N=1 Supersymmetric Liouville Equation

Z. Popowicz

Institute of Theoretical Physics, University of Wroclaw, pl. Maxa Borna 9
WROCŁAW

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Supersymmetric Liouville Equation

$$S = \int dz d\bar{z} d\theta d\bar{\theta} \Big(\mathcal{D}\Phi \bar{\mathcal{D}}\Phi + ie^{\Phi} \Big)$$

$$\begin{aligned} \mathcal{D} &= \partial_\theta + \theta\partial_z, \quad \bar{\mathcal{D}} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}} \\ \mathcal{Q} &= \partial_\theta - \theta\partial_z, \quad \bar{\mathcal{Q}} = \partial_{\bar{\theta}} - \bar{\theta}\partial_{\bar{z}} \\ \Phi &= \phi + i\theta\psi + i\bar{\theta}\bar{\psi} + i\theta\bar{\theta}F \end{aligned}$$

The equation of motion in the component

$$\begin{aligned} \phi_{z,\bar{z}} &= Fe^{\phi} + i\psi\bar{\psi} \\ \psi_{\bar{z}} &= -ie^{\phi}\bar{\psi}, \qquad \bar{\psi}_z = ie^{\phi}\psi \\ F &= e^{\phi} \end{aligned}$$

references: M. Chaichian, P. Kulish (1978) E. Ivanov, S. Krivonos (1984) N=2 and N=4 super Liouville

Why it is interesting susy generalization ?

1.) Connection with superstrings and four dimensional supergauge theory.

Energy stress tensor $T(z)$ and $T(\bar{z})$ and superconformal current $S(z)$ and $S(\bar{z})$

$$\begin{aligned} T(z) &= -\frac{1}{2}(\partial\phi\partial\phi + \psi\partial\psi) = \sum_{n \varepsilon Z} z^{n-2} L_{-n} \\ S(z) &= i(\psi\partial\phi - 2\partial\psi) = \sum_{k \varepsilon Z + \frac{1}{2}} z^{k-\frac{3}{2}} S_{-k} \end{aligned}$$

We obtain Neveue - Schwartz algebra (super Virasoro) assuming the OPE for $T(z), S(z)$

$$\begin{aligned} [L_m, L_n] &= (mn)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n}, \\ [L_m, S_k] &= \frac{m-2k}{2} S_{m+k}, \\ \{S_k, S_l\} &= 2L_{k+l} + \frac{c}{3}(k^2 - \frac{1}{4})\delta_{k+l}, \qquad c = 12 + \frac{3}{2} \end{aligned}$$

This susy generalization follows from the following observation

it means that one goes to the zero-curvature representation on some superalgebra containing the algebra of an analogous representation of the original bosonic equation as an even subalgebra. In doing so, the systems with the simple as well as extended supersymmetries can be obtained

E.A. Ivanov, S.O. Krivonos *U(1)- supersymmetric extension of the Liouville equation*

Comment:

We have no any quaranty that all properties of Liouville equation are transformed to susy case

What we can say on other properties of Liouville equation ?

The reciprocal link of Liouville and Hunter - Saxton equation

Hunter - Saxton Equation

$$u_{x,t} = \frac{1}{2}u_x^2 + uu_{xx}$$

Change variables $x, t \Rightarrow y(x,t), \tau(x,t)$ as

$$\begin{aligned} dy &= vdx + (v\partial^{-2}v^2)dt, \qquad d\tau = dt \\ \partial_x &= v\partial_y, \qquad \partial_t = \partial_\tau + (v\partial_x^{-2}v^2)\partial_y, \\ v^2 &= u_{xx}, \qquad (logv)_{y,\tau} = v. \end{aligned}$$

Is it possible to construct the supersymmetric reciprocal transformation ?

