Fusion coefficients for tensor powers

B_n-case

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Supersymmetries and Quantum Symmetries, SQS-2011
Joint Institute for Nuclear Research, July 18-23, 2011

Outline

1. Motivation
   - Previous Work

2. Our Approach
   - Main Results

3. Other approach to fusion.
   - Basic Properties

Motivation

Hamiltonian $\mathcal{H}$ of an integrable spin chain with p sites is defined on the tensor product of state spaces at sites: $\otimes^p \mathbb{C}^n$.

If the hamiltonian is invariant w.r.t. a Lie algebra (or a quantum group) $\mathfrak{g}$

$$[\mathcal{H}, a] = 0, \quad a \in \mathfrak{g}$$

then its spectrum has an obvious multiplet structure.

The latter one is connected with the decomposition of space of states of the system into a direct sum of spaces of irreps of $\mathfrak{g}$:

$H = \bigoplus_\nu L_\nu \otimes \mathbb{C}^{m(\nu)}$, where $m(\nu)$ are the multiplicities of the corresponding irreps.

In this report we shall consider the finite dimensional Lie algebras of series B_n

Combinatorial Studies. Klimyk formulas

There are numerous combinatorial studies of the problem [Kirillov & Reshetikhin 1996] usually based on the complicated path counting. In most of these studies the simply laced algebras are considered [Kleber 1996].

There is also a series of works [Hatayama, Kuniba et al 1998] dealing with fermonic formulas for all the classical series of quantum algebras and using the crystal basis approach.

Practical computations with these formulas are highly difficult.

The problem in details

Simple Lie algebras B_n , rank$= n$.

Simple roots: $\alpha_i|_{i=1..n}$

The second fundamental weight: ω_2

The fundamental module: L^{ω_2}.

Consider the tensor powers $(L^{\omega_2})^{\otimes p}|_{p\in Z_+}$ and their decompositions

$$(L^{\omega_2})^{\otimes p} = \sum_{\nu} m(\nu, p) L^{\nu}. \quad (1)$$

Our aim is to find multiplicities $m(\nu, p)$ as a function of ν and p .

To solve this problem we study the properties of singular element of a module L^{μ} : $\Psi^{L^{\mu}} = \sum_{w\in W} det(w) e^{w(\mu+\rho)-\rho}$ We define the multiplicity function on the set of singular weights. That allows us to find the explicit expression for $m(\nu, p)$ using the Weyl symmetry properties of the singular element.

Simplifying the recursion

Our problem is equivalent to the problem of reduction of irreducible module $(L^{\mu})^{\otimes p}$ of algebra $\oplus^{p}\mathfrak{g}$ to the diagonal $\mathfrak{g}$. Let us consider $\mathfrak{g} \oplus \mathfrak{g} \downarrow \mathfrak{g}_{\mathrm{diag}}$.

The Weyl formula $L^{\mu} \otimes L^{\nu}$:

$$\mathrm{ch}\left(L^{\mu}\right)\mathrm{ch}\left(L^{\nu}\right)_{\downarrow P_{diag}} = \sum_{\xi \in P_{diag}} m_{\xi}^{\mu\nu}\mathrm{ch}\left(L^{\xi}\right),$$

$$\left(\frac{\Psi^{(\mu)}\Psi^{(\nu)}}{\Psi^{(0)}\Psi^{(0)}}\right)_{\downarrow P_{diag}} = \sum_{\xi \in P_{diag}} m_{\xi}^{\mu\nu}\frac{\Psi_{diag}^{(\xi)}}{\Psi_{diag}^{(0)}}.$$

Using

$$\left(\Psi^{(\mu)}\Psi^{(\nu)}\right)_{\downarrow P_{diag}} = \Psi_{\downarrow P_{diag}}^{(\mu)}\Psi_{\downarrow P_{diag}}^{(\nu)} = \Psi_{diag}^{(\mu)}\Psi_{diag}^{(\nu)},$$

obtain

$$\left(\Psi_{diag}^{(0)}\right)^{-1}\Psi_{diag}^{(\mu)}\Psi_{diag}^{(\nu)} = \sum_{\xi \in P_{diag}} m_{\xi}^{\mu\nu}\Psi_{diag}^{(\xi)}.$$

