Extremal vectors for Verma type factor-representations of $U_q(sl(3,\mathbb{C}))$

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Representations of $U_q\big(\mathrm{sl}(3)\big)$ – ordering $\mathbf{E}_{32}^{n_1}\mathbf{E}_{31}^{n_2}\mathbf{E}_{21}^{n_3}$

Algebra $U_q\big(\mathrm{gl}(3)\big)$ is generated by elements $\mathbf{L}_i = q^{\mathbf{E}_{ii}}$, $i = 1, 2, 3$, $\mathbf{L}_i^{-1}$, $\mathbf{E}_{12}$, $\mathbf{E}_{23}$, $\mathbf{E}_{21}$ a $\mathbf{E}_{32}$ which fulfill the relations

$$
\begin{aligned}
&\mathbf{L}_i\mathbf{L}_j = \mathbf{L}_j\mathbf{L}_i\,,\\
&\mathbf{L}_i\mathbf{E}_{jk} = q^{\delta_{ij}-\delta_{ik}}\mathbf{E}_{jk}\mathbf{L}_i\,,\\
&\big[\mathbf{E}_{i,i+1},\mathbf{E}_{j+1,j}\big] = \frac{\mathbf{L}_i\mathbf{L}_{i+1}^{-1} - \mathbf{L}_i^{-1}\mathbf{L}_{i+1}}{q-q^{-1}}\,\delta_{ij}\,,\\
&\mathbf{E}_{12}\mathbf{E}_{23}^2 - (q+q^{-1})\mathbf{E}_{23}\mathbf{E}_{12}\mathbf{E}_{23} + \mathbf{E}_{23}^2\mathbf{E}_{12} = 0\,,\\
&\mathbf{E}_{23}\mathbf{E}_{12}^2 - (q+q^{-1})\mathbf{E}_{12}\mathbf{E}_{23}\mathbf{E}_{12} + \mathbf{E}_{12}^2\mathbf{E}_{23} = 0\,,\\
&\mathbf{E}_{21}\mathbf{E}_{32}^2 - (q+q^{-1})\mathbf{E}_{32}\mathbf{E}_{21}\mathbf{E}_{32} + \mathbf{E}_{32}^2\mathbf{E}_{21} = 0\,,\\
&\mathbf{E}_{32}\mathbf{E}_{21}^2 - (q+q^{-1})\mathbf{E}_{21}\mathbf{E}_{32}\mathbf{E}_{21} + \mathbf{E}_{21}^2\mathbf{E}_{32} = 0\,.
\end{aligned}
$$

It is a Chevalley basis for this algebra.

We will need a Cartan-Weyl basis so we define

$$\mathbf{E}_{13} = \mathbf{E}_{12}\mathbf{E}_{23} - q^{-1}\mathbf{E}_{23}\mathbf{E}_{12}\,, \qquad \mathbf{E}_{31} = \mathbf{E}_{32}\mathbf{E}_{21} - q\mathbf{E}_{21}\mathbf{E}_{32}\,.$$

than we obtain

$$\mathbf{E}_{21}\mathbf{E}_{32} = q^{-1}\mathbf{E}_{32}\mathbf{E}_{21} - q^{-1}\mathbf{E}_{31}\,, \qquad \mathbf{E}_{21}\mathbf{E}_{31} = q\mathbf{E}_{31}\mathbf{E}_{21}\,, \qquad \mathbf{E}_{31}\mathbf{E}_{32} = q\mathbf{E}_{32}\mathbf{E}_{31}\,.$$

and PBW-theorem is valid for this algebra.

Now we denote

$$|n_1, n_2\rangle = \mathbf{E}_{32}^{n_1}\mathbf{E}_{31}^{n_2}\,, \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\,.$$

then we have

$$\mathbf{E}_{32}|n_1, n_2\rangle = |n_1 + 1, n_2\rangle\,, \qquad \mathbf{E}_{31}|n_1, n_2\rangle = q^{n_1}|n_1, n_2 + 1\rangle\,.$$

By induction we obtain

$$\mathbf{E}_{21}\mathbf{E}_{32}^{n} = q^{-n}\mathbf{E}_{32}^{n}\mathbf{E}_{21} - q^{-1}[n]_q\mathbf{E}_{32}^{n-1}\mathbf{E}_{31}\,.$$

and futher

$$\mathbf{E}_{21}|n_1, n_2\rangle = -q^{-1}[n_1]_q|n_1 - 1, n_2 + 1\rangle + q^{-n_1+n_2}|n_1, n_2\rangle\mathbf{E}_{21}\,.$$

It is simple to obtain

$$\begin{aligned}
\mathbf{L}_1|n_1, n_2\rangle &= q^{-n_2}|n_1, n_2\rangle\mathbf{L}_1\,,\\
\mathbf{L}_2|n_1, n_2\rangle &= q^{-n_1}|n_1, n_2\rangle\mathbf{L}_2\,,\\
\mathbf{L}_3|n_1, n_2\rangle &= q^{n_1+n_2}|n_1, n_2\rangle\mathbf{L}_3\,.
\end{aligned}$$

To compute $\mathbf{E}_{12}$ we need relation

$$\mathbf{E}_{12}\mathbf{E}_{31}^{n} = \mathbf{E}_{31}^{n}\mathbf{E}_{12} - q[n]_q\mathbf{E}_{32}\mathbf{E}_{31}^{n-1}\mathbf{L}_1\mathbf{L}_2^{-1}\,.$$

Using this relation we have

$$\mathbf{E}_{12}|n_1,n_2\rangle = -q[n_2]_q|n_1+1,n_2-1\rangle\mathbf{L}_1\mathbf{L}_2^{-1} + |n_1,n_2\rangle\mathbf{E}_{12}\,.$$

To compute the action of $\mathbf{E}_{23}$ we use the relations

$$\begin{aligned}
\mathbf{E}_{23}\mathbf{E}_{32}^{n} &= \mathbf{E}_{32}^{n}\mathbf{E}_{23} + [n]_q\mathbf{E}_{32}^{n-1}\,\frac{q^{-n+1}\mathbf{L}_2\mathbf{L}_3^{-1} - q^{n-1}\mathbf{L}_2^{-1}\mathbf{L}_3}{q-q^{-1}}\,,\\
\mathbf{E}_{23}\mathbf{E}_{31}^{n} &= \mathbf{E}_{31}^{n}\mathbf{E}_{23} + q^{n-1}[n]_q\mathbf{E}_{31}^{n-1}\mathbf{E}_{21}\mathbf{L}_2^{-1}\mathbf{L}_3
\end{aligned}$$

and we obtain

$$\begin{aligned}
\mathbf{E}_{23}|n_1,n_2\rangle &= \frac{q^{-n_1-n_2+1}[n_1]_q}{q-q^{-1}}\,|n_1-1,n_2\rangle\mathbf{L}_2\mathbf{L}_3^{-1} - \frac{q^{n_1+n_2-1}[n_1]_q}{q-q^{-1}}\,|n_1-1,n_2\rangle\mathbf{L}_2^{-1}\mathbf{L}_3 + \\
&\quad + q^{n_2-1}[n_2]_q|n_1,n_2-1\rangle\mathbf{E}_{21}\mathbf{L}_2^{-1}\mathbf{L}_3 + |n_1,n_2\rangle\mathbf{E}_{23}\,.
\end{aligned}$$