Classical Harry Dym Hierarchy

$$w_t = (w^{-\frac{1}{2}})_{xxx}, \quad u_t = \partial(\partial^2 u^{-2} - 3u_x^2 u^{-4})$$

where $u = \sqrt{w}$. It is a Tri - Hamiltonian system

$$\begin{aligned}
w_t &= P_1 \frac{\delta H_{-1}}{\delta w} = P_2 \frac{\delta H_{-2}}{\delta w} = P_3 \frac{\delta H_{-3}}{\delta w} \\
P_1 &= \partial^3, \qquad P_2 = 2\sqrt{w}\partial\sqrt{w}, \qquad P_3 = P_2 P_1^{-1} P_2 \\
H_{-1} &= 2\int dx\ \sqrt{w}, \qquad H_{-2} = \frac{1}{8}\int dx\ w^{-\frac{5}{2}} w_x^2 \\
H_{-3} &= \frac{1}{8}\int dx\ (16 w_{xx}^2 w^{-\frac{7}{2}} - 35 w_x^4 w^{-\frac{11}{2}})
\end{aligned}$$

P_i is the so called Poisson tensor or Hamiltonian operator which is connected with the Poisson bracket as

$$\{w(x), w(y)\}_i = P_i \delta(x - y) \tag{1}$$

Lax representation

$$L = w^{-2}\partial^2, \quad \Rightarrow \quad L_t = \left[L_2^{\frac{3}{2}}, L\right]$$

Inverse Hierarchy

$$\begin{aligned}
w_{t_1} &= P_1\frac{\delta H_0}{\delta w} = P_2\frac{\delta H_{-1}}{\delta w} = P_3\frac{\delta H_{-2}}{\delta w} = 0 \\
w_{t_2} &= P_1\frac{\delta H_1}{\delta w} = P_2\frac{\delta H_0}{\delta w} = w_x \\
w_\tau &= P_1\frac{\delta H_2}{\delta w} = P_2\frac{\delta H_1}{\delta w} = P_3\frac{\delta H_0}{\delta w} = 2w\partial^{-1}w + w_x\partial^{-2}w \\
H_0 &= \int dx\ w, \qquad H_1 = \frac{1}{2}\int dx\ (\partial^{-1}w)^2, \\
H_2 &= \frac{1}{2}\int dx\ (\partial^{-2}w)(\partial^{-1}w)^2
\end{aligned}$$

The last equation is the Hunter - Saxton equation which is connected with the Liouville equation by reciprocal transformation

Even and Odd Susy Harry Dym Equation

J.C.Brunelli, A. Das, Z. Popowicz (2003)
Q.P.Liu, Z. Popowicz, Kai Tian (2010)

A.) Tri - Hamiltonian structure $\Longrightarrow$ Even
B.) Lax representation $\Longrightarrow$ Odd

Even Structure in Quantum Field Theory

$$\begin{aligned} [Boson, Boson] &\quad \Rightarrow \quad Boson \\ [Boson, Fermion] &\quad \Rightarrow \quad Fermion \\ [Fermion, Fermion] &\quad \Rightarrow \quad Boson \end{aligned}$$

Odd Structure in Quantum Field Theory

$$\begin{aligned} [Boson, Boson] &\quad \Rightarrow \quad Fermion \\ [Boson, Fermion] &\quad \Rightarrow \quad Boson \\ [Fermion, Fermion] &\quad \Rightarrow \quad Fermion \end{aligned}$$

Buttin first idea ,
Leites, Kupershmidt, Soroka, Batalin, Vikovski, Khudaverian, Volkov, Pashnev, Frydryszak
Leites: in the classical supersymmetrical models can occurs the even as well as odd Hamiltonian structures.

A.) Even case Susy Harry Dym equation

$$
\begin{aligned}
W_t &= K_1 \frac{\delta H_{-1}}{\delta W} = K_2 \frac{\delta H_{-2}}{\delta W} = K_3 \frac{\delta H_{-3}}{\delta W} = \\
&= \frac{1}{4}\partial^2 \Big[-4W(\mathcal{D}W)^{-\frac{1}{2}} + 3W(\mathcal{D}W_x)(\mathcal{D}W)^{-\frac{5}{2}} \Big] \\
K_1 &= \mathcal{D}\partial, \qquad K_3 = K_2 K_1^{-1} K_2 \\
K_2 &= \frac{1}{2}\Big[W\partial + 2\partial W + (\mathcal{D}W)\mathcal{D} \Big]
\end{aligned}
$$

where $W = \xi + \theta w$ and K_2 corresponds to the centerless susy $N = 1$ Virasoro algebra

$$
\begin{aligned}
H_{-1} &= \int dx d\theta \; W(\mathcal{D}W)^{-\frac{1}{2}} \\
H_{-2} &= \int dx d\theta \left[W_x (\mathcal{D}W_x)(\mathcal{D}W)^{-\frac{5}{2}} - 15 W W_x W_{xx} (\mathcal{D}W)^{-\frac{7}{2}} \right]
\end{aligned}
$$

and H_{-3} is complicated.

