SQS'11, Dubna, July 18 - 23, 2011

Photon-Graviton Amplitudes from the Effective Action

1. Relations between gauge and gravity amplitudes.
2. Properties of the QED N photon amplitudes.
3. Worldline approach to Einstein-Maxwell theory.
4. One-loop effective action in Einstein-Maxwell theory.
5. The graviton - photon - photon amplitude and its properties.
6. Outlook.

Relations between gauge and gravity amplitudes

H. Kawai, D.C. Lewellen, S.H.H. Tye, "A Relation between Tree Amplitudes of Closed and Open Strings" (Nucl. Phys. B 269, 1, 1986):

$$(gravity\, amplitude) \sim (gauge\, amplitude)^2$$

From the factorization of vertex operators,

$$V^{\text{closed}} = V^{\text{open}}_{\text{left}} \bar{V}^{\text{open}}_{\text{right}}$$

These string relations induce also relations in field theory.
For example, at four and five point,

$$
\begin{aligned}
M_4(1,2,3,4) &= -is_{12}A_4(1,2,3,4)A_4(1,2,4,3) \\
M_5(1,2,3,4,5) &= is_{12}s_{34}A_5(1,2,3,4,5)A_5(2,1,4,3,5) \\
&\quad +is_{13}s_{24}A_5(1,3,2,4,5)A_5(3,1,4,2,5)
\end{aligned}
$$

M_n = tree-level graviton amplitudes
A_n= (colour-stripped) tree-level gauge theory amplitudes $s_{ij} = (k_i + k_j)^2$

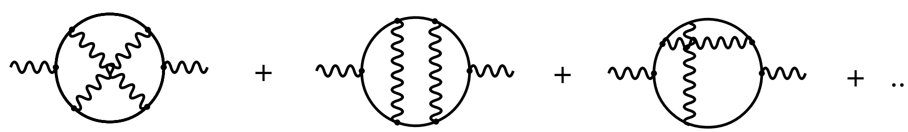
Tree level relations $\rightarrow$ loop level relations by unitarity.

See the review by Z. Bern, "Perturbative Quantum Gravity and its Relation to Gauge Theory", Living Reviews in Relativity 5, 5 (2002).

Those relations are particularly useful for the SUSY case, and for the study of infrared divergences. Lots of recent activity....

The possible finiteness of $N = 8$ Supergravity involves large-scale cancellations between Feynman diagrams whose origin is not fully understood yet. In this respect, gravity amplitudes are more similar to QED amplitudes than to nonabelian amplitudes, since colour factors greatly reduce the potential for cancellations between diagrams.

E.g., the three-loop photon propagator (J. Rosner 1967)



- Extensive cancellations for this type of sums of diagrams (K. Johnson, M. Baker and R. Willey; P. Cvitanovic; D. Broadhurst.....).
- They are clearly related to gauge invariance.
- Even for QED, little is known about the influence of these cancellations on the large-order behaviour of the QED perturbation series.

Gravity - inspired studies of the structure of QED amplitudes

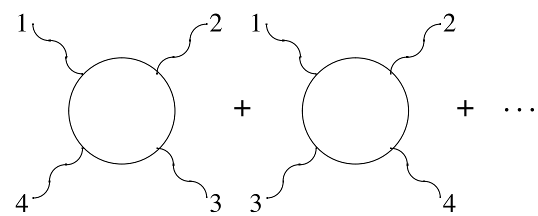
- S.D. Badger, N.E.J. Bjerrum-Bohr, and P. Vanhove, JHEP **0902**:038 (2009).
- A. Brandhuber, G. Travaglini, M. Vincon, arXiv:0908.1306 [hep-th].
- S.D. Badger and J.M. Henn, Phys. Lett. **B 692** 143 (2010).

In the following, we will study more generally the mixed photon – graviton amplitudes (ongoing work with F. Bastianelli, O. Corradini, J.M. Dávila).

Properties of the QED N photon amplitudes

(Scalar and Spinor QED - no real differences)

4 photon amplitude:



Sum of photon scattering diagrams.

First calculation by R. Karplus and M. Neumann 1950.

Massless QED

Use a helicity basis, and the spinor helicity formalism:

$$\varepsilon_\mu^\pm(k) \;=\; \pm\frac{\langle q^\mp|\gamma_\mu|k^\mp\rangle}{\sqrt{2}\langle q^\mp|k^\pm\rangle}$$

Massless 6 photon amplitude:

T. Binoth, G. Heinrich, T. Gehrmann, P. Mastrolia, PLB 649, 422 (2007).

Four independent helicity components:

$$A_6(++++++)$$
$$A_6(-+++++)$$
$$A_6(--++++)$$
$$A_6(---+++)$$

(CP invariance, no need to order legs in the abelian case.)