Similarly, for $\mathbf{E}_{13}$ we use

$$\begin{aligned}\mathbf{E}_{13}\mathbf{E}_{32}^{n} &= \mathbf{E}_{32}^{n}\mathbf{E}_{13} + q^{-n}[n]_q\mathbf{E}_{32}^{n-1}\mathbf{E}_{12}\mathbf{L}_2\mathbf{L}_3^{-1}\,,\\ \mathbf{E}_{13}\mathbf{E}_{31}^{n} &= \mathbf{E}_{31}^{n}\mathbf{E}_{13} + [n]_q\mathbf{E}_{31}^{n-1}\,\frac{q^{-n+1}\mathbf{L}_1\mathbf{L}_3^{-1} - q^{n-1}\mathbf{L}_1^{-1}\mathbf{L}_3}{q-q^{-1}}\end{aligned}$$

Using these relations we obtain

$$\begin{aligned}\mathbf{E}_{13}|n_1,n_2\rangle &= \frac{q^{-2n_1-n_2+1}[n_2]_q}{q-q^{-1}}\,|n_1,n_2-1\rangle\mathbf{L}_1\mathbf{L}_3^{-1} - \frac{q^{n_2-1}[n_2]_q}{q-q^{-1}}\,|n_1,n_2-1\rangle\mathbf{L}_1^{-1}\mathbf{L}_3 + \\ &\quad + q^{-n_1-n_2}[n_1]_q|n_1-1,n_2\rangle\mathbf{E}_{12}\mathbf{L}_2\mathbf{L}_3^{-1} + |n_1,n_2\rangle\mathbf{E}_{13}\,.\end{aligned}$$

Now we define

$$\mathbf{K}_1 = \mathbf{L}_1\mathbf{L}_2^{-1}\,, \qquad \mathbf{K}_2 = \mathbf{L}_2\mathbf{L}_3^{-1}\,, \qquad \mathbf{E}_{23}\,, \mathbf{E}_{13} \mapsto 0$$

and we obtain $U_q(\mathrm{sl}(3))$:

$$\begin{aligned}
\mathbf{E}_{32}|n_1,n_2\rangle &= |n_1+1,n_2\rangle\,, \\
\mathbf{E}_{31}|n_1,n_2\rangle &= q^{n_1}|n_1,n_2+1\rangle\,, \\
\mathbf{E}_{21}|n_1,n_2\rangle &= -q^{-1}[n_1]_q|n_1-1,n_2+1\rangle + q^{-n_1+n_2}|n_1,n_2\rangle\mathbf{E}_{21}\,, \\
\mathbf{K}_1|n_1,n_2\rangle &= q^{n_1-n_2}|n_1,n_2\rangle\mathbf{K}_1\,, \\
\mathbf{K}_2|n_1,n_2\rangle &= q^{-2n_1-n_2}|n_1,n_2\rangle\mathbf{K}_2\,, \\
\mathbf{E}_{12}|n_1,n_2\rangle &= -q[n_2]_q|n_1+1,n_2-1\rangle\mathbf{K}_1 + |n_1,n_2\rangle\mathbf{E}_{12}\,, \\
\mathbf{E}_{23}|n_1,n_2\rangle &= \frac{q^{-n_1-n_2+1}[n_1]_q}{q-q^{-1}}\,|n_1-1,n_2\rangle\mathbf{K}_2 - \frac{q^{n_1+n_2-1}[n_1]_q}{q-q^{-1}}\,|n_1-1,n_2\rangle\mathbf{K}_2^{-1} + \\
&\quad + q^{n_2-1}[n_2]_q|n_1,n_2-1\rangle\mathbf{E}_{21}\mathbf{K}_2^{-1}\,, \\
\mathbf{E}_{13}|n_1,n_2\rangle &= \frac{q^{-2n_1-n_2+1}[n_2]_q}{q-q^{-1}}\,|n_1,n_2-1\rangle\mathbf{K}_1\mathbf{K}_2 - \frac{q^{n_2-1}[n_2]_q}{q-q^{-1}}\,|n_1,n_2-1\rangle\mathbf{K}_1^{-1}\mathbf{K}_2^{-1} + \\
&\quad + q^{-n_1-n_2}[n_1]_q|n_1-1,n_2\rangle\mathbf{E}_{12}\mathbf{K}_2\,,
\end{aligned}$$

where $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{E}_{12}$ a $\mathbf{E}_{21}$ form a subalgebra $U_q(\mathrm{gl}(2))$ with Casimir operator

$$\mathbf{K}_1\mathbf{K}_2^2 = \mathbf{L}_1\mathbf{L}_2\mathbf{L}_3^{-2}\,.$$

One representation of this subalgebra one can get putting $|n_3\rangle = \mathbf{E}_{21}^{n_3}$. Then

$$\mathbf{E}_{21}|n_3\rangle = |n_3+1\rangle\,, \qquad \mathbf{K}_1|n_3\rangle = q^{-2n_3}|n_3\rangle\mathbf{K}_1\,, \qquad \mathbf{K}_2|n_2\rangle = q^{n_3}|n_3\rangle\mathbf{K}_2$$

and the relation

$$\mathbf{E}_{12}\mathbf{E}_{21}^{n} = \mathbf{E}_{21}^{n}\mathbf{E}_{12} + [n]_q\mathbf{E}_{21}^{n-1}\,\frac{q^{-n+1}\mathbf{K}_1 - q^{n-1}\mathbf{K}_1^{-1}}{q-q^{-1}}$$

help us getting the action of E_{12}:

$$\mathbf{E}_{12}|n_3\rangle = \frac{q^{-n_3+1}[n_3]_q}{q-q^{-1}}\,|n_3-1\rangle\mathbf{K}_1 - \frac{q^{n_3-1}[n_2]_q}{q-q^{-1}}\,|n_3-1\rangle\mathbf{K}_1^{-1} + |n_3\rangle\mathbf{E}_{12}\,.$$

Now we factorize this putting

$$\mathbf{E}_{12}\mapsto 0\,, \qquad \mathbf{K}_1\mapsto q^{\lambda_1}\,, \qquad \mathbf{K}_2\mapsto q^{\lambda_2}\,,$$

and we get the following representation:

$$\begin{aligned}
\mathbf{E}_{21}|n_3\rangle &= |n_3+1\rangle\,, \qquad \mathbf{K}_1|n_3\rangle = q^{\lambda_1-2n_3}|n_3\rangle\,, \qquad \mathbf{K}_2|n_3\rangle = q^{\lambda_2+n_3}|n_3\rangle\,,\\
\mathbf{E}_{12}|n_3\rangle &= [n_3]_q[\lambda_1-n_3+1]_q|n_3-1\rangle\,,
\end{aligned}$$

where λ_1, $\lambda_2 \in \mathbb{C}$ a $n_3 \in \mathbb{N}_0$.

