A_∞-algebra on self-conjugate Young tableaux

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A_∞-algebras

An A_∞-algebra over $\mathbb{K}$ is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

endowed with a family of graded $\mathbb{K}$-linear maps

$$m_n : A^{\otimes n} \to A, \qquad deg(m_n) = 2 - n \qquad n \geq 1$$

satisfying the Stasheff identities **SI(n)** for $n \geq 1$

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0 \qquad \mathbf{SI(n)}$$

sum over all trees with $n = r + s + t$ leafs.

$$r \geq 0 \qquad t \geq 0 \qquad s \geq 1$$

Homotopy transfer theorem

Theorem (Kadeishvili)

Let A be an A_∞-algebra and let HA be the cohomology ring of A. There is an A_∞- algebra structure on HA with $m_1 = 0$ and m_2 induced by the multiplication on A, constructed from the A_∞ structure of A, such that there is a quasi-isomorphism of A_∞-algebras $HA \to A$ lifting the identity of HA.

Apply Homotopy transfer theorem to physically interesting algebras

Parastatistics Algebra and para-Fock space

Parastatistics algebra (parafermionic)

$$\begin{array}{rclrcl} [[a_i^\dagger, a_j], a_k^\dagger] & = & 2\delta_{jk} a_i^\dagger & [[a_i^\dagger, a_j], a_k] & = & -2\delta_{ik} a_j \\ [[a_i^\dagger, a_j^\dagger], a_k^\dagger] & = & 0 & [[a_i, a_j], a_k] & = & 0 \end{array} \quad (1)$$

Para-Fock space $\mathcal{F} = \bigoplus_{r\geq 0} \mathcal{F}_r$ (space of states)
built on an unique vacuum state $|0\rangle$ such that $a_i|0\rangle = 0$

$$a_{i_1}^\dagger \dots a_{i_r}^\dagger |0\rangle \in \mathcal{F}_r$$

Basis-free definition of $\mathcal{F}$ generated in $V = \bigoplus_{i\in I} \mathbb{K} a_i^\dagger$

$$PS(V) = T(V)/([[V, V]_\otimes, V]_\otimes)$$

where $(\mathfrak{I})$ stands for a twosided ideal generated by $\mathfrak{I}$

Young Tableaux and Parastatistics Algebra $PS(V)$

Let $\mathfrak{g}$ be the 2-step nilpotent Lie algebra generated in V having the Lie bracket

$$[u,v] = \begin{cases} u \wedge v & \in \Lambda^2 V \\ 0 & \text{otherwise} \end{cases}$$

The algebra $PS(V)$ is Universal Enveloping Algebra of $\mathfrak{g}(2)$

$$PS(V) = U\mathfrak{g} = U(V \oplus \Lambda^2 V) \tag{2}$$

Lemma

Let $S^\lambda(V)$ be the Schur module associated with Young diagram λ. The algebra $PS(V)$ is a $GL(V)$-model, i.e., every irreducible polynomial $GL(V)$-representations appears once and exactly once

$$PS(V) = \bigoplus_\lambda S^\lambda(V)$$

The Young tableaux label the basis of $PS(V)$.

$U\mathfrak{g}$ - Universal Enveloping Algebra of a Lie algebra $\mathfrak{g}$
Chevalley-Eilenberg complex $C_\bullet(\mathfrak{g}) = (C_p, d_p)$

$$C_p = U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g} \qquad d_p : C_p \to C_{p-1} \tag{3}$$

$$d_p(u \otimes x_1 \wedge \ldots \wedge x_p) = \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_p \tag{4}$$

$$+ \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_p \tag{5}$$

Theorem (Chevalley-Eilenberg)

The chain complex $C(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K}$ is a resolution.

vector space V over $\mathbb{K}$ ($char\mathbb{K} = 0$)
Exemple: Abelian Lie algebra generated in V, $[V, V] = 0$
Koszul complex

$$\cdots \to SV \otimes \Lambda^p V \to SV \otimes \Lambda^{p-1} V \to \cdots \tag{6}$$

$$\cdots \to S\,V \otimes V \to S\,V \to \mathbb{K} \to 0 \tag{7}$$

Chevalley-Eilenberg for 2-nilpotent $\mathfrak{g} = V \oplus \Lambda^2 V$

The Poincaré-Birkhoff-Witt theorem for $PS(V) = U\mathfrak{g}$ yields

$$PS(V) = U(V \oplus \Lambda^2 V) \cong S(V \oplus \Lambda^2 V) \tag{8}$$

The Chevalley-Eilenberg complex of $\mathfrak{g} = V \oplus \Lambda^2 V$ is acyclic

$$C(\mathfrak{g}) = \bigoplus_{p \geq 0} PS(V) \otimes \Lambda^p \mathfrak{g} \tag{9}$$

$$C(\mathfrak{g}) = \bigoplus_{p \geq 0} S(V \oplus \Lambda^2 V) \otimes \Lambda^p (V \oplus \Lambda^2 V) \tag{10}$$