Mahlon's theorem: $A_N(+++\ldots++) = A_N(-++\cdots++) = 0$

for $N > 4$ (G. Mahlon, Phys. Rev. D 49, 2197 (1994)).

Closed formula for the "two $-$" case:

$$A_N(1^-,2^-,3^+,\ldots,N^+) = -\frac{i(-e\sqrt{2})^N}{8\pi^2}\sum_{\mathrm{Perm}(3\ldots N)}\sum_{j=4}^{N}\frac{[\langle 1\mid 3\rangle\langle j\mid 3\rangle^*\langle j\mid 2\rangle]^2}{\langle 3\mid 4\rangle\langle 4\mid 5\rangle\cdots\langle n\mid 3\rangle}\frac{\Lambda(3,\ldots,j)}{(2k_3\cdot k_j)^2},$$

where $q(i,j) \equiv k_2{+}k_i{+}\cdots{+}k_j$ and

$$\begin{aligned}\Lambda(3,\ldots,j) \;=\;& \mathrm{Li}_2\left[1-\frac{q^2(3,j)q^2(4,j-1)}{q^2(4,j)q^2(3,j-1)}\right] - \mathrm{Li}_2\left[1-\frac{q^2(3,j)}{q^2(4,j)}\right] - \mathrm{Li}_2\left[1-\frac{q^2(4,j-1)}{q^2(3,j-1)}\right]\\ &- \mathrm{Li}_2\left[1-\frac{q^2(3,j)}{q^2(3,j-1)}\right] - \mathrm{Li}_2\left[1-\frac{q^2(4,j-1)}{q^2(4,j)}\right] - \frac{1}{2}\ln^2\left[\frac{q^2(3,j-1)}{q^2(4,j)}\right]\end{aligned}$$

(G. Mahlon, FermilabConf94/421-T, hep-ph/9412350.)

S. Badger, N.E.J. Bjerrum-Bohr, P. Vanhove, JHEP 0902:038 (2009):

$N \geq 8$ **photon amplitudes involve only box functions (no triangles).**

(Using both the worldline formalism and unitarity methods.)

Similar to the "no triangle" property of $\mathcal{N} = 8$ supergravity

(important for possible finiteness).

Low energy limit of massive photon amplitudes

L.C. Martin, C. S., V.M. Villanueva, NPB 668, 335 (2003).

Low energy = large mass limit: All photon energies small compared to the electron mass, $\omega_i \ll m$.

Information on the N photon amplitudes in this limit contained in the Euler-Heisenberg resp. Weisskopf Lagrangians:

Spinor QED

$$\mathcal{L}^{(EH)}_{\rm spin}(F) \;=\; -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{1}{3}(a^2-b^2)T^2 - 1\right]$$

(W. Heisenberg and H. Euler, 1936)

Scalar QED

$$\mathcal{L}^{(EH)}_{\rm scal}(F) \;=\; \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{1}{6}(a^2-b^2)T^2 - 1\right]$$

(V. Weisskopf, 1936)

Here T is the proper-time of the loop particle and a, b are defined by $a^2 - b^2 = B^2 - E^2, \quad ab = \mathbf{E}\cdot\mathbf{B}$.

N - photon amplitudes in the low energy limit

$$
\begin{aligned}
A^{(EH)}_{\rm spin}[\varepsilon_1^+;\dots;\varepsilon_K^+;\varepsilon_{K+1}^-;\dots;\varepsilon_N^-] &= -\frac{m^4}{8\pi^2}\Bigl(\frac{2ie}{m^2}\Bigr)^N (N-3)! \\
&\times \sum_{k=0}^{K}\sum_{l=0}^{N-K}(-1)^{N-K-l}\frac{B_{k+l}B_{N-k-l}}{k!l!(K-k)!(N-K-l)!}\chi_K^+\chi_{N-K}^-
\end{aligned}
$$

$$
\begin{aligned}
A^{(EH)}_{\rm scal}[\varepsilon_1^+;\dots;\varepsilon_K^+;\varepsilon_{K+1}^-;\dots;\varepsilon_N^-] &= \frac{m^4}{16\pi^2}\Bigl(\frac{2ie}{m^2}\Bigr)^N (N-3)! \\
&\times \sum_{k=0}^{K}\sum_{l=0}^{N-K}(-1)^{N-K-l}\frac{\bigl(1-2^{1-k-l)}\bigr)\bigl(1-2^{1-N+k+l}\bigr)B_{k+l}B_{N-k-l}}{k!l!(K-k)!(N-K-l)!}\chi_K^+\chi_{N-K}^-
\end{aligned}
$$

The B_k are Bernoulli numbers.