Recurrence

Thus we have

$$\sum_{\xi \in P_{diag}} m_{\xi}^{\mu\nu} \Psi_{diag}^{(\xi)} = N\left(L_{diag}^{(\mu)}\right) \Psi_{diag}^{(\nu)} = \Psi_{diag}^{(\mu)} N\left(L_{diag}^{(\nu)}\right) = \sum_{\xi \in P_{diag}} M_{\xi}^{\mu\nu} e^{\xi}.$$

Here on the r.h.s is the decomposition of the initial formal element in the formal algebra.

Place ξ in the closure of the main Weyl chamber $\overline{C^{(0)}}$.
The values of $M_{\xi}^{\mu\nu}$ are the desired multiplicities $m_{\xi}^{\mu\nu}$ of L^{ξ}.

Recurrence 2

Now put $\mu = \omega$, $\nu = (p-1)\,\omega$

$$\sum_{\xi \in P} M_{\xi}^{\otimes^{p}\omega} e^{\xi} = N\left(L^{(\omega)}\right)\psi^{\left(\otimes^{(p-1)}\omega\right)}$$

$M_{\xi}^{\otimes^{p}\omega}$ defines the singular element $\psi^{\left(\otimes^{(p)}\omega\right)}$.
For the case of $\mathfrak{g} = B_n$ and $\omega = \omega_2$ we have the system of relations:

$$\begin{aligned} M_{\xi}^{\otimes^{p}\omega} - M_{\xi-\omega}^{\otimes^{(p-1)}\omega} &= \\ M_{\xi-\omega+\alpha_1}^{\otimes^{(p-1)}\omega} &+ M_{\xi-\omega+\alpha_1+\alpha_2}^{\otimes^{(p-1)}\omega} + \dots \\ \dots &+ M_{\xi-\omega+\alpha_1+\alpha_2+\dots+\alpha_n}^{\otimes^{(p-1)}\omega} \end{aligned} \quad (*)$$

How to solve the system

To solve these relations step by step?

We propose to find the solution using the Weyl symmetry of the singular element.

The function $M^{\otimes^p \omega}$ is

- zero outside the orbit of the highest weight of $L^{\otimes^p \omega}$,
- zero on the boundaries of the Weyl chambers,
- anti-invariant with respect to the Weyl group transformations.
- subject to a natural boundary condition, $M^{\otimes^p \omega}_{p\omega} = 1$.

Below these conditions are illustrated for B_2.