Combining with previous relations we finally get the Verma module of $U_q(\mathrm{sl}(3))$ which is given by the following relations:

$$
\begin{aligned}
\mathbf{E}_{32}|n_1,n_2,n_3\rangle &= |n_1+1,n_2,n_3\rangle\,,\\
\mathbf{E}_{31}|n_1,n_2,n_3\rangle &= q^{n_1}|n_1,n_2+1,n_3\rangle\,,\\
\mathbf{E}_{21}|n_1,n_2,n_3\rangle &= q^{-n_1+n_2}|n_1,n_2,n_3+1\rangle - q^{-1}[n_1]_q|n_1-1,n_2+1,n_3\rangle\,,\\
\mathbf{K}_1|n_1,n_2,n_3\rangle &= q^{\lambda_1+n_1-n_2-2n_3}|n_1,n_2,n_3\rangle\,,\\
\mathbf{K}_2|n_1,n_2,n_3\rangle &= q^{\lambda_2-2n_1-n_2+n_3}|n_1,n_2,n_3\rangle\,,\\
\mathbf{E}_{12}|n_1,n_2,n_3\rangle &= [n_3]_q[\lambda_1-n_3+1]_q|n_1,n_2,n_3-1\rangle - q^{\lambda_1-2n_3+1}[n_2]_q|n_1+1,n_2-1,n_3\rangle\,,\\
\mathbf{E}_{23}|n_1,n_2,n_3\rangle &= [n_1]_q[\lambda_2-n_1-n_2+n_3+1]_q|n_1-1,n_2,n_3\rangle +\\
&\quad +q^{-\lambda_2+n_2-n_3-1}[n_2]_q|n_1,n_2-1,n_3+1\rangle\,,\\
\mathbf{E}_{13}|n_1,n_2,n_3\rangle &= q^{-n_1}[n_2]_q[\lambda_1+\lambda_2-n_1-n_2-n_3+1]_q|n_1,n_2-1,n_3\rangle +\\
&\quad +q^{\lambda_1-n_1-n_2+n_3}[n_1]_q[n_3]_q[\lambda_1-n_3+1]_q|n_1-1,n_2,n_3-1\rangle\,,
\end{aligned}
$$

where $\lambda_1,\ \lambda_2 \in \mathbb{C}$ a $n_1,\ n_2,\ n_3 \in \mathbb{N}_0$.

For $\lambda_1 = J \in \mathbb{N}_0$ we get

$$\begin{aligned}
\mathbf{E}_{32}|n_1, n_2, n_3\rangle &= |n_1+1, n_2, n_3\rangle\,,\\
\mathbf{E}_{31}|n_1, n_2, n_3\rangle &= q^{n_1}|n_1, n_2+1, n_3\rangle\,,\\
\mathbf{E}_{21}|n_1, n_2, n_3\rangle &= q^{-n_1+n_2}|n_1, n_2, n_3+1\rangle - q^{-1}[n_1]_q|n_1-1, n_2+1, n_3\rangle\,,\\
\mathbf{K}_1|n_1, n_2, n_3\rangle &= q^{J+n_1-n_2-2n_3}|n_1, n_2, n_3\rangle\,,\\
\mathbf{K}_2|n_1, n_2, n_3\rangle &= q^{\lambda_2-2n_1-n_2+n_3}|n_1, n_2, n_3\rangle\,,\\
\mathbf{E}_{12}|n_1, n_2, n_3\rangle &= [n_3]_q[J-n_3+1]_q|n_1, n_2, n_3-1\rangle - q^{J-2n_3+1}[n_2]_q|n_1+1, n_2-1, n_3\rangle\,,\\
\mathbf{E}_{23}|n_1, n_2, n_3\rangle &= [n_1]_q[\lambda_2-n_1-n_2+n_3+1]_q|n_1-1, n_2, n_3\rangle +\\
&\quad +q^{-\lambda_2+n_2-n_3-1}[n_2]_q|n_1, n_2-1, n_3+1\rangle\,,\\
\mathbf{E}_{13}|n_1, n_2, n_3\rangle &= q^{-n_1}[n_2]_q[J+\lambda_2-n_1-n_2-n_3+1]_q|n_1, n_2-1, n_3\rangle +\\
&\quad +q^{J-n_1-n_2+n_3}[n_1]_q[n_3]_q[J-n_3+1]_q|n_1-1, n_2, n_3-1\rangle\,,
\end{aligned}$$

where

$$J \in \mathbb{N}_0\,, \qquad \lambda_2 \in \mathbb{C}\,, \qquad n_1,\, n_2 \in \mathbb{N}_0\,, \qquad n_3 = 0, 1, \ldots, J\,.$$

Now let us move to the extremal vectors. We have three positive roots of $\mathrm{sl}(3)$, namely

$$\alpha_1 = (2,-1)\,, \qquad \alpha_2 = (-1,2)\,, \qquad \alpha_3 = \alpha_1 + \alpha_2 = (1,1)\,,$$

Root α_1 corresponds $\mathbf{E}_{12}$, ie. n_3, root α_2 corresponds $\mathbf{E}_{23}$, ie. n_1 and root α_3 corresponds $\mathbf{E}_{13}$, ie. n_2.
For the vector $\mathbf{v}_0 = |0,0,0\rangle$ we have

$$\mathbf{E}_{12}\mathbf{v}_0 = \mathbf{E}_{23}\mathbf{v}_0 = \mathbf{E}_{13}\mathbf{v}_0 = 0\,, \qquad \mathbf{K}_1\mathbf{v}_0 = q^{\lambda_1}\mathbf{v}_0\,, \qquad \mathbf{K}_2\mathbf{v}_0 = q^{\lambda_2}\mathbf{v}_0\,.$$

