Unfolded Scalar Supermultiplet

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`D.P. and M.A. Vasiliev, [arXiv:1012.2903[hep-th]].`

Plan of the Talk

- Unfolded equations: definitions, properties, examples
- Analysis of unfolded equations, σ_-–cohomology technics
- $\mathcal{N} = 1$ $d = 4$ unfolded scalar supermultiplet
- σ_--spectral sequences

Proca field

$$\partial^\mu\partial_\mu A_\nu - \partial^\mu\partial_\nu A_\mu + m^2 A_\nu = 0 \quad \Rightarrow \quad \partial^\nu A_\nu = 0,$$

$$\partial^\mu\partial_\mu A_\nu - (1+\alpha)\partial^\mu\partial_\nu A_\mu + m^2 A_\nu = 0 \quad \not\Rightarrow \quad \partial^\nu A_\nu = 0,$$

Scalar electrodynamics

$$S_A = \int d^d x(-F_{\mu\nu}F^{\mu\nu} - A^\mu j_\mu) \quad \Rightarrow \quad \frac{\delta S_A}{\delta A^\nu} \sim \partial^\mu F_{\mu\nu} - j_\nu = 0 \quad \Rightarrow$$

requires conserved current $\partial^\mu j_\mu = 0$.

$$S_\phi = \int d^d x(\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi) \quad \Rightarrow \quad \partial^\mu j_\mu = 0,$$

where $\quad j_\mu = \phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*.$

$$\frac{\delta(S_A + S_\phi)}{\delta\phi} = 0 \quad \Rightarrow \quad \partial^\mu j_\mu \neq 0.$$

Unfolded equations

Generalized curvatures R^α for fields W^α

$$R^\alpha(x) \stackrel{def}{=} dW^\alpha(x) + G^\alpha(W(x)),$$

$$G^\alpha(W^\beta) \stackrel{def}{=} \sum_{n=1}^{\infty} f^\alpha_{\beta_1\ldots\beta_n} W^{\beta_1} \ldots W^{\beta_n}.$$

Compatibility condition

$$G^\beta(W)\frac{\delta^L G^\alpha(W)}{\delta W^\beta} \equiv 0 \quad \Rightarrow \quad dR^\alpha \equiv R^\beta \frac{\delta G^\alpha}{\delta W^\beta}.$$

Does not depend on base space

Equations:

$$R^\alpha(x) = 0.$$

Gauge invariance

$$\delta W^\alpha = d\varepsilon^\alpha - \varepsilon^\beta \frac{\delta^L G^\alpha(W)}{\delta W^\beta}.$$

Examples

Zero curvature equation in YM theory

Field: 1-form $\hat{\Omega}_0 = \Omega_0^a \hat{T}_a \in g$, g — Lie algebra, $\hat{T}_a$ — generators.

$$\hat{R} \stackrel{def}{=} d\hat{\Omega}_0 + \hat{\Omega}_0 \hat{\Omega}_0 = 0.$$

Let g be Poincare algebra $(\hat{P}_a, \hat{M}_{ab})$

$$\hat{\Omega}_0 = e_0{}^a \hat{P}_a + \omega_0{}^{ab} \hat{M}_{ab}, \quad \hat{R} = R_L{}^{ab} \hat{M}_{ab} + T^a \hat{P}_a,$$

$$R_L{}^{ab} = d\omega_0^{ab} + \omega_0{}^{ac} \omega_{0c}{}^b = 0,$$

$$T^a = de_0^a + \omega_0{}^{ab} e_{0b} = 0.$$

Describes background geometry of Minkowski space.

Examples

Covariant constancy equation

Fields: set of p-forms C^i, $W = \Omega_0 + C + \dots$.

$$R^i \stackrel{def}{=} dC^i + \Omega_0^a (T_a)^i{}_j C^j = 0.$$

Representation

$$\hat{A} = A^a \hat{T}_a \quad \to \quad A^i{}_j = A^a (T_a)^i{}_j.$$

$C \in$ representation space,

$$R^i = 0 \quad \Leftrightarrow \quad D_{\Omega_0} C^i = 0.$$

Examples

Massless scalar field

Cartesian coordinates

$$e_{0m}{}^{a} = \delta^{a}_{m}, \quad \omega_{0m}{}^{ab} = 0, \quad D^{L} = d.$$

