(2+1)Dirac Darboux transformation

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Darboux transformation method

The Darboux transformation method [1]based on the intertwining relation:

$$LH_0 = H_1 L,$$

where L is transformation operator, H_0, H_1 are one-dimensional Hamiltonians of the Shrödinger equation.

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[1]G. Darboux,*Compt. Rend. Acad. Sci. Paris*, **94**, 1343; 1456 (1882) [physics/9908003].
G. Darboux,*Compt. Leçons sur la théorie générale des surfaces et les application géométriques du calcul infinitésimate*, Paris: Guatier–Villar et Fils, 522 (1889).

• For application the Darboux transformation method the initial Shrödinger equation must be analytically solvable

$$H_0\psi = E\psi.$$

• Then the transformed Shrödinger equation will the analytically solvable too

$$H_1\tilde{\psi} = E\tilde{\psi},$$

where $\tilde{\psi} = L\psi$.

• The Darboux transformation of Shrödinger equation is related with supersymmetric quantum mechanics[2] in the two supercharges case Q_1, Q_2.

• In charge terms $Q = (Q_1 + iQ_2)/\sqrt{2}$, $Q^+ = (Q_1 - iQ_2)/\sqrt{2}$ the algebra

$$\{Q_i, Q_j\} = \delta_{ij} H, \quad i, j = 1, 2, ...N, \tag{1}$$

$$[Q_i, H] = 0, \tag{2}$$

[2] E. Witten, *Nucl. Phys.* B, **185**, 513 (1981).
E. Witten, *Nucl. Phys.* B, **202**, 253 (1982).

can be represeted by the following relations [3]:

$$H = \{Q, Q^+\}, \quad Q^2 = (Q^+)^2 = 0, \tag{3}$$

$$[Q, H] = [Q^+, H] = 0, \tag{4}$$

where

$$Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} 0 & L^+ \\ 0 & 0 \end{pmatrix}, \tag{5}$$

$$H \equiv \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}. \tag{6}$$

The charges Q, Q^+ are commutates with Hamiltonian H, thus spectrum is degenerated .

[3] V.F. Marchenko, *Inverse dispersion problem*, Kharkov, KhSU(1960)

• The Darboux transformation are applied to generalized Dirac equation in two dimensions[4].

• At present report we apply the Darboux transformation to (2+1)–dimensional Dirac equation.

[4] E. Pozdeeva, A. Schulze-Halberg, Darboux transformations for a generalized Dirac equation in two dimensions, Journal of mathematical physics 51, 113501 (2010)

Dirac equation.

- The non-stationary (3+1) Dirac equation has the form:

$$i\hbar\frac{\partial\psi(t,x,y,z)}{\partial t} = -i\hbar c\alpha_i\frac{\partial\psi(t,x,y,z)}{\partial x_i} - W,$$

where $-W = mc^2\beta - V$.
For simplicity we work in unit $\hbar = c = 1$

$$i\frac{\partial\psi(t,x,y,z)}{\partial t} = -i\alpha_i\frac{\partial\psi(t,x,y,z)}{\partial x_i} - W,$$

where $-W = m\beta - V$.
Here β, α_i are matrixes in standard Dirac-Pauli representation.
We suppose that the potential V as well as the wave function ψ not depend on the z. Thus we suppose that the particles move on plane.

The intertwine relation for (2+1) Dirac equation

- Let us consider the intertwine relation

$$LH_0 = H_1 L \tag{7}$$

where

$$L = A\partial_y + B\partial_x + \Omega, \tag{8}$$

$$H_0 = i\partial_t + i\alpha_x\partial_x + i\alpha_y\partial_y + W_0, \tag{9}$$

$$H_1 = i\partial_t + i\alpha_x\partial_x + i\alpha_y\partial_y + W_1. \tag{10}$$

Substitute the operators forms to the intertwining relations we obtain conditions to matrixes A, B

$$
\begin{aligned}
&[A, i\alpha_1] + [B, i\alpha_2] = 0, \\
&[A, i\alpha_2] = 0, \\
&[B, i\alpha_1] = 0.
\end{aligned}
$$

These relations can be satisfied if

$$
\begin{aligned}
A &= f(t,x,y)\alpha_2 + g_A(t,x,y)I, \\
B &= f(t,x,y)\alpha_1 + g_B(t,x,y)I.
\end{aligned}
$$