The variables $\chi_K^{\pm}$ are written, in spinor helicity notation,

$$\chi_K^+ = \frac{(\frac{K}{2})!}{2^{\frac{K}{2}}}\Bigl\{[12]^2[34]^2\cdots[(K-1)K]^2 + \text{ all permutations}\Bigr\}$$
$$\chi_K^- = \frac{(\frac{K}{2})!}{2^{\frac{K}{2}}}\Bigl\{\langle 12\rangle^2\langle 34\rangle^2\cdots\langle (K-1)K\rangle^2 + \text{ all permutations}\Bigr\}$$

These variables appear naturally in the low energy limit. Since they require even numbers of positive and negative helicity polarizations, we get a

Selection rule ("double Furry theorem"):

$$A^{(EH)}_{\text{spin,scal}}[\varepsilon_1^+;\ldots;\varepsilon_K^+;\varepsilon_1^-;\ldots;\varepsilon_L^-] = 0$$

unless both K and L are even. *This rule holds to all loop orders.*

For the MHV ("all +" or "all −") case:

$$A^{(EH)}_{\text{spin}}[\varepsilon_1^+;\dots;\varepsilon_N^+] = -2A^{(EH)}_{\text{scal}}[\varepsilon_1^+;\dots;\varepsilon_N^+]$$

Corresponds to a self-dual background, in which the Dirac operator has a quantum-mechanical supersymmetry (M.J.Duff, C.J. Isham, PLB 86, 157 (1979); G.V. Dunne, H. Gies, C.S., JHEP 0211:032 (2002)).

Worldline approach to Einstein-Maxwell theory

Worldline representation of the one-loop N photon M graviton amplitude:

Photon vertex operator:

$$
\begin{aligned}
V^A_{\rm scal}[k,\varepsilon] &= \int_0^T d\tau\, \varepsilon\cdot\dot{x}(\tau)\,\mathrm{e}^{ik\cdot x(\tau)} \\
V^A_{\rm spin}[k,\varepsilon] &= \int_0^T d\tau\,\Big(\varepsilon\cdot\dot{x}(\tau)+2i\varepsilon\cdot\psi k\cdot\psi\Big)\,\mathrm{e}^{ik\cdot x(\tau)}
\end{aligned}
$$

Graviton vertex operator:

$$
\begin{aligned}
V^h_{\rm scal}[k,\varepsilon] &= \varepsilon_{\mu\nu}\int_0^T d\tau\,\Big(\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)+a^\mu(\tau)a^\nu(\tau)+b^\mu(\tau)c^\nu(\tau)+4\bar{\xi}(\delta^{\mu\nu}k^2-k^\mu k^\nu)\Big)\,\mathrm{e}^{ik\cdot x(\tau)} \\
V^h_{\rm spin}[k,\varepsilon] &= \varepsilon_{\mu\nu}\int_0^T d\tau\,\Big(\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)+a^\mu(\tau)a^\nu(\tau)+b^\mu(\tau)c^\nu(\tau) \\
&\qquad +2(\psi^\mu(\tau)\dot{\psi}^\nu(\tau)+\alpha^\mu(\tau)\alpha^\nu(\tau)+i\dot{x}^\mu(\tau)\psi^\nu(\tau)\psi(\tau)\cdot k)\,\mathrm{e}^{ik\cdot x(\tau)}
\end{aligned}
$$

Correlators:

$$
\begin{aligned}
\langle x^\mu(\tau_1) x^\nu(\tau_2)\rangle &= -\delta^{\mu\nu}\, G_B(\tau_1,\tau_2) \\
\langle \psi^\mu(\tau_1) \psi^\nu(\tau_2)\rangle &= \frac{1}{2}\delta^{\mu\nu}\, G_F(\tau_1,\tau_2)
\end{aligned}
$$

G_B, G_F are the 'worldline Green's functions'

$$
\begin{aligned}
G_B(\tau_1,\tau_2) &= \mid \tau_1 - \tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{T} - \frac{T}{6} \\
G_F(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2)
\end{aligned}
$$