Symmetry properties of M for B_2

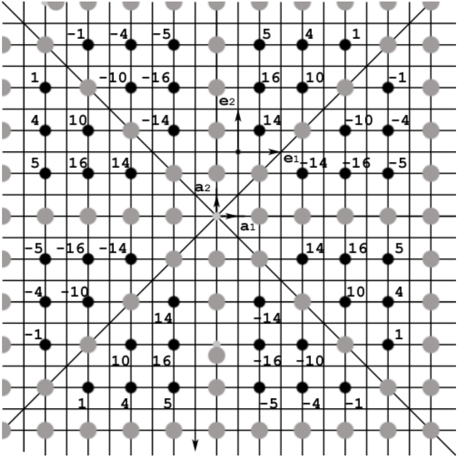

Figure: singular element $\Psi^{\left(\otimes^{(p)}\omega\right)}$ for $p = 5$

Explicit form of factors for B_2

- Zeros outside the orbit of the highest weight of $L^{\otimes^p \omega}$
  $\longrightarrow A \frac{1}{\left(\frac{p-a_1+3}{2}\right)!}$,
  here $\{a_i\}$ are the coordinates of a weight in the basis of symmetry center and $A \in \mathbb{R}$
  Notice that each elementary factor here has power one,
  and this is what the recurrent relations (*) dictate.
- Zeros on the boundaries of the Weyl chambers
  $\longrightarrow a_1, a_2, (a_1 + a_2), (a_1 - a_2)$
- Weyl anti-invariance
  $\longrightarrow\ a_1 a_2 \left(a_1 + a_2\right)\left(a_1 - a_2\right)$,
  $\longrightarrow A \frac{1}{\left(\frac{p-a_1+3}{2}\right)!} \frac{1}{\left(\frac{p-a_2+3}{2}\right)!} \frac{1}{\left(\frac{p+a_1+3}{2}\right)!} \frac{1}{\left(\frac{p+a_1+3}{2}\right)!}$,
- The condition $M_{p\omega}^{\otimes^p \omega} = 1$
  $\longrightarrow \frac{p!(p+2)!\ a_1 a_2 (a_1+a_2)(a_1-a_2)}{\left(\frac{p-a_1+3}{2}\right)!\left(\frac{p-a_2+3}{2}\right)!\left(\frac{p+a_1+3}{2}\right)!\left(\frac{p+a_2+3}{2}\right)!}$.

General solution for B_n

Using this algorithm we can construct the general solution for B_n:

$$M^{\otimes^p \omega}_{(B_n)\{a_i\}} = \prod_{k=0}^{n-1} \frac{(p+2k)!}{2^{2k}\left(\frac{p+a_{k+1}+2n-1}{2}\right)!\left(\frac{p-a_{k+1}+2n-1}{2}\right)!} \prod_{l=1}^{n} a_l \prod_{i<j} \left(a_i^2 - a_j^2\right).$$

Here all the variables are positive and integral.

The indexes $i, j = 1, \ldots, n$

Properties

Properties of the solution $M^{\otimes^p \omega}$.

- It can be directly checked that the obtained function $M_\xi^{\otimes^p \omega}$ satisfies the initial set of recursion relations (*).
- On the weight lattice P the function $M_\xi^{\otimes^p \omega}$ gives polynomials of p.
- The function $M_\xi^{\otimes^p \omega}$ accumulates an infinite set of curious properties of tensor power decompositions: it tells us that there exists an infinite number of families of modules whose multiplicities (in the decomposition) are described by one and the same function of p.
- If we consider a set of weights on a lattice for algebra B_n that have nonzero multiplicities M we would be able to obtain one more curious property: The expression for M for the boundary of that set is the exact expression for multiplicities for B_{n-1}. Note that this is true for the boundary of maximal dimension.

Crossections and polynomials

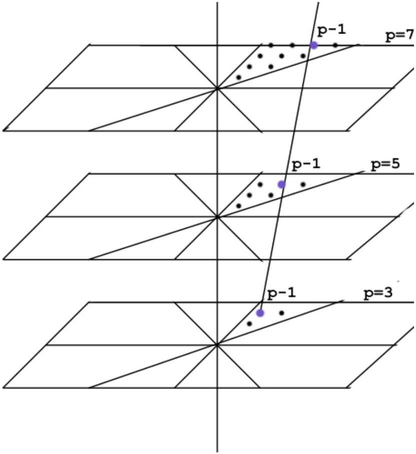

Figure: An infinite number of families of modules with multiplicies described by the same function

Summary

- We apply our algorithm to algebras of type B_n. Again it demonstrates that the properties of Weyl symmetry allow us to find an explicit expression for the multiplicities for the case of second fundamental module.
- This algorithm is applicable to other algebras. In previous work we obtained the multiplicities for the first fundamental module for A_n. [Kulish,Lyakhovsky,Postnova 2011]
- The essential point is that we consider the spinor fundamental module and not the relation between the rank of the algebra and the dimension of a module.
- The obtained expression allows to study asymptotics as $p \to \infty$
- When the modules are considered other than the (second) fundamental ones the solution can be described by polynomials of p but they are not necessarily factorized: the power of $M_{\xi}^{\otimes^p \omega}$ may be higher than the number of "integral" zeroes corresponding to the weight ξ. The answer can be certainly obtained recursively but the general expression becomes overcomplicated. This situation occurs, for example, for the first fundamental module in B_2.

For Further Reading I

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