\sum_{\substack{\mathbf{r},\mathbf{s},\\ \mathbf{r}+\mathbf{s}+\mathbf{1}=\mathbf{n}}}(-1)^s\check{\bar{\psi}}_x^{(\mathbf{r})}\gamma^{\mathbf{1}}\check{\psi}_y^{(\mathbf{s})}+\sum_{\substack{\mathbf{r},\mathbf{s}\\ \mathbf{r}+\mathbf{s}+\widetilde{11'}=\mathbf{n}}}(-1)^s\check{A}_{\mathbf{1}x}^{(\mathbf{r})}\check{A}_{\mathbf{1}'y}^{(\mathbf{s})}$$

$$\check{\Phi}_x=\frac{\partial}{\partial\Phi_x}$$

One vertex lever (two point function)

The first non-trivial contribution comes from the first term of the expansion of interaction exponent,

$$\int_y \left[e^{\check{D}_{y0}} \right] V_y = \sum_k \frac{1}{k!} \left[D_{y0}^{(\mathbf{n}_1)} \dots D_{y0}^{(\mathbf{n}_k)} \right] \check{s}_{y0}^{(\mathbf{n}_1)} \dots \check{s}_{y0}^{(\mathbf{n}_k)} V_y$$



Two loops $\Rightarrow k = 3$.

Evaluation of divergencies: 1-vertex level; two loops

$$\Delta_{(\mathbf{n})(\mathbf{m})(\mathbf{k})} \equiv \left[D_x^{(\mathbf{n})} D_x^{(\mathbf{m})} D_x^{(\mathbf{k})}\right] = \frac{1}{(4\pi)^3}\left[\partial_{(\mathbf{n})}\frac{1}{x}\partial_{(\mathbf{m})}\frac{1}{x}\partial_{(\mathbf{k})}\frac{1}{x}\right]$$

From general considerations we have,

$$\Delta_{(\mathbf{n}),(\mathbf{m}),(\mathbf{k})}(x) = \frac{1}{(4\pi)^3}\sum_{\mathbf{r},s} F^{(\mathbf{r}),s}_{(\mathbf{n}),(\mathbf{m}),(\mathbf{k})}\partial_{(\mathbf{r})}\partial^{2s}\delta(x),$$

where $F^{(\mathbf{r}),s}_{(\mathbf{n}),(\mathbf{m}),(\mathbf{k})}$ are numerical coefficients, defined by

$$F^{(\mathbf{r}),s}_{(\mathbf{n}),(\mathbf{m}),(\mathbf{k})} = f^{(\mathbf{r}),s}_{nmk}\left[\int_x \frac{\mathbf{x}^{(\mathbf{n})+(\mathbf{m})+(\mathbf{k})+(\mathbf{r})}}{x^{3+2(n+m+k-s)}}\right]$$

where

$$f^{(\mathbf{r}),s}_{nmk} = (-1)^{n+m+k}(2n-1)!!(2m-1)!!(2k-1)!!\alpha^{(\mathbf{r}),s}$$

and the factors $\alpha^{(\mathbf{n}),r}$ are the trace-reduced coefficients of Taylor expansion,

$$V_x = \sum_{(\mathbf{n}),r}\alpha^{(\mathbf{n}),r}\mathbf{x}^{(\mathbf{n})}x^{2r}\partial_{(\mathbf{n})}\partial^{2r}V_0.$$

One-vertex level coefficients

The evaluation of Fs in dimensional regularization scheme and with IR cut-off μ, produces the following relevant contribution

$$\frac{2^{-(n+m+k+1)}(n+m+k)!\pi^{D/2}}{\Gamma\left(\frac{D+n+m+k}{2}\right)} g^{(\mathbf{n})+(\mathbf{m})+(\mathbf{k})}\mu^{-\epsilon}\Gamma(\epsilon)$$

This yields,

$$F^{(\mathbf{r}),s}_{(\mathbf{n}),(\mathbf{m}),(\mathbf{k})} = -\delta_{n+m+k-r-2s,0} f^{(\mathbf{r}),s}_{nmk} g^{(\mathbf{n})+(\mathbf{m})+(\mathbf{k})} \frac{2^{-\frac{n+m+k}{2}}\pi}{(n+m+k+1)!!}$$

The Dilatation operator at one vertex level is given by,

$$H_{1-\text{vertex}} = \frac{1}{(4\pi)^3} \sum_{\{\mathbf{nmk}\}} F^{(\mathbf{r}),s}_{(\mathbf{n}),(\mathbf{m}),(\mathbf{k})} \partial_{(\mathbf{r})} \partial^{2s} \check{s}_{(\mathbf{n})} \check{s}_{(\mathbf{m})} \check{s}_{(\mathbf{k})} [V]$$