We denote $\lambda = (\lambda_1, \lambda_2)$ and

$$\mathcal{V}^\lambda = U_q\big(\mathrm{sl}(3)\big)\mathbf{v}_0\,.$$

Now

$$\mathcal{V}^\lambda = \bigoplus_\mu \mathcal{V}^\lambda_\mu\,,$$

where $\mu = (\mu_1, \mu_2)$ and

$$\mathcal{V}^\lambda_\mu = \left\{\mathbf{v} \in \mathcal{V}^\lambda\,;\ \mathbf{K}_1\mathbf{v} = q^{\mu_1}\mathbf{v}\,,\ \ \mathbf{K}_2\mathbf{v} = q^{\mu_2}\mathbf{v}\right\}.$$

In order to vector

$$|n_1, n_2, n_3\rangle = \mathbf{E}_{32}^{n_1}\mathbf{E}_{31}^{n_2}\mathbf{E}_{21}^{n_3}$$

be an element from the space $\mathcal{V}_{\mu}^{\lambda}$, we get the condition

$$\mu = \lambda - n_1\alpha_2 - n_2\alpha_3 - n_3\alpha_1 \,,$$

or equivalently

$$\mu_1 = \lambda_1 + n_1 - n_2 - 2n_3 \,, \qquad \mu_2 = \lambda_2 - 2n_1 - n_2 + n_3 \,.$$

Let us now denote

$$\delta = \tfrac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) = (1,1)\,, \qquad \lambda = \widehat{\lambda} - \delta\,, \qquad \mu = \widehat{\mu} - \delta\,,$$

that means

$$\lambda_1 = \widehat{\lambda}_1 - 1\,, \qquad \lambda_2 = \widehat{\lambda}_2 - 1\,, \qquad \mu_1 = \widehat{\mu}_1 - 1\,, \qquad \mu_2 = \widehat{\mu}_2 - 1\,.$$

If the vector $\mathbf{v}_\mu^\lambda$ is an extremal vector from the space $\mathcal{V}_\mu^\lambda$, i. e. if $\mathbf{v}_\mu^\lambda$ is nonzero vector such that

$$\mathbf{E}_{12}\mathbf{v}_\mu^\lambda = \mathbf{E}_{23}\mathbf{v}_\mu^\lambda = \mathbf{E}_{13}\mathbf{v}_\mu^\lambda = 0\,,$$

we must have

$$\widehat{\mu} = s_w\left(\widehat{\lambda}\right)\,,$$

for some w which is an element of the Weyl group of the algebra $\mathrm{sl}(3)$.

The Weyl group has six elements, namely the unit element and the elements

$$\begin{aligned}
s_{\alpha_1}\left(\widehat{\lambda}_1, \widehat{\lambda}_2\right) &= \left(-\widehat{\lambda}_1, \widehat{\lambda}_1 + \widehat{\lambda}_2\right)\,, & s_{\alpha_2}\left(\widehat{\lambda}_1, \widehat{\lambda}_2\right) &= \left(\widehat{\lambda}_1 + \widehat{\lambda}_2, -\widehat{\lambda}_2\right)\,,\\
s_{\alpha_2}s_{\alpha_1}\left(\widehat{\lambda}_1, \widehat{\lambda}_2\right) &= \left(\widehat{\lambda}_2, -\widehat{\lambda}_1 - \widehat{\lambda}_2\right)\,, & s_{\alpha_1}s_{\alpha_2}\left(\widehat{\lambda}_1, \widehat{\lambda}_2\right) &= \left(-\widehat{\lambda}_1 - \widehat{\lambda}_2, \widehat{\lambda}_1\right)\,,\\
s_{\alpha_3}\left(\widehat{\lambda}_1, \widehat{\lambda}_2\right) &= \left(-\widehat{\lambda}_2, -\widehat{\lambda}_1\right)\,.
\end{aligned}$$

Now we examine each of these cases separately.

1. Case $\widehat{\mu} = s_{\alpha_1}(\widehat{\lambda})$

In this case the

$$\widehat{\mu}_1 = -\widehat{\lambda}_1\,, \qquad \widehat{\mu}_2 = \widehat{\lambda}_1 + \widehat{\lambda}_2\,,$$

or

$$\mu_1 = -\lambda_1 - 2\,, \qquad \mu_2 = \lambda_1 + \lambda_2 + 1\,.$$

The condition "$\mu = \lambda - n_1\alpha_2 - n_2\alpha_3 - n_3\alpha_1$" then implies that in this case we have

$$-\lambda_1 - 2 = \lambda_1 + n_1 - n_2 - 2n_3\,, \qquad \lambda_1 + \lambda_2 + 1 = \lambda_2 - 2n_1 - n_2 + n_3\,,$$

or

$$n_1 + n_2 = 0\,, \qquad \lambda_1 = n_3 - 1\,.$$

Thus $\lambda_1 = N_1 \in \mathbb{N}_0$, λ_2 is arbitrary and for the extremal vector we get immediately

$$\mathbf{v}^{(N_1,\lambda_2)}_{(-N_1-2,N_1+\lambda_2+1)} = |0, 0, N_1 + 1\rangle\,.$$

2. Case $\widehat{\mu} = s_{\alpha_2}(\widehat{\lambda})$

In this case we have

$$\widehat{\mu}_1 = \widehat{\lambda}_1 + \widehat{\lambda}_2\,, \qquad \widehat{\mu}_2 = -\widehat{\lambda}_2\,,$$

or equivalently

$$\mu_1 = \lambda_1 + \lambda_2 + 1\,, \qquad \mu_2 = -\lambda_2 - 2\,.$$

For $\mathbf{v}$ to be in $\mathcal{V}_\mu^\lambda$ we must have

$$\lambda_1 + \lambda_2 + 1 = \lambda_1 + n_1 - n_2 - 2n_3\,, \qquad -\lambda_2 - 2 = \lambda_2 - 2n_1 - n_2 + n_3\,,$$

or

$$n_2 + n_3 = 0\,, \qquad \lambda_2 = n_1 - 1\,.$$

Thus $\lambda_2 = N_2 \in \mathbb{N}_0$, λ_1 arbitrary and for the extremal vector we obtain

$$\mathbf{v}^{(\lambda_1, N_2)}_{(N_2+\lambda_1+1, -N_2-2)} = |N_2 + 1, 0, 0\rangle\,.$$

3. Case $\widehat{\mu} = s_{\alpha_2} s_{\alpha_1}(\widehat{\lambda})$

In this case we have

$$\widehat{\mu}_1 = \widehat{\lambda}_2 , \qquad \widehat{\mu}_2 = -\widehat{\lambda}_1 - \widehat{\lambda}_2 ,$$

or equivalently

$$\mu_1 = \lambda_2 , \qquad \mu_2 = -\lambda_1 - \lambda_2 - 3 .$$

Now there must be

$$\lambda_2 = \lambda_1 + n_1 - n_2 - 2n_3 , \qquad -\lambda_1 - \lambda_2 - 3 = \lambda_2 - 2n_1 - n_2 + n_3 ,$$

or

$$\lambda_1 = n_2 + n_3 - 1 , \qquad \lambda_2 = n_1 - n_3 - 1 , \qquad \lambda_1 + \lambda_2 = n_1 + n_2 - 2 .$$