Resolution of PS-module $\mathbb{K}$

There exists a minimal free resolution $\mathbf{P}$
such that $H_\bullet(\mathbb{K} \otimes_{PS} \mathbf{P}) = \mathbb{K} \otimes_{PS} \mathbf{P}$

$$\mathbf{P}: \qquad 0 \to P_d \to \cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{K}$$

where $P_n = PS \otimes E_n$ and $E_n = \mathrm{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$
Free resolution of the left PS-module $\mathbb{K}$: $C(\mathfrak{g}) \to \mathbb{K}$

$$\cdots \to PS(V) \otimes \Lambda^2\mathfrak{g} \to PS(V) \otimes \Lambda\mathfrak{g} \to PS(V) \to \mathbb{K}$$

Homology groups $H_\bullet(\mathfrak{g}, \mathbb{K}) \cong \mathrm{Tor}_\bullet^{PS}(\mathbb{K}, \mathbb{K})$
$E_n = H_n(\mathfrak{g}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{PS} C(\mathfrak{g})) \cong \mathrm{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$

Homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ as a $GL(V)$-module

$$H_\bullet(\mathfrak{g}, \mathbb{K}) \cong H_\bullet(\mathbb{K} \otimes_{PS} C(\mathfrak{g})) \cong \mathrm{Tor}_\bullet^{PS}(\mathbb{K}, \mathbb{K})$$

$$\Lambda^p \mathfrak{g} = \sum_{r+s=p} \Lambda^r(\Lambda^2 V) \otimes \Lambda^s(V) \tag{11}$$

$$\partial_p(w \otimes e_1 \wedge \ldots \wedge e_s) = \sum_{i<j} (-1)^{i+j} w \wedge (e_i \wedge e_j) \otimes e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_s$$

Theorem (Jozefiak and Weyman)

The homology of the chain complex $(\Lambda^n \mathfrak{g}, \partial_n)$ decomposes into irreducible $GL(V)$-modules as follows

$$H_n(\Lambda \mathfrak{g}) \cong \mathrm{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})(V) \cong \bigoplus_{\lambda : \lambda = \lambda'} S^\lambda(V) \tag{12}$$

the sum is over self-conjugate λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

Representation theory meaning of Littlewood formula

Jozefiak and Weyman(1985)
Representation-theory interpretation of Littlewood formula

$$\prod_i (1 - x_i) \prod_{i<j} (1 - x_i x_j) = \sum_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} s_\lambda(x) \qquad (13)$$

sum over self-conjugated Young diagrams $\lambda = \lambda'$
Character of the chain complex **P**

$$\mathbf{P} = \bigoplus_{n\geq 0} \bigoplus_{\substack{\lambda : \lambda = \lambda' \\ \frac{1}{2}(|\lambda| + r(\lambda)) = n}} S(V \oplus \Lambda^2 V) \otimes S^\lambda(V)) \qquad (14)$$

$$1 = \prod_i \frac{1}{(1 - x_i)} \prod_{i<j} \frac{1}{(1 - x_i x_j)} \sum_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} s_\lambda(x) \qquad (15)$$

Poincaré duality

The algebra $PS(V) = U\mathfrak{g}$ is an Artin-Schelter algebra: Applying to **P** the functor $\mathrm{Hom}_{PS}(\bullet, PS)$ one gets another resolution of $\mathbb{K}$ as a right PS-module.

$$\mathbb{K} \leftarrow E_d^* \otimes PS \leftarrow \ldots \leftarrow E_1^* \otimes PS \leftarrow PS \leftarrow 0 \qquad (16)$$

where $E_n^* = H^n(\mathfrak{g}, \mathbb{K}) = H^n(\Lambda\mathfrak{g}^*) \cong \mathrm{Ext}^n_{PS}(\mathbb{K}, \mathbb{K})$.

$$H^\bullet(\mathfrak{g}, \mathbb{K}) = H^\bullet(\mathrm{Hom}_{\mathfrak{g}}(C(\mathfrak{g}), \mathbb{K})) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$$

Corollary

The cohomology of the cochain complex $(\Lambda^n\mathfrak{g}^*, \delta_n)$ decomposes into irreducible $GL(V^*)$-modules as follows

$$H^n(\Lambda\mathfrak{g}^*) \cong \mathrm{Ext}^n_{PS}(\mathbb{K}, \mathbb{K})(V^*) \cong \bigoplus_{\lambda:\lambda=\lambda'} S^\lambda(V^*) \qquad (17)$$

the sum is over self-conjugate λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

Lie algebras (finite) and DG algebras

$$(\mathfrak{g}, [\,,\,]) \qquad \longleftrightarrow \qquad (\Lambda^n \mathfrak{g}^*, \delta_n)$$

Apply the Homotopy transfer theorem to the DG algebras

$$(\Lambda^n \mathfrak{g}^*, \delta_n) \quad \text{and} \quad H^\bullet(\Lambda^n \mathfrak{g}^*, \delta_n)$$

Corollary

The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) = H^\bullet(\Lambda^n \mathfrak{g}^*)$ of $\mathfrak{g} = V \otimes \Lambda^2 V$ is an A_∞-algebra with elements identified with the self-conjugated Young Tableaux.

THANK YOU for your attention!