Two-vertex level

Two vertex level is given by the second term of the expansion of interaction exponent

$$H_{2-\mathrm{vertex}} = \frac{1}{2!}\int_x\int_y \left[\mathrm{e}^{\check{\Phi}_x\cdot\mathbf{D}_{xy}\cdot\check{\Phi}_y+\check{\Phi}_x\cdot\mathbf{D}_x\cdot\check{\Phi}+\check{\Phi}_y\cdot\mathbf{D}_y\cdot\check{\Phi}}\right] V_x V_y$$

Restrict to the two loop part

$$H_{2-\mathrm{vertex}} = \int_x\int_y \Big(\tfrac{1}{2}\Delta_{(\mathbf{n}_1)(\mathbf{n}_2);(\mathbf{m});(\mathbf{k})}(x,y)\check{s}_x^{(\mathbf{n}_1)}\check{s}_x^{(\mathbf{n}_2)}\check{s}_y^{(\mathbf{m})}\check{s}_{xy}^{(\mathbf{k})}$$

$$+\tfrac{1}{4}\Delta_{(\mathbf{n});(\mathbf{m});(\mathbf{k}_1)(\mathbf{k}_2)}(x,y)\check{s}_x^{(\mathbf{n})}\check{s}_y^{(\mathbf{m})}\check{s}_{xy}^{(\mathbf{k}_1)}\check{s}_{xy}^{(\mathbf{k}_2)}\Big) V_x V_y,$$

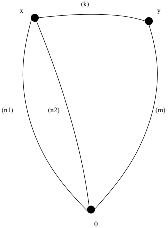
with

$$\Delta_{(\mathbf{n}_1)(\mathbf{n}_2);(\mathbf{m});(\mathbf{k})}(x,y) = \frac{1}{(4\pi)^4}\left[\partial_{(\mathbf{n}_1)}\frac{1}{x}\partial_{(\mathbf{n}_2)}\frac{1}{x}\partial_{(\mathbf{m})}\frac{1}{y}\partial^x_{(\mathbf{k})}\frac{1}{|x-y|}\right],$$

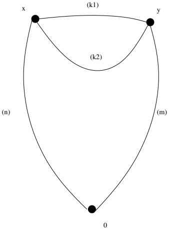
$$\Delta_{(\mathbf{n});(\mathbf{m});(\mathbf{k}_1)(\mathbf{k}_2)}(x,y) = \frac{1}{(4\pi)^4}\left[\partial_{(\mathbf{n})}\frac{1}{x}\partial_{(\mathbf{m})}\frac{1}{y}\partial^x_{(\mathbf{k}_1)}\frac{1}{|x-y|}\partial^x_{(\mathbf{k}_2)}\frac{1}{|x-y|}\right],$$

Two-vertex level

The corresponding Master Feynman diagrams are



$$\Delta_{(\mathbf{n_1})(\mathbf{n_2});(\mathbf{m});(\mathbf{k})}(x,y)$$



$$\Delta_{(\mathbf{n});(\mathbf{m});(\mathbf{k_1})(\mathbf{k_2})}(x,y)$$

Two-vertex level

Duality relates these factors

$$\Delta_{(\mathbf{n});(\mathbf{m});(\mathbf{k}_1)(\mathbf{k}_2)}(x,y) = (-1)^m \Delta_{(\mathbf{k}_1)(\mathbf{k}_2);(\mathbf{m});(\mathbf{n})}(x-y,-y)$$

Evaluation of the general structure reveals

$$\Delta_{(\mathbf{n}_1),(\mathbf{n}_2);(\mathbf{m});(\mathbf{k})}(x,y) = \frac{1}{(4\pi)^4} \sum_{\substack{\mathbf{p},r \\ \mathbf{s},t}} F^{(\mathbf{p}),r;(\mathbf{s}),t}_{(\mathbf{n}_1),(\mathbf{n}_2);(\mathbf{m});(\mathbf{k})} \partial_{(\mathbf{p})}\partial^{2r}\delta(x)\partial_{(\mathbf{s})}\partial^{2t}\delta(y)$$