If

$$n_2 = n_3 = 0 , \quad \lambda_1 = -1 , \quad \lambda_2 = n_1 - 1 = N_2 \in \mathbb{N}_0 ,$$

then we get the same extremal vector $\mathbf{v}^{(-1,N_2)}_{(N_2,-N_2-2)} = |N_2 + 1, 0, 0\rangle$ as in the previous case.
If

$$n_1 = n_2 = 0 , \quad \text{is} \quad \lambda_1 = n_3 - 1 = N_1 \in \mathbb{N}_0 , \quad \lambda_2 = -n_3 - 1 = -N_1 - 2$$

then we get the extremal vector $\mathbf{v}^{(N_1,-N_1-2)}_{(-N_1-2,-1)} = |0, 0, N_1 + 1\rangle$ as in the case 1.

Finally let

$$\lambda_1 = n_2 + n_3 - 1 = N_1 \in \mathbb{N}_0\,,$$
$$\lambda_2 = n_1 - n_3 - 1 = K_2 \in \mathbb{Z}\,,$$
$$\lambda_1 + \lambda_2 + 1 = n_1 + n_2 - 1 = N_3 \in \mathbb{N}_0\,.$$

and denote

$$n_1 = N_3 - n + 1\,, \qquad n_2 = n\,, \qquad n_3 = N_1 - n + 1\,,$$

where

$$0 \le n \le \min(N_3 + 1, N_1 + 1) = R_{21}\,.$$

Then the extremal vector may exists and has the following form:

$$\mathbf{v}^{(N_1,K_2)}_{(K_2,-N_3-2)} = \sum_{n=0}^{R_{21}} c_n |N_3 - n + 1, n, N_1 - n + 1\rangle\,.$$

4. Case $\widehat{\mu} = s_{\alpha_1} s_{\alpha_2}\big(\widehat{\lambda}\big)$

In this case we have

$$\widehat{\mu}_1 = -\widehat{\lambda}_1 - \widehat{\lambda}_2\,, \qquad \widehat{\mu}_2 = \widehat{\lambda}_1\,,$$

or equivalently

$$\mu_1 = -\lambda_1 - \lambda_2 - 3\,, \qquad \mu_2 = \lambda_1\,.$$

Now for the vector $\mathbf{v}$ to be in $\mathcal{V}_\mu^\lambda$ we must have

$$-\lambda_1 - \lambda_2 - 3 = \lambda_1 + n_1 - n_2 - 2n_3\,, \qquad \lambda_1 = \lambda_2 - 2n_1 - n_2 + n_3\,,$$

or equivalently

$$\lambda_1 = -n_1 + n_3 - 1\,, \quad \lambda_2 = n_1 + n_2 - 1\,, \quad \lambda_1 + \lambda_2 = n_2 + n_3 - 2\,.$$

Now if

$$n_1 = n_2 = 0\,, \quad \lambda_2 = -1\,, \quad \lambda_1 = n_3 - 1 = N_1 \in \mathbb{N}_0\,,$$

then we get the extremal vector $\mathbf{v}^{(N_1,-1)}_{(-N_1-2,N_1)} = |0,0,N_1+1\rangle$ as in the case 1.

If

$$n_2 = n_3 = 0\,, \quad \lambda_2 = n_1 - 1 = N_2\,, \quad \lambda_1 = -n_1 - 1 = -N_2 - 2\,,$$

we get $\mathbf{v}^{(-N_2-2,N_2)}_{(-1,-N_2-2)} = |N_2+1,0,0\rangle$ as in the case 2.

After all, if

$$\lambda_1 = -n_1 + n_3 - 1 = K_1 \in \mathbb{Z}\,,$$

$$\lambda_2 = n_1 + n_2 - 1 = N_2 \in \mathbb{N}_0\,,$$

$$\lambda_1 + \lambda_2 + 1 = N_3 \in \mathbb{N}_0\,,$$

we denote

$$n_1 = N_2 - n + 1\,, \qquad n_2 = n\,, \qquad n_3 = N_3 - n + 1\,,$$

where

$$0 \leq n \leq \min(N_2+1, N_3+1) = R_{12}\,.$$

Then the possible extremal vector is of the form

$$\mathbf{v}^{(K_1,N_2)}_{(-N_3-2,K_1)} = \sum_{n=0}^{R_{12}} c_n |N_2 - n + 1, n, N_3 - n + 1\rangle\,.$$

5. Case $\widehat{\mu} = s_{\alpha_3}(\widehat{\lambda})$

In this case we have

$$\widehat{\mu}_1 = -\widehat{\lambda}_2\,, \qquad \widehat{\mu}_2 = -\widehat{\lambda}_1\,,$$

or equivalently

$$\mu_1 = -\lambda_2 - 2\,, \qquad \mu_2 = -\lambda_1 - 2\,.$$

There must be:

$$-\lambda_2 - 2 = \lambda_1 + n_1 - n_2 - 2n_3\,, \qquad -\lambda_1 - 2 = \lambda_2 - 2n_1 - n_2 + n_3\,,$$

or

$$n_1 = n_3\,, \quad \lambda_1 + \lambda_2 = n_1 + n_2 - 2\,, \quad \text{i.e.} \quad \lambda_1 + \lambda_2 + 1 = n_1 + n_2 - 1 = N_3 \in \mathbb{N}_0.$$

We denote

$$n_1 = n_3 = N_3 - n + 1\,, \quad n_2 = n\,, \quad \text{where} \quad 0 \leq n \leq N_3 + 1\,.$$

Then the extremal vector can have a form

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{N_3+1} c_n |N_3 - n + 1, n, N_3 - n + 1\rangle\,.$$

Summary – possible extremal vectors for Verma module $\mathcal{V}^{(\lambda_1,\lambda_2)}$:

(1) If $\lambda_1\,, \lambda_2,\,, \lambda_1+\lambda_2+1 \notin \mathbb{N}_0$, extremal vector does not exist.

(2) If $\lambda_1 = N_1 \in \mathbb{N}_0$ a $\lambda_1+\lambda_2+1 \notin \mathbb{N}_0$ there is just one extremal vector of the form

$$\mathbf{v}^{(N_1,\lambda_2)}_{(-N_1-2,N_1+\lambda_2+1)} = |0,0,N_1+1\rangle\,.$$

Here we have $\widehat{\mu} = s_{\alpha_1}(\widehat{\lambda})$.