Even Negative hierarchy

$$
\begin{aligned}
W_{t_1} &= K_1 \frac{\delta H_0}{\delta W} = K_2 \frac{\delta H_{-1}}{\delta W} = K_3 \frac{\delta H_{-2}}{\delta W} = 0 \\
W_{t_2} &= K_1 \frac{\delta H_1}{\delta W} = K_2 \frac{\delta H_0}{\delta W} = W_x \\
H_0 &= \int dx d\theta \; W, \qquad H_1 = \int dx d\theta \; (\mathcal{D}^{-3} W) W
\end{aligned}
$$

Even Susy Hunter - Saxton equation

$$
\begin{aligned}
W_\tau &= K_1 \frac{\delta H_2}{\delta W} = K_2 \frac{\delta H_1}{\delta W} = K_3 \frac{\delta H_0}{\delta W} = \\
&= -\frac{3}{2} W(\mathcal{D}^{-1} W) - W_x(\mathcal{D}^{-3} W) W - \frac{1}{2}(\mathcal{D} W)\partial^{-1} W \\
H_2 &= \int dx d\theta \; (\mathcal{D}^{-1} W)(\partial^{-1} W)(\mathcal{D}^{-3} W)
\end{aligned}
$$

In terms of $U = \mathcal{D} W$ we have

$$
\begin{aligned}
U_t &= 2U_x \partial^{-2} U + 4U\partial^{-1} U - (\mathcal{D}U)\partial^{-2}(\mathcal{D}U) \\
(U^{\frac{1}{4}})_t &= \mathcal{D}\Big[U^{\frac{1}{4}} \partial^{-2}(\mathcal{D}U) + \frac{1}{8}(\mathcal{D}U)^{\frac{1}{4}} \partial^{-2} U\Big]
\end{aligned}
$$

Susy Reciprocal Transformation

Q.P.Liu, Z.Popowicz, K.Tian (2010)

Theorem: If G is a conserved quantity such that

$$\frac{\partial G}{\partial t} = \mathcal{D}\Psi$$

and exists potential Ω such that

$$\mathcal{D}\Omega = 2G\Psi$$

then the Susy recipocal transformation is

$$\begin{aligned}
(x, \theta, t) \quad &\Rightarrow \quad (y, \rho, \tau) \quad \Rightarrow \quad (y(x, \theta, t), \rho(y, x, \theta), \tau) \\
\hat{\mathcal{D}} \quad &= \quad \frac{\partial}{\partial \rho} + \rho \frac{\partial}{\partial y}, \qquad\qquad \mathcal{D} = G\hat{\mathcal{D}} \\
\frac{\partial}{\partial t} \quad &= \quad \frac{\partial}{\partial \tau} + \Omega \frac{\partial}{\partial y} + \Psi \hat{\mathcal{D}}
\end{aligned}$$

Even Susy Liouville Equation

$$\Omega = 2U^{\frac{1}{2}}\partial^{-2}U$$
$$\partial_{y,\tau}\log U = 4U^{\frac{3}{2}} - 8((\hat{\mathcal{D}}U^{\frac{1}{4}})\hat{\mathcal{D}}^{-1}U^{\frac{1}{4}}\partial^{-1}U^{\frac{3}{2}})_y$$

B. Odd Susy Harry - Dym Equation

The Lax representation A.C.Brunelli, A. Das, Z. Popowicz (2003)

$$L = V^{-1}\mathcal{D}V^{-1}\partial, \qquad L_t = \left[L_{2}^{\frac{3}{2}}, L\right]$$
$$V_t = \frac{1}{8}\left[\partial^3 V^{-2} - 3\mathcal{D}\partial(\mathcal{D}V)V_x V^{-4}\right]$$

where $V = u + \theta\xi$ is a superbosonic function.