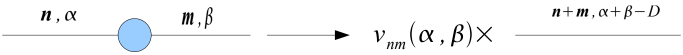
$$F^{(\mathbf{p}),r;(\mathbf{s}),t}_{(\mathbf{n}_1),(\mathbf{n}_2);(\mathbf{m});(\mathbf{k})} = f_{n_1 n_2 mk} \int_x \int_y \left[\frac{\mathbf{x}^{(\mathbf{n}_1)+(\mathbf{n}_2)+(\mathbf{p})}}{x^{2+2(n_1+n_2-r)}} \frac{\mathbf{y}^{(\mathbf{m})+(\mathbf{s})}}{y^{1+2(m-t)}} \frac{(\mathbf{x}-\mathbf{y})^{(\mathbf{k})}}{|x-y|^{1+2k}} \right].$$

where,

$$f_{n_1 n_2 mk} = (-1)^{n_1+n_2+m+k}(2n_1-1)!!(2n_2-1)!!(2m-1)!!(2k-1)!!\alpha^{(\mathbf{p}),r}\alpha^{(\mathbf{s}),t}$$

Two-vertex level coefficients

We can apply a "trick" to simplify
the three-point coefficient function. Method of 'Uniqueness' [Kazakov et al.].



or

$$\int_y \frac{(\mathbf{x}-\mathbf{y})^{\mathbf{m}}}{|x-y|^{\beta}} \frac{\mathbf{y}^{\mathbf{n}}}{y^{\alpha}} = v_{nm}(\alpha,\beta) \frac{\mathbf{x}^{\mathbf{n}+\mathbf{m}}}{x^{\alpha+\beta-D}}$$

$$v_{nm}(\alpha,\beta) = \pi^{D/2} 2^{-2(n+m)} \frac{\Gamma(\frac{D-\alpha}{2}+n)\Gamma(\frac{D-\beta}{2}+m)\Gamma(\frac{\alpha+\beta-D}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})\Gamma(D-\frac{\alpha+\beta}{2}+m+n)}$$

Then we can remove one integration and reduce the regularized three point function to two point functions $\times$ some factor.
No additional singularities compared to derivative-free exchange!

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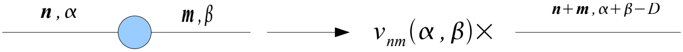

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Two-vertex level result

The two-point function coefficients are given by

$$F^{(\mathbf{p}),r;(\mathbf{s}),t}_{(\mathbf{n}_1),(\mathbf{n}_2);(\mathbf{m});(\mathbf{k})} = -\delta_{(n_1+n_2+m+k),(p+s+2r+2t)} f_{n_1 n_2 m k} \times v_{k,m+s}(1+2(m-t),1+2k)\beta^{(\mathbf{n}_1)+(\mathbf{n}_2)+(\mathbf{m})+(\mathbf{k})+(\mathbf{p})+(\mathbf{s})}\Big|_{D=3}.$$

where,

$$\beta^{\mathbf{n}} = \int_{\hat{\mathbf{x}}^2=1} \mathrm{d}\hat{\mathbf{x}}\, \hat{\mathbf{x}}^{\mathbf{n}} = \frac{2^{-(n+1)} n! \pi^{D/2}}{\Gamma\left(\frac{D+n}{2}\right)} g^{\mathbf{n}},$$

with $g^{\mathbf{n}} = 0$ for odd n while for even n it is the symmetrized product of metric tensors,

$$g^{\mathbf{n}} \mapsto \frac{1}{|S_n|} \sum_{p \in S_n} g^{\mu_{p(1)}\mu_{p(2)}} \dots g^{\mu_{p(n-1)}\mu_{p(n)}}.$$

Three-vertex level

The three-vertex contribution is given by,

$$H_{3V} = \int_1 \int_2 \int_3 \left(3\left[\check{D}_1 \check{D}_2 \check{D}_{12} \check{D}_{13} \check{D}_{23}\right] + 3\left[\check{D}_1 \check{D}_2 \check{D}_3 \check{D}_{12} \check{D}_{23}\right]\right) V_1 V_2 V_3$$

$$\check{D}_{xy} \equiv \sum_{\mathbf{n}} D_{xy}^{(\mathbf{n})} \check{s}^{(\mathbf{n})}$$

Therefore, we face to the evaluation of two scale factors,

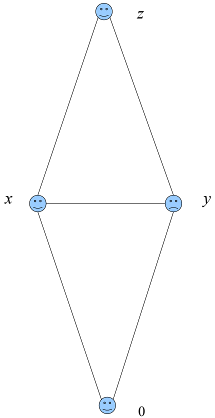
$$\Delta_{(\mathbf{n}),(\mathbf{m});(\mathbf{k}),(\mathbf{l}),(\mathbf{p})}(x,y,z) = \left[\partial_{(\mathbf{n})} \frac{1}{x} \partial_{(\mathbf{m})} \frac{1}{y} \partial_{(\mathbf{k})} \frac{1}{|x-y|} \partial_{(\mathbf{l})} \frac{1}{|x-z|} \partial_{(\mathbf{p})}^{y} \frac{1}{|y-z|}\right],$$

$$\Delta_{(\mathbf{n}),(\mathbf{m}),(\mathbf{k});(\mathbf{l}),(\mathbf{p})}(x,y,z) = \left[\partial_{(\mathbf{n})} \frac{1}{x} \partial_{(\mathbf{m})} \frac{1}{y} \partial_{(\mathbf{k})} \frac{1}{z} \partial_{(\mathbf{l})} \frac{1}{|x-y|} \partial_{(\mathbf{p})}^{y} \frac{1}{|y-z|}\right].$$