The ghost fields a, b, c, α have to do with the nontrivial path integral measure in curved space,

$$\mathcal{D}x = Dx \prod_{0 \le \tau < T} \sqrt{\det g_{\mu\nu}(x(\tau))}$$

Their correlators involve only δ functions

$$\begin{aligned} \langle a^\mu(\tau_1)\, a^\nu(\tau_2)\rangle &= 2\delta(\tau_1 - \tau_2)\, \delta^{\mu\nu} \\ \langle b^\mu(\tau_1)\, c^\nu(\tau_2)\rangle &= -4\delta(\tau_1 - \tau_2)\, \delta^{\mu\nu} \\ \langle \alpha^\mu(\tau_1)\, \alpha^\nu(\tau_2)\rangle &= \delta(\tau_1 - \tau_2)\, \delta^{\mu\nu}, \end{aligned}$$

and essentially remove singularities.

One-loop effective action in Einstein-Maxwell theory

Effective action for the low energy limit of the N photon – one graviton amplitude (F. Bastianelli, J.M. Dávila, C.S., JHEP 03 (2009) 086; J. M. Dávila, C.S., CQG 27 075007 (2010)).

$$
\begin{aligned}
\mathcal{L}_{\text{spin}}^{(EH)(R)} \;=\; & -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\mathrm{det}^{-1/2}\left[\frac{\tan(FT)}{FT}\right] \\
& \times\Biggl\{1+\frac{iT^2}{8}F_{\mu\nu;\alpha\beta}\,\mathcal{G}_{B11}^{\alpha\beta}\Bigl(\dot{\mathcal{G}}_{B11}^{\mu\nu}-2\,\mathcal{G}_{F11}^{\mu\nu}\Bigr) \\
& +\frac{iT^2}{8}\left(F_{\mu\nu;\beta\alpha}+F_{\mu\nu;\alpha\beta}\right)\dot{\mathcal{G}}_{B11}^{\mu\beta}\mathcal{G}_{B11}^{\nu\alpha}+\frac{T}{3}R_{\alpha\beta}\,\mathcal{G}_{B11}^{\alpha\beta} \\
& -\frac{iT^2}{24}F_{\lambda\nu}R^{\lambda}{}_{\alpha\beta\mu}\,\Bigl(\dot{\mathcal{G}}_{B11}^{\nu\mu}\,\mathcal{G}_{B11}^{\alpha\beta}+\dot{\mathcal{G}}_{B11}^{\alpha\mu}\,\mathcal{G}_{B11}^{\nu\beta}+\dot{\mathcal{G}}_{B11}^{\beta\mu}\,\mathcal{G}_{B11}^{\nu\alpha}+4\,\mathcal{G}_{F11}^{\mu\nu}\,\mathcal{G}_{B11}^{\alpha\beta}\Bigr) \\
& +\frac{T}{12}R_{\mu\alpha\beta\nu}\Bigl(\dot{\mathcal{G}}_{B11}^{\mu\alpha}\dot{\mathcal{G}}_{B11}^{\beta\nu}+\dot{\mathcal{G}}_{B11}^{\mu\beta}\dot{\mathcal{G}}_{B11}^{\alpha\nu}+\Bigl(\ddot{\mathcal{G}}_{B11}^{\mu\nu}-2g^{\mu\nu}\delta(0)\Bigr)\mathcal{G}_{B11}^{\alpha\beta} \\
& +\dot{\mathcal{G}}_{B11}^{\alpha\beta}\,\mathcal{G}_{F11}^{\mu\nu}+\dot{\mathcal{G}}_{B11}^{\nu\beta}\,\mathcal{G}_{F11}^{\mu\alpha}-\mathcal{G}_{B11}^{\alpha\beta}\,\Bigl(\dot{\mathcal{G}}_{F11}^{\mu\nu}-2g^{\mu\nu}\delta(0)\Bigr)\Bigr) \\
& -\frac{1}{6}T^3F_{\alpha\beta;\gamma}\,F_{\mu\nu;\delta}\,\int_0^1 d\tau_1\Bigl(\dot{\mathcal{G}}_{B12}^{\alpha\nu}\,\dot{\mathcal{G}}_{B12}^{\beta\mu}\,\mathcal{G}_{B12}^{\gamma\delta}+\dot{\mathcal{G}}_{B12}^{\alpha\nu}\,\mathcal{G}_{B12}^{\beta\delta}\,\dot{\mathcal{G}}_{B12}^{\gamma\mu}+\frac{3}{2}\,\mathcal{G}_{B12}^{\gamma\delta}\,\mathcal{G}_{F12}^{\alpha\mu}\,\mathcal{G}_{F12}^{\beta\nu}\Bigr)\Biggr\}
\end{aligned}
$$