Odd Hamiltonian Structure

This equation is the Bi - Hamiltonian only *not tri*

$$
\begin{aligned}
V_t &= P_2 \frac{\partial H_{-2}}{\partial V} = P_3 \frac{\partial H_{-3}}{\partial V} \\
P_2 &= \mathcal{D}^3, \qquad \Pi = V^{\frac{1}{2}} \mathcal{D} V^{\frac{1}{2}} \mathcal{D} V \\
P_3 &= -\Pi \mathcal{D}^5 \Pi^{\star}, \qquad H_{-2} = \frac{1}{8} \int dx d\theta \ (\mathcal{D} V) V_x V \\
H_{-3} &= \int dx d\theta \ (\mathcal{D} V) \Big[4 V_{xxx} V^5 - 30 V_{xx} V_x V^6 + 35 V_x^3 V^7 \Big]
\end{aligned}
$$

Odd negative hierarchy of Susy Harry Dym Equation

$$
\begin{aligned}
V_{t_1} &= P_2 \frac{\partial H_{-1}}{\partial V} = P_3 \frac{\partial H_{-2}}{\partial V} = 0 \\
H_{-1} &= \int dx d\theta \ V^{-1} \mathcal{D}^{-1} V \\
V_{t_2} &= P_2 \frac{\delta H_0}{\partial V} = V_x, \qquad H_0 = \int dx d\theta \ V \mathcal{D}^{-1} V
\end{aligned}
$$

Odd Susy Hunter - Saxton Equation

The next flow in the negative hierarchy is

$$\begin{aligned} V_t &= P_2 \frac{\partial H_1}{\partial V} = P_3 \frac{\partial H_0}{\partial V} \\ &= \frac{1}{2}\Big[\partial V \mathcal{D}^{-3} V \partial^{-1} \mathcal{D} V - (\mathcal{D}V)\partial^{-1} V \partial^{-1} \mathcal{D} V\Big] \\ H_1 &= \int dx d\theta \, \Big[2(\mathcal{D}^{-1}V) V \mathcal{D}^{-3} - (\mathcal{D}V)(\mathcal{D}^{-3}V)\partial^{-1}\Big] V \mathcal{D}^{-1} V \end{aligned}$$

Let us rewrite this equation as

$$(V^{\frac{1}{2}})_t = \frac{\partial}{\partial x}(V^{\frac{1}{2}} \mathcal{D}^{-3} V \mathcal{D}^{-1} V) + \mathcal{D}(V^{-\frac{1}{2}}(\mathcal{D}V)\mathcal{D}^{-3} V \mathcal{D}^{-1} V)$$

Odd Susy Liouville Equation

Using previous theorem we have

$$
\begin{aligned}
\Omega &= 2V\mathcal{D}^{-3}V\mathcal{D}^{-1}V, \qquad \mathcal{D} = V^{\frac{1}{2}}\hat{\mathcal{D}}, \qquad G = V^{\frac{1}{2}} \\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau} + 2GV(\mathcal{D}^{-3}V\mathcal{D}^{-1}V)\frac{\partial}{\partial y} + (V^{-\frac{1}{2}}(\mathcal{D}V)\mathcal{D}^{-3}V\mathcal{D}^{-1}V + \\
&\qquad V^{-\frac{1}{2}}\partial^{-1}V\mathcal{D}^{-1}V)\hat{\mathcal{D}} \\
V_\tau &= 2V\hat{\mathcal{D}}^{-1}V^{-\frac{1}{2}}\hat{\mathcal{D}}^{-1}V^{\frac{1}{2}}
\end{aligned}
$$

Introducing $V = (\hat{\mathcal{D}}\Lambda)^2$ we obtain **Susy Odd Liouville equation**

$$
\frac{\partial}{\partial \tau}\hat{\mathcal{D}} \ln(\hat{\mathcal{D}}\Lambda) = (\hat{\mathcal{D}}\Lambda)\Lambda, \qquad \hat{\mathcal{D}} = \partial_\rho + \rho\partial_y
$$

In components $\Lambda = \xi + \rho u$ we have

$$
\frac{\partial^2}{\partial y \partial \tau} \ln u = u^2 + \xi_y \xi, \qquad (u^{-1}\xi)_\tau = u\xi
$$