(3) If $\lambda_2 = N_2 \in \mathbb{N}_0$ a $\lambda_1+\lambda_2+1 \notin \mathbb{N}_0$ there is just one extremal vector of the form

$$\mathbf{v}^{(\lambda_1,N_2)}_{(N_2+\lambda_1+1,-N_2-2)} = |N_2+1,0,0\rangle\,.$$

Here we have $\widehat{\mu} = s_{\alpha_2}(\widehat{\lambda})$.

(4) If $\lambda_1 + \lambda_2 + 1 = N_3 \in \mathbb{N}_0$ a $\lambda_1\,, \lambda_2 \notin \mathbb{N}_0$, there can exists the extremal vector of the form

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{N_3+1} c_n |N_3 - n + 1, n, N_3 - n + 1\rangle\,.$$

Here one has $\widehat{\mu} = s_{\alpha_3}(\widehat{\lambda})$.

(5) If $\lambda_1 = N_1 \in \mathbb{N}_0$, $\lambda_1 + \lambda_2 + 1 = N_3 \in \mathbb{N}_0$ a $\lambda_2 \notin \mathbb{N}_0$, ie. $\lambda_2 = N_3 - N_1 - 1 < 0$, there can exist the extremal vectors

$$\begin{aligned}
\mathbf{v}^{(N_1,N_3-N_1-1)}_{(-N_1-2,N_3)} &= |0, 0, N_1 + 1\rangle\,, \\
\mathbf{v}^{(N_1,N_3-N_1-1)}_{(N_3-N_1-1,-N_3-2)} &= \sum_{n=0}^{N_3+1} c_n |N_3 - n + 1, n, N_1 - n + 1\rangle\,, \\
\mathbf{v}^{(N_1,N_3-N_1-1)}_{(N_1-N_3-1,-N_1-2)} &= \sum_{n=0}^{N_3+1} c_n |N_3 - n + 1, n, N_3 - n + 1\rangle\,.
\end{aligned}$$

The first vector corresponds to the case $\widehat{\mu} = s_{\alpha_1}(\widehat{\lambda})$, the second to the case $\widehat{\mu} = s_{\alpha_2} s_{\alpha_1}(\widehat{\lambda})$ and the third to the case $\widehat{\mu} = s_{\alpha_3}(\widehat{\lambda})$.

(6) If $\lambda_2 = N_2 \in \mathbb{N}_0$, $\lambda_1 + \lambda_2 + 1 = N_3 \in \mathbb{N}_0$ and $\lambda_1 \notin \mathbb{N}_0$, i. e. $\lambda_1 = N_3 - N_2 - 1 < 0$, there can exist the extremal vectors

$$\begin{aligned}
\mathbf{v}^{(N_3-N_2-1,N_2)}_{(N_3,-N_2-2)} &= |N_2+1,0,0\rangle\,, \\
\mathbf{v}^{(N_3-N_2-1,N_2)}_{(-N_3-2,N_3-N_2-1)} &= \sum_{n=0}^{N_3+1} c_n |N_2-n+1,n,N_3-n+1\rangle\,, \\
\mathbf{v}^{(N_3-N_2-1,N_2)}_{(-N_2-2,N_2-N_3-1)} &= \sum_{n=0}^{N_3+1} c_n |N_3-n+1,n,N_3-n+1\rangle\,.
\end{aligned}$$

The first vector corresponds to the case $\widehat{\mu} = s_{\alpha_2}(\widehat{\lambda})$, the second to the case $\widehat{\mu} = s_{\alpha_1} s_{\alpha_2}(\widehat{\lambda})$ and the third to the case $\widehat{\mu} = s_{\alpha_3}(\widehat{\lambda})$.

(7) If $\lambda_1 = N_1 \in \mathbb{N}_0$ a $\lambda_2 = N_2 \in \mathbb{N}_0$, there can exist the extremal vectors

$$\begin{aligned}
\mathbf{v}^{(N_1,N_2)}_{(-N_1-2,N_1+N_2+1)} &= |0,0,N_1+1\rangle\,, \\
\mathbf{v}^{(N_1,N_2)}_{(N_1+N_2+1,-N_2-2)} &= |N_2+1,0,0\rangle\,, \\
\mathbf{v}^{(N_1,N_2)}_{(N_2,-N_1-N_2-3)} &= \sum_{n=0}^{N_1+1} c_n |N_1+N_2-n+2, n, N_1-n+1\rangle\,, \\
\mathbf{v}^{(N_1,N_2)}_{(-N_1-N_2-3,N_1)} &= \sum_{n=0}^{N_2+1} c_n |N_2-n+1, n, N_1+N_2-n+2\rangle\,, \\
\mathbf{v}^{(N_1,N_2)}_{(-N_2-2,-N_1-2)} &= \sum_{n=0}^{N_1+N_2+2} c_n |N_1+N_2-n+2, n, N_1+N_2-n+2\rangle\,.
\end{aligned}$$

The first vector corresponds to the case $\widehat{\mu} = s_{\alpha_1}(\widehat{\lambda})$, the second to the case $\widehat{\mu} = s_{\alpha_2}(\widehat{\lambda})$, the third to the case $\widehat{\mu} = s_{\alpha_2}s_{\alpha_1}(\widehat{\lambda})$, the fourth to the case $\widehat{\mu} = s_{\alpha_1}s_{\alpha_2}(\widehat{\lambda})$ and the fifth one to the case $\widehat{\mu} = s_{\alpha_3}(\widehat{\lambda})$.

The existence of the extremal vectors

For the Verma module $\mathcal{V}^{(\lambda_1,\lambda_2)}$ we obtain

$$
\begin{aligned}
\mathbf{E}_{12}|n_1,n_2,n_3\rangle &= [n_3]_q[\lambda_1-n_3+1]_q|n_1,n_2,n_3-1\rangle - q^{\lambda_1-2n_3+1}[n_2]_q|n_1+1,n_2-1,n_3\rangle\,,\\
\mathbf{E}_{23}|n_1,n_2,n_3\rangle &= [n_1]_q[\lambda_2-n_1-n_2+n_3+1]_q|n_1-1,n_2,n_3\rangle + \\
&\quad +q^{-\lambda_2+n_2-n_3-1}[n_2]_q|n_1,n_2-1,n_3+1\rangle\,,\\
\mathbf{E}_{13}|n_1,n_2,n_3\rangle &= q^{-n_1}[n_2]_q[\lambda_1+\lambda_2-n_1-n_2-n_3+1]_q|n_1,n_2-1,n_3\rangle + \\
&\quad +q^{\lambda_1-n_1-n_2+n_3}[n_1]_q[n_3]_q[\lambda_1-n_3+1]_q|n_1-1,n_2,n_3-1\rangle\,.
\end{aligned}
$$

With the help of these formulas, we now examine the cases (4), (5), (6) and (7) to obtain the coefficients c_n.