Field-dependent worldline correlators:

$$
\begin{aligned}
\langle x^\mu(\tau_1)x^\nu(\tau_2)\rangle &= -\mathcal{G}_B^{\mu\nu}(\tau_1,\tau_2)\\
\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= \frac{1}{2}\mathcal{G}_F^{\mu\nu}(\tau_1,\tau_2)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_B(\tau_1,\tau_2) &= \frac{T}{2(\mathcal{Z})^2}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}} + i\mathcal{Z}\dot{G}_{B12} - 1\right)\\
\mathcal{G}_F(\tau_1,\tau_2) &= G_{F12}\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}}{\cos(\mathcal{Z})}
\end{aligned}
$$

Here $\mathcal{Z}_{\mu\nu} = eTF_{\mu\nu}$.

Expand out in powers of $F_{\mu\nu}$, do the integrals, reduce number of terms using Bianchi identities....

Two photon - one graviton level

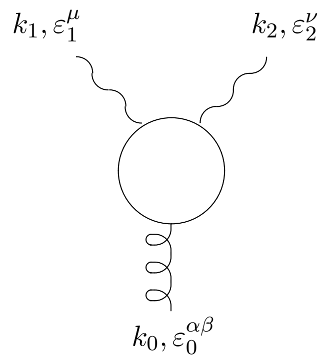
$$
\begin{aligned}
\mathcal{L}^{h\gamma\gamma}_{\text{scal}} &= \frac{1}{360\, m^2(4\pi)^2}\Big[5(6\xi-1)RF^2_{\mu\nu} + 4R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} - 6R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \\
&\qquad -2(\nabla^{\alpha}F_{\alpha\mu})^2 - 8(\nabla_{\alpha}F_{\mu\nu})^2 - 12F_{\mu\nu}\Box F^{\mu\nu}\Big] \\
\mathcal{L}^{h\gamma\gamma}_{\text{spin}} &= \frac{1}{180\, m^2(4\pi)^2}\Big[5RF^2_{\mu\nu} - 4R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} - 9R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \\
&\qquad +2(\nabla^{\alpha}F_{\alpha\mu})^2 - 7(\nabla_{\alpha}F_{\mu\nu})^2 - 18F_{\mu\nu}\Box F^{\mu\nu}\Big]
\end{aligned}
$$

Four photon - one graviton level

$$
\begin{aligned}
\mathcal{L}_{\rm spin}^{h,4\gamma} \;=\;& -\frac{1}{8\,\pi^2}\frac{1}{m^6}\Bigg[-\frac{1}{432}R(F_{\mu\nu})^4+\frac{7}{1080}R\,{\rm tr}[F^4]-\frac{1}{945}R_{\alpha\beta}(F^4)^{\alpha\beta}\\
& -\frac{1}{540}R_{\alpha\beta}(F^2)^{\alpha\beta}(F_{\gamma\delta})^2+\frac{4}{135}R_{\alpha\mu\beta\nu}(F^3)^{\alpha\mu}F^{\beta\nu}+\frac{1}{108}R_{\alpha\mu\beta\nu}F^{\alpha\mu}F^{\beta\nu}(F_{\gamma\delta})^2\\
& +\frac{7}{270}(F^3)^{\mu\nu}\,\Box F_{\mu\nu}+\frac{1}{108}F^{\mu\nu}\Box F_{\mu\nu}(F_{\gamma\delta})^2+\frac{1}{270}F_{\mu\nu;\alpha\beta}(F^2)^{\alpha\beta}F^{\mu\nu}\\
& -\frac{1}{540}(F_{\alpha\beta;\gamma})^2(F_{\mu\nu})^2-\frac{1}{945}F_{\mu\nu;\alpha}\,F^{\mu\nu}{}_{;\beta}(F^2)^{\alpha\beta}-\frac{11}{945}F_{\alpha\beta;\gamma}F_{\mu}{}^{\beta;\gamma}(F^2)^{\alpha\mu}\\
& -\frac{2}{189}F_{\alpha\beta;\gamma}F_{\mu\nu;}{}^{\gamma}F^{\alpha\mu}F^{\beta\nu}-\frac{2}{189}F_{\alpha\beta;\gamma}F_{\mu}{}^{\alpha}{}_{;\delta}F^{\beta\mu}F^{\gamma\delta}\Bigg]
\end{aligned}
$$