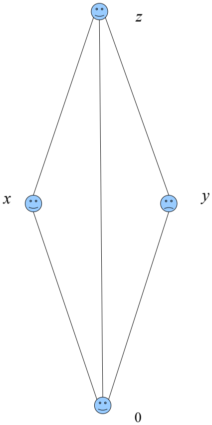
. . . again duality relates these factors. . .

Three-vertex level

Graphically the Master diagrams are



$\Delta_{(\boldsymbol{n}),(\boldsymbol{m});(\boldsymbol{k}),(\boldsymbol{l}),(\boldsymbol{p})}(x,y,z)$



$\Delta_{(\boldsymbol{n}),(\boldsymbol{m}),(\boldsymbol{k});(\boldsymbol{l}),(\boldsymbol{p})}(x,y,z)$

Three-vertex level

Duality relates

$$\Delta_{(\mathbf{n}),(\mathbf{m}),(\mathbf{k});(\mathbf{l}),(\mathbf{p})}(x,y,z)=\Delta_{(\mathbf{n}),(\mathbf{l});(\mathbf{k}),(\mathbf{m}),(\mathbf{p})}(-z,-y,-x)$$

The general structure is given by

$$\Delta_{(\mathbf{n}),(\mathbf{m});(\mathbf{k}),(\mathbf{l}),(\mathbf{p})}(x,y,z)= \frac{1}{(4\pi)^5}\sum_{\substack{\mathbf{p},r\\ \mathbf{s},t\\ \mathbf{u},v}} F^{(\mathbf{p}),r;(\mathbf{s}),t;(\mathbf{u}),v}_{(\mathbf{n}),(\mathbf{m});(\mathbf{k}),(\mathbf{l}),(\mathbf{p})}\partial_{(\mathbf{p})}\partial^{2r}\delta(x)\partial_{(\mathbf{s})}\partial^{2t}\delta(y)\partial_{(\mathbf{u})}\partial^{2v}\delta(z),$$

$$F^{(\mathbf{p}),r;(\mathbf{s}),t;(\mathbf{u}),v}_{(\mathbf{n}),(\mathbf{m});(\mathbf{k}),(\mathbf{l}),(\mathbf{q})}=f^{(\mathbf{p}),r;(\mathbf{s}),t;(\mathbf{u}),v}_{nmklq} \times\int_x\int_y\int_z\left[\frac{\mathbf{x}^{(\mathbf{n})+(\mathbf{p})}}{x^{1+2(n-r)}}\frac{\mathbf{y}^{(\mathbf{m})+(\mathbf{s})}}{y^{1+2(m-t)}}\frac{\mathbf{z}^{(\mathbf{k})+(\mathbf{u})}}{z^{1+2(k-v)}}\frac{(\mathbf{x}-\mathbf{z})^{(\mathbf{l})}}{|x-z|^{1+2l}}\frac{(\mathbf{y}-\mathbf{z})^{(\mathbf{q})}}{|y-z|^{1+2q}}\right]$$

Three-vertex level results

Applying the chain rule twice, we can remove two integrations and reduce the four point integral to a two point one

$$\left[\frac{\mathbf{x}^{(\mathbf{n})+(\mathbf{p})}}{x^{1+2(n-r)}}\frac{\mathbf{y}^{(\mathbf{m})+(\mathbf{s})}}{y^{1+2(m-t)}}\frac{\mathbf{z}^{(\mathbf{k})+(\mathbf{u})}}{z^{1+2(k-v)}}\frac{(\mathbf{x}-\mathbf{z})^{(\mathbf{l})}}{|x-z|^{1+2l}}\frac{(\mathbf{y}-\mathbf{z})^{(\mathbf{q})}}{|y-z|^{1+2q}}\right]=$$
$$(-1)^{q+l}v_{q,m+s}(1+2q,1+2(m-t))v_{l,n+p}(1+2l,1+2(n-r))$$
$$\times\left[\int_z\frac{\mathbf{z}^{(\mathbf{n})+(\mathbf{m})+(\mathbf{k})+(\mathbf{l})+(\mathbf{q})+(\mathbf{s})+(\mathbf{p})+(\mathbf{u})}}{z^{5+2(n+m+k+l+q-r-t-v)-2D}}\right]$$

which we know how to evaluate!