Thus, we can consider only the following equations:

$$
\begin{aligned}
&iB_t + i\alpha_1 B_x + i\alpha_2 B_y + W_1 B - BW_0 + [i\alpha_1, \Omega] = 0, \\
&iA_t + i\alpha_1 A_x + i\alpha_2 A_y + W_1 A - AW_0 + [i\alpha_2, \Omega] = 0, \\
&i\Omega_t + i\alpha_1 \Omega_x + i\alpha_2 \Omega_y + W_1 \Omega - \Omega W_0 - AW_{0y} - BW_{0x} = 0.
\end{aligned}
$$

Simplest choosing of A and B
Standard Darboux transformation

Let us consider the case: $A = 0$, $B = I$. In this case system of equations has the form:

$$D = -[i\alpha_1, \Omega],$$
$$[i\alpha_2, \Omega] = 0, \quad \text{or} \quad \alpha_2\Omega = \Omega\alpha_2$$
$$i\Omega_t + i\alpha_1\Omega_x + i\alpha_2\Omega_y + W_1\Omega - \Omega W_0 - W_{0x} = 0.$$

Let us make linearization. Suppose

$$\Omega = -u_x u^{-1}$$

and substitute it into the equation to Ω using the property of Ω:

$$[u^{-1}(iu_t + i\alpha_1 u_x + i\alpha_2 u_y + W_0)]_x = 0$$

Analogously, if $A = I$, $B = 0$, then we obtain the following system of equations:

$$D = -[i\alpha_2, \Omega],$$
$$[i\alpha_1, \Omega] = 0, \quad \text{or} \quad \alpha_1\Omega = \Omega\alpha_1$$
$$i\Omega_t + i\alpha_1\Omega_x + i\alpha_2\Omega_y + W_1\Omega - \Omega W_0 - W_{0y} = 0.$$

Let us try to make linearization

$$\Omega = -u_y u^{-1}$$

and substitute it into the equation to Ω using the another property of Ω:

$$[u^{-1}(iu_t + i\alpha_1 u_x + i\alpha_2 u_y + W_0)]_y = 0$$

The considered simplifications can be useful for the construction $W_i = \alpha_3\Phi(x, y)$ type potentials for the Dirac equation. The potentials of this type can correspond to a magnetic field.

Choosing of $A = I$ and $B = I$

- Let us consider the case then $A = I$, $B = I$:

$$L = \partial_x + \partial_y + \Omega,$$

then from

$$\begin{aligned} &iB_t + i\alpha_1 B_x + i\alpha_2 B_y + W_1 B - BW_0 + [i\alpha_1, \Omega] = 0, \\ &iA_t + i\alpha_1 A_x + i\alpha_2 A_y + W_1 A - AW_0 + [i\alpha_2, \Omega] = 0, \\ &i\Omega_t + i\alpha_1 \Omega_x + i\alpha_2 \Omega_y + W_1 \Omega - \Omega W_0 - AW_{0y} - BW_{0x} = 0. \end{aligned}$$

we obtain the following equations system:

$$\begin{aligned} &D = [\Omega, i\alpha_1], \\ &D = [\Omega, i\alpha_2], \\ &i\Omega_t + i\alpha_1 \Omega_x + i\alpha_2 \Omega_y + W_1 \Omega - \Omega W_0 - W_{0y} - W_{0x} = 0 \end{aligned}$$

From potential difference expressions, evidently that Ω has the form:

$$\Omega = \Omega_I I + \Omega_1 \alpha_1 + \Omega_2 \alpha_2,$$

where $\Omega_2 = -\Omega_1$.