The graviton – photon – photon amplitude and its properties

F. Bastianelli, O. Corradini, J.M. Dávila, C.S., in preparation



Effective action $\rightarrow$ on-shell amplitude with graviton polarizations

$$
\begin{aligned}
\varepsilon^{++}_{0\mu\nu}(k_0) &= \varepsilon^{+}_{\mu}(k_0)\varepsilon^{+}_{\nu}(k_0) \\
\varepsilon^{--}_{0\mu\nu}(k_0) &= \varepsilon^{-}_{\mu}(k_0)\varepsilon^{-}_{\nu}(k_0)
\end{aligned}
$$

Nonvanishing only

$$
\begin{aligned}
A_{spin}^{(++;++)} &= \frac{\kappa\, e^2}{90(4\pi)^2 m^2}\,[01]^2\,[02]^2 \\
A_{spin}^{(--;--)} &= \frac{\kappa\, e^2}{90(4\pi)^2 m^2}\,\langle 01\rangle^2\,\langle 02\rangle^2
\end{aligned}
$$

$$
\begin{aligned}
A_{spin}^{(++;++)} &= (-2)\,A_{scal}^{(++;++)} \\
A_{spin}^{(--;--)} &= (-2)\,A_{scal}^{(--;--)}
\end{aligned}
$$

Simple relation to the four photon amplitudes:

$$
\begin{aligned}
A^{++++}[k_1,k_2,k_3,k_4] &\sim [12]^2[34]^2+[13]^2[24]^2+[14]^2[23]^2\,,\\
A^{+++-}[k_1,k_2,k_3,k_4] &= 0\,,\\
A^{++--}[k_1,k_2,k_3,k_4] &\sim [12]^2\langle 34\rangle^2\,,\\
A^{+---}[k_1,k_2,k_3,k_4] &= 0\,,\\
A^{----}[k_1,k_2,k_3,k_4] &\sim \langle 12\rangle^2\langle 34\rangle^2+\langle 13\rangle^2\langle 24\rangle^2+\langle 14\rangle^2\langle 23\rangle^2\,,
\end{aligned}
$$

Replacing $k_1 \to k_0, k_2 \to k_0$ in the 4 photon amplitudes,

$$
\begin{array}{rcl}
A^{++++}[k_0,k_0,k_3,k_4] \sim & 2[03]^2[04]^2 & \sim A^{++;++}[k_0,k_3,k_4]\\
A^{+++-}[k_0,k_0,k_3,k_4] = & 0 & = A^{++;+-}[k_0,k_3,k_4]\\
A^{++--}[k_0,k_0,k_3,k_4] = & 0 & = A^{++;--}[k_0,k_3,k_4]\\
A^{--++}[k_0,k_0,k_3,k_4] = & 0 & = A^{--;++}[k_0,k_3,k_4]\\
A^{--+-}[k_0,k_0,k_3,k_4] = & 0 & = A^{--;+-}[k_0,k_3,k_4]\\
A^{----}[k_0,k_0,k_3,k_4] \sim & 2\langle 03\rangle^2\langle 04\rangle^2 & \sim A^{--;--}[k_0,k_3,k_4]
\end{array}
$$

For the three-point case, have also checked the (off-shell) Ward identities:

Gauge Ward identity:

$$k_{i\alpha_i} A^{\mu\nu,\alpha_1\ldots\alpha_N}[k_0,\ldots,k_N] = 0\,, \qquad i = 1,\ldots,N$$

Gravitational Ward identity:

$$\begin{aligned} 2k_{0\mu} A^{\mu\nu,\alpha_1\ldots\alpha_N}[k_0,\ldots,k_N] \;&=\; -\sum_{i=1}^{N} A^{\mu\alpha_1\ldots\widehat{\alpha}_i\ldots\alpha_N}[k_0+k_i,k_1,\ldots,\widehat{k}_i,\ldots,k_N] \\ &\qquad \times(\delta_\mu^{\alpha_i} k_i^\nu - \eta^{\alpha_i\nu} k_{i\mu}) \end{aligned}$$

Outlook

- On to the 4 photon - 1 graviton case...
- In the low energy limit, is the information on the N photon – M graviton amplitude contained in the $N+2M$ photon amplitude?