All in one

$$F^{(\mathbf{p}),r;(\mathbf{s}),t;(\mathbf{u}),v}_{(\mathbf{n}),(\mathbf{m});(\mathbf{k}),(\mathbf{l}),(\mathbf{q})}=\delta_{n+m+k+l+q,2(r+t+v)+5}f^{(\mathbf{p}),r;(\mathbf{s}),t;(\mathbf{u}),v}_{nmklq}(-1)^{q+l}$$
$$\times\, v_{q,m+s}(1+2q,1+2(m-t))v_{l,n+p}(1+2l,1+2(n-r))$$
$$\times\,\beta^{(\mathbf{n})+(\mathbf{m})+(\mathbf{k})+(\mathbf{l})+(\mathbf{q})+(\mathbf{s})+(\mathbf{p})+(\mathbf{u})}\Big|_{D=3}$$

Summary of the computation

$$\Delta_{2-\text{loop}} = H_{1-\text{vertex}} + H_{2-\text{vertx}} + H_{3-\text{vertex}}$$

$$H_{1-\text{vertex}} = -\frac{1}{3!(4\pi)^3} \sum_{\substack{n,m,k \\ r,s}} (-1)^r F^{(r),s}_{(n),(m),(k)} \partial_{(r)} \partial^{2s} \check{s}^{(n)}_{y0} \check{s}^{(m)}_{y0} \check{s}^{(k)}_{y0} (V_y) \Big|_{y=0} ,$$

$$H_{2-\text{vertex}} = \frac{1}{2(4\pi)^4} \Big\{ \sum (-1)^{p+s} F^{(p)r;(s)t}_{(n_1)(n_2);(m);(k)} (\partial_{(p)} \partial^{2r})_x (\partial_{(s)} \partial^{2t})_y \check{s}^{(n_1)}_x \check{s}^{(n_2)}_x \check{s}^{(m)}_y \check{s}^{(k)}_{xy}$$
$$+ \tfrac{1}{2} \sum (-1)^{m+p+s} F^{(p)r;(s)t}_{(k_1)(k_2);(m);(n)} (\partial_{(p)} \partial^{2r})_x (\partial_{(s)} \partial^{2t})_y \check{s}^{(n)}_x \check{s}^{(n_2)}_x \check{s}^{(m)}_y \check{s}^{(k_1)}_{xy} \check{s}^{(k_2)}_{xy} \Big\} V_x V_y \big|_{x=y=0}$$

$$H_{3-\text{vertex}} = -\frac{1}{2(4\pi)^5} \Big\{ (-1)^{p+r+s} F^{(p),r;(s),t;(u),v}_{(1),(2);(12),(13),(23)} (\partial_{(p)} \partial^{2r})_1 (\partial_{(s)} \partial^{2t})_2 (\partial_{(u)} \partial^{2v})_3 \check{s}_1 \check{s}_2 \check{s}_{12} \check{s}_{13} \check{s}_{23}$$
$$+ (-1)^p F^{(p),r;(s),t;(u),v}_{(1),(12);(2),(3),(23)} (\partial_{(p)} \partial^{2r})_1 (\partial_{(s)} \partial^{2t})_2 (\partial_{(u)} \partial^{2v})_3 \check{s}_1 \check{s}_2 \check{s}_{12} \check{s}_3 \check{s}_{23}) \Big\} V_1 V_2 V_3 \big|_{1=2=3=0}$$

A faster way to obtain the dilatation operator?

Alternatively, consider the dilatation operator as Nöther charge corresponding to classical dilatations:

- classical dimension $\longmapsto$ mass
- The large mass limit $\longleftrightarrow$ perturbative expansion of the dilatation operator
- Indeed, the massive Yang–Mills mechanics reproduces the one-loop dilatation operator in the scalar sector of $\mathcal{N} = 4$ SYM

Conclusion & Outlook

- Starting from a renormalizable theory one can obtain a model for which the (imaginary) time evolution coincides with RG-flows of given theory
- At least at the one-loop level the same result can be obtained by a slow roll limit of a massive extension of the model
- The approach is purely constructive, we didn't prove any existence theorem, positivity of the norms, unitarity etc.
- In the case of conformal theories the scheme coincides with the standard AdS/CFT correspondence
- For a general renormalizable gauge theory the large N limit is expected to lead to a *local* geometrical model
- Better parametrization for the LGICO would do a better job