Case (4) i. e. $\lambda_1 + \lambda_2 + 1 = N_3 \in \mathbb{N}_0$ *a* $\lambda_1,\ \lambda_2 \notin \mathbb{N}_0$:

The extremal vectors are of the form

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{N_3+1} c_n |N_3 - n + 1, n, N_3 - n + 1\rangle$$

and fulfil the equations $\mathbf{E}_{12}\mathbf{v} = \mathbf{E}_{23}\mathbf{v} = 0$. This gives us the condition

$$q^{-\lambda_1-2\lambda_2+2n-1}[n+1]_q c_{n+1} + [\lambda_2 - n + 1]_q[\lambda_1 + \lambda_2 - n + 2]_q c_n = 0\,, \quad n = 0, 1, \ldots, N_3\,.$$

The solution is

$$c_n = (-1)^n q^{n(\lambda_1+2\lambda_2-n+2)}[n]_q! \begin{bmatrix} \lambda_2 + 1 \\ n \end{bmatrix}_q \begin{bmatrix} \lambda_1 + \lambda_2 + 2 \\ n \end{bmatrix}_q .$$

For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ let us denote

$$\begin{bmatrix} \lambda \\ 0 \end{bmatrix}_q = 1\,, \qquad \begin{bmatrix} \lambda \\ n \end{bmatrix}_q = \frac{[\lambda]_q}{[n]_q} \begin{bmatrix} \lambda - 1 \\ n - 1 \end{bmatrix}_q = \frac{[\lambda]_q[\lambda-1]_q \ldots [\lambda - n + 1]_q}{[n]_q!}\,.$$

The extremal vector is then given by a formula

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{\lambda_1+\lambda_2+2} (-1)^n q^{n(\lambda_1+2\lambda_2-n+2)}[n]_q! \begin{bmatrix} \lambda_2 + 1 \\ n \end{bmatrix}_q \begin{bmatrix} \lambda_1 + \lambda_2 + 2 \\ n \end{bmatrix}_q \times$$
$$\times|\lambda_1 + \lambda_2 - n + 2, n, \lambda_1 + \lambda_2 - n + 2\rangle\,.$$

Summary

Let Verma module $\mathcal{V}^{(\lambda_1,\lambda_2)}$ is generated by the vector $\mathbf{v}_0$.

1. If $\lambda_1,\ \lambda_2,\ \lambda_1+\lambda_2+1 \notin \mathbb{N}_0$, there is only one extremal vector $\mathbf{v}_0$.
2. If $\lambda_1 \in \mathbb{N}_0$ a $\lambda_1+\lambda_2+1 \notin \mathbb{N}_0$, There are just two extremal elements: $\mathbf{v}_0$ and

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)} = \mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0\ .$$

3. If $\lambda_2 \in \mathbb{N}_0$ a $\lambda_1+\lambda_2+1 \notin \mathbb{N}_0$, there are just two extremal elements: $\mathbf{v}_0$ and

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)} = \mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0\ .$$

4. If $\lambda_1+\lambda_2+1 \in \mathbb{N}_0$ and $\lambda_1,\ \lambda_2 \notin \mathbb{N}_0$, there are just two extremal elements: $\mathbf{v}_0$ and

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{\lambda_1+\lambda_2+2} (-1)^n q^{n(\lambda_1+2\lambda_2-n+2)}[n]_q! \begin{bmatrix}\lambda_2+1\\ n\end{bmatrix}_q \begin{bmatrix}\lambda_1+\lambda_2+2\\ n\end{bmatrix}_q \times$$
$$\times \mathbf{E}_{32}^{\lambda_1+\lambda_2-n+2}\mathbf{E}_{31}^{n}\mathbf{E}_{21}^{\lambda_1+\lambda_2-n+2}\mathbf{v}_0\ .$$

5. If $\lambda_1 \in \mathbb{N}_0$ and $\lambda_2 = -1$, there are just three extremal elements: $\mathbf{v}_0$,

$$\mathbf{v}^{(\lambda_1,-1)}_{(-\lambda_1-2,\lambda_1)} = \mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0 \,,$$
$$\mathbf{v}^{(\lambda_1,-1)}_{(-1,-\lambda_1-2)} = \mathbf{E}_{32}^{\lambda_1+1}\mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0 = \mathbf{E}_{32}^{\lambda_1+1}\mathbf{v}^{(\lambda_1,-1)}_{(-\lambda_1-2,\lambda_1)} \,.$$

6. If $\lambda_2 \in \mathbb{N}_0$ and $\lambda_1 = -1$, there are just three extremal elements: $\mathbf{v}_0$,

$$\mathbf{v}^{(-1,\lambda_2)}_{(\lambda_2,-\lambda_2-2)} = \mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0 \,,$$
$$\mathbf{v}^{(-1,\lambda_2)}_{(-\lambda_2-2,-1)} = \mathbf{E}_{21}^{\lambda_2+1}\mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0 = \mathbf{E}_{21}^{\lambda_2+1}\mathbf{v}^{(-1,\lambda_2)}_{(\lambda_2,-\lambda_2-2)} \,.$$

7. If $\lambda_1 \in \mathbb{N}$ a $\lambda_2 = -2, -3, \ldots, -\lambda_1 - 1$, there are just four extremal elements: $\mathbf{v}_0$,

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)} = \mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0 \,,$$
$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{\lambda_1+\lambda_2+2}(-1)^n q^{n(\lambda_1+2\lambda_2-n+2)}[n]_q! \begin{bmatrix}\lambda_2+1\\ n\end{bmatrix}_q \begin{bmatrix}\lambda_1+\lambda_2+2\\ n\end{bmatrix}_q \times$$
$$\times \mathbf{E}_{32}^{\lambda_1+\lambda_2-n+2}\mathbf{E}_{31}^{n}\mathbf{E}_{21}^{\lambda_1+\lambda_2-n+2}\mathbf{v}_0 \,,$$
$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_2,-\lambda_1-\lambda_2-3)} = \mathbf{E}_{32}^{\lambda_1+\lambda_2+2}\mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0 = \mathbf{E}_{32}^{\lambda_1+\lambda_2+2}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)} \sim$$
$$\sim \mathbf{E}_{21}^{-\lambda_2-1}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} \,,$$