If suppose $\Omega = -u_x u^{-1}$ and use the equation to Ω we get:

$$-u\left(\left(u^{-1}(iu_t + i\alpha_1 u_x + i\alpha_2 u_y + W_0 u)\right)_x - u^{-1} D u_y\right) u^{-1} = W_{0y}.$$

The equation can be represented in the form

$$\left(u^{-1}(iu_t + i\alpha_1 u_x + i\alpha_2 u_y + W_0 u)\right)_x - u^{-1} D u_y = -u^{-1} W_{0y} u$$

or in the form

$$\left(u^{-1}(iu_t + i\alpha_1 u_x + i\alpha_2 u_y + W_0 u)\right)_x = -u^{-1}(W_{0y} + D u_y u^{-1}) u.$$

• The integration of last equation gives the following:

$$iu_t + i\alpha_1 u_x + i\alpha_2 u_y + W_0 u = -u \int u^{-1}(W_{0y} + Du_y u^{-1}) u dx,$$

where we can suppose

$$\Lambda(t, y) = \int u^{-1}(W_{0y} + Du_y u^{-1}) u dx$$

and consider the equation:

$$iu_t + i\alpha_1 u_x + i\alpha_2 u_y + W_0 u = -u\Lambda(t, y).$$

Now suppose that:

$$u = \hat{u}\Gamma(t, y),$$

and obtain the initial Dirac equation with right part:

$$i\hat{u}_t + i\alpha_1\hat{u}_x + i\alpha_2\hat{u}_y + W_0\hat{u} = -\hat{u}(\Lambda + i\Gamma_t\Gamma^{-1} + i\alpha_2\Gamma_y\Gamma^{-1})$$

Here we can choose matrix function Γ such way that ,

$$i\Gamma_t\Gamma^{-1} + i\alpha_2\Gamma_y\Gamma^{-1} = -\Lambda(t, y) = -\int u^{-1}(W_{0y} + Du_y u^{-1})u dx$$

and from where evidently:

$$i\hat{u}_t + i\alpha_1\hat{u}_x + i\alpha_2\hat{u}_y + W_0\hat{u} = 0.$$

- Thus, in considered case

$$\Omega = \Omega_I I + \Omega_1\alpha_1 + \Omega_2\alpha_2, \quad \Omega_2 = -\Omega_1, \quad \Omega = -u_x u^{-1}$$

$$D = [\Omega, i\alpha_1] = [\Omega, i\alpha_2],$$

$$u = \hat{u}\Gamma(t, y)$$

$$i\Gamma_t\Gamma^{-1} + i\alpha_2\Gamma_y\Gamma^{-1} = -\Lambda(t, y) = -\int \hat{u}^{-1}(W_{0y} + D\hat{u}_y\hat{u}^{-1})\hat{u}dx$$

where $\hat{u}$ is matrix solution to initial Dirac equation:

$$i\hat{u}_t + i\alpha_1\hat{u}_x + i\alpha_2\hat{u}_y + W_0\hat{u} = 0.$$

Such as $u_x u^{-1} = \hat{u}_x\hat{u}^{-1}$, the potential difference $W_1 - W_0 = D$ can be represented it in the form: $D = [-\hat{u}_x\hat{u}^{-1}, i\alpha_2]$.

Analogically we can consider the $\Omega = -u_y u^{-1}$ case:

$$\Omega = \Omega_I I + \Omega_1 \alpha_1 + \Omega_2 \alpha_2, \quad \Omega_2 = -\Omega_1, \quad \Omega = -u_y u^{-1}$$

$$D = [\Omega, i\alpha_1] = [\Omega, i\alpha_2],$$

$$u = \hat{u}\Gamma(t, x),$$

$$i\Gamma_t \Gamma^{-1} + i\alpha_1 \Gamma_x \Gamma^{-1} = -\Lambda(t, x) = -\int \hat{u}^{-1}(W_{0x} + D\hat{u}_x \hat{u}^{-1})\hat{u} dy,$$

where $\hat{u}$ is solution to initial Dirac equation:

$$i\hat{u}_t + i\alpha_1 \hat{u}_x + i\alpha_2 \hat{u}_y + W_0 \hat{u} = 0.$$

Such as $u_y u^{-1} = \hat{u}_y \hat{u}^{-1}$, the potential difference $W_1 - W_0 = D$ can be represented it in the form: $D = [-\hat{u}_y \hat{u}^{-1}, i\alpha_2]$

Summarizing

•In this report the application of Darboux transformation to two-dimensional non-stationary Dirac equation firstly considered.

• Also the intertwine operator $L = A\partial_x + B\partial_y + \Omega$ with two nonzero coefficients A, B firstly applied to DE.

• The transformation potential can be constructed using the matrix solution of initial Dirac equation.

Thak you for attention