8. If $\lambda_2 \in \mathbb{N}$ a $\lambda_1 = -2, -3, \ldots, -\lambda_2 - 1$, there are just four extremal elements: $\mathbf{v}_0$,

$$\begin{aligned}
\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)} &= \mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0\,,\\
\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} &= \sum_{n=0}^{\lambda_1+\lambda_2+2}(-1)^n q^{n(\lambda_1+2\lambda_2-n+2)}[n]_q! \begin{bmatrix}\lambda_2+1\\ n\end{bmatrix}_q \begin{bmatrix}\lambda_1+\lambda_2+2\\ n\end{bmatrix}_q \times\\
&\qquad \times \mathbf{E}_{32}^{\lambda_1+\lambda_2-n+2}\mathbf{E}_{31}^{n}\mathbf{E}_{21}^{\lambda_1+\lambda_2-n+2}\mathbf{v}_0\,,\\
\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-\lambda_2-3,\lambda_1)} &= \mathbf{E}_{32}^{-\lambda_1-1}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} \sim \mathbf{E}_{21}^{\lambda_1+\lambda_2+2}\mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0 =\\
&= \mathbf{E}_{21}^{\lambda_1+\lambda_2+2}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)}\,.
\end{aligned}$$

9. If $\lambda_1,\ \lambda_2 \in \mathbb{N}_0$, there are six extremal elements: $\mathbf{v}_0$,

$$\begin{aligned}
\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)} &= \mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0\,,\\
\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)} &= \mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0\,,\\
\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_2,-\lambda_1-\lambda_2-3)} &= \mathbf{E}_{32}^{\lambda_1+\lambda_2+2}\mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0 = \mathbf{E}_{32}^{\lambda_1+\lambda_2+2}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)}\,,\\
\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-\lambda_2-3,\lambda_1)} &= \mathbf{E}_{21}^{\lambda_1+\lambda_2+2}\mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0 = \mathbf{E}_{21}^{\lambda_1+\lambda_2+2}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)}\,,\\
\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} &= \sum_{n=0}^{\lambda_2+1}(-1)^n q^{n(\lambda_1+2\lambda_2-n+2)}[n]_q! \begin{bmatrix}\lambda_2+1\\ n\end{bmatrix}_q \begin{bmatrix}\lambda_1+\lambda_2+2\\ n\end{bmatrix}_q \times\\
&\qquad \times \mathbf{E}_{32}^{\lambda_1+\lambda_2-n+2}\mathbf{E}_{31}^{n}\mathbf{E}_{21}^{\lambda_1+\lambda_2-n+2}\mathbf{v}_0\,.
\end{aligned}$$

Extremal vectors for factor-representation where $\lambda_1 = J \in \mathbb{N}_0$

In this case we have $n_1, n_2 \in \mathbb{N}_0$, $n_3 = 0, 1, 2, \ldots, J$ and $n_1 + n_2 + n_3 > 0$.
1. For the case $\widehat{\mu} = s_{\alpha_1}(\widehat{\lambda})$ by the same calculations as before we obtain

$$n_1 = n_2 = 0\,, \qquad J = n_3 - 1\,,$$

and extremal vectors do not exist.
2. When $\widehat{\mu} = s_{\alpha_2}(\widehat{\lambda})$, we have

$$n_2 + n_3 = 0\,, \qquad \lambda_2 = n_1 - 1 = N_2 \in \mathbb{N}_0$$

and the extremal vectors is

$$\mathbf{v}^{(J,N_2)}_{(N_2+J,-N_2-2)} = |N_2+1,0,0\rangle\,.$$

3. When $\widehat{\mu} = s_{\alpha_2}s_{\alpha_1}(\widehat{\lambda})$, we have

$$J = n_2 + n_3 - 1\,, \qquad \lambda_2 = n_1 - n_3 - 1 \in \mathbb{Z}\,, \qquad J + \lambda_2 + 1 = n_1 + n_2 - 1 = N_3 \in \mathbb{N}_0\,.$$

In this case we define

$$n_1 = N_3 - n + 1\,, \quad n_2 = n\,, \quad n_3 = J - n + 1\,, \quad \text{where} \quad 1 \le n \le \min(J+1, N_3+1) = R\,.$$

Possible extremal vector can have a form

$$\mathbf{v}^{(J,N_3-J-1)}_{(N_3-J-1,-N_3-2)} = \sum_{n=1}^{R} c_n |N_3 - n + 1, n, J - n + 1\rangle\,.$$

4. For $\widehat{\mu} = s_{\alpha_1} s_{\alpha_2}\left(\widehat{\lambda}\right)$ we obtain the condition $J = -n_1 + n_3 - 1$, which can not be fulfilled because $n_3 \leq J$. In this case there is no extremal vector.
5. When $\widehat{\mu} = s_{\alpha_3}\left(\widehat{\lambda}\right)$, we have

$$n_1 = n_3\,, \quad J + \lambda_2 + 1 = n_1 + n_2 - 1 = N_3 \in \mathbb{N}_0\,.$$

We introduce

$$n_1 = n_3 = N_3 - n + 1\,, \quad n_2 = n\,, \quad \text{where} \quad S = \max(0, N_3 - J + 1) \leq n \leq N_3 + 1\,,$$

and the possible extremal vector must have the following form:

$$\mathbf{v}^{(J, N_3 - J - 1)}_{(J - N_3 - 1, -J - 2)} = \sum_{n=S}^{N_3+1} c_n |N_3 - n + 1, n, N_3 - n + 1\rangle\,.$$

Summary: The extremal vectors for factor representation with $\lambda_1 = J \in \mathbb{N}_0$:

1. If $J + \lambda_2 + 1 \notin \mathbb{N}_0$ there is only one extremal vector $\mathbf{v}_0$.
2. If $\lambda_2 = -1$ there is only one extremal vector $\mathbf{v}_0$.
3. If $J \in \mathbb{N}$ and $\lambda_2 = -2, -3, \dots, -J-1$, there are two extremal vectors: $\mathbf{v}_0$ and

$$\mathbf{v}^{(J,\lambda_2)}_{(-\lambda_2-2,-J-2)} = \sum_{n=\lambda_2+2}^{J+\lambda_2+2} (-1)^n q^{n(J+2\lambda_2-n+2)} [n]_q! \begin{bmatrix} \lambda_2+1 \\ n \end{bmatrix}_q \begin{bmatrix} J+\lambda_2+2 \\ n \end{bmatrix}_q \times$$
$$\times \mathbf{E}_{32}^{J+\lambda_2-n+2} \mathbf{E}_{31}^{n} \mathbf{E}_{21}^{J+\lambda_2-n+2} \mathbf{v}_0 \, .$$

4. If $\lambda_2 \in \mathbb{N}_0$, there are two extremal vectors: $\mathbf{v}_0$ and

$$\mathbf{v}^{(J,\lambda_2)}_{(J+\lambda_2+1,-\lambda_2-2)} = \mathbf{E}_{32}^{\lambda_2+1} \mathbf{v}_0 \, .$$

Thank you for your attention.

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