Fields: 0-forms $C^{a(k)}$, $C_{b}{}^{ba(k-2)} = 0$.
Curvatures:

$$R^{a(k)} \stackrel{def}{=} dC^{a(k)} + e_{0}{}^{b} C_{b}{}^{a(k)}.$$

The first and the second equations

$$\partial_{a} C(x) + C_{a}(x) = 0,$$

$$\partial_{b} C_{a}(x) + C_{ab}(x) = 0$$

entail

$$C_{b}{}^{b}(x) = 0 \quad \Rightarrow \quad \partial^{a}\partial_{a} C(x) = 0.$$

σ_-–cohomology technics

Unfolded equations for p-forms C

$$R \stackrel{def}{=} (d + \sum \sigma)C = 0,$$

σ — algebraic operators.

$$\delta C = (d + \sum \sigma)\varepsilon,$$

$$I = (d + \sum \sigma)R \equiv 0$$

— gauge symmetries and Bianchi identities.
How to analyse them? How to identify dynamical fields and dynamical equations?

σ_-–cohomology technics

The choice of dynamical fields is not unique!

Example

$$\frac{\partial}{\partial x}B(x) + A(x) = 0, \quad \frac{\partial}{\partial x}A(x) + B(x) = 0.$$

Introduce $\mathbb{Z}$-grade $\mathcal{G}$: diagonalizable on the space of fields, bounded below.

$$R \overset{def}{=} (d + \sigma_-)C = 0,$$

$$\delta C = (d + \sigma_-)\varepsilon,$$

$$I = (d + \sigma_-)R \equiv 0,$$

σ_- — the only algebraic operator, lowers grade.
Compatibility conditions $\Rightarrow \quad (\sigma_-)^2 = 0.$

σ_-–cohomology technics.

Dynamical fields

Expressing fields in terms of fields of lower grade by means of

$$(d+\sigma_-)C = 0, \qquad \{dC^{k-1} + \sigma_-^1 C^k = 0\}$$

we get $C \notin \mathrm{Ker}(\sigma_-) \Rightarrow C$ — auxiliary.
Gauge transformation

$$\delta C = (d+\sigma_-)\varepsilon$$

allows to fix $C = 0$ if $C \in \mathrm{Im}(\sigma_-)$.
Result: dynamical fields C_d

$$C_d \in \frac{\mathrm{Ker}(\sigma_-)}{\mathrm{Im}(\sigma_-)} = H(\sigma_-).$$

σ_-–cohomology technics. The result

Dynamical fields C_d

$$C_d \in \frac{\mathrm{Ker}_p(\sigma_-)}{\mathrm{Im}_p(\sigma_-)} = H_p(\sigma_-),$$

Dynamical equations $R_d = 0$

$$R_d \in \frac{\mathrm{Ker}_{p+1}(\sigma_-)}{\mathrm{Im}_{p+1}(\sigma_-)} = H_{p+1}(\sigma_-),$$

Differential gauge symmetries ε_d (cannot be fixed by algebraic gauge)

$$\varepsilon_d \in \frac{\mathrm{Ker}_{p-1}(\sigma_-)}{\mathrm{Im}_{p-1}(\sigma_-)} = H_{p-1}(\sigma_-),$$

where p — rank of C as a differential form.

`[O.V. Shaynkman and M.A. Vasiliev '00]`

$H(\sigma_-)$-analysis for massless scalar field

Curvatures:

$$R^{a(k)} \stackrel{def}{=} dC^{a(k)} + e_0{}^b C_b{}^{a(k)}.$$

Grade $\mathcal{G}$ counts number of tensor indices, σ_- — contraction with frame e_0^b, $[\mathcal{G}, \sigma_-] = -\sigma_-$.

Dymanical field

only $C \in \mathrm{Ker}_0(\sigma_-)$, $\mathrm{Im}_0(\sigma_-) = 0 \Rightarrow C \in \mathrm{H}_0(\sigma_-)$.

Dynamical equation

has the form $R_d \sim e_0^a t$, where t is 0-form.
Indeed, $e_0{}^b e_{0b} t = 0$, $e_0^a t \notin \mathrm{Im}_1(\sigma_-) \Rightarrow e_0^a t \in \mathrm{H}_1(\sigma_-)$. Then $t \sim R_m{}^m = \partial_m C^m = 0$ yields dynamical equation.

`[O.V. Shaynkman and M.A. Vasiliev '00]`

Flat superspace

Background fields: 1-form $\Omega_0 \in \mathcal{N}=1$, $d=4$ SUSY:

$$\Omega_0 = e_0{}^a P_a + \omega_0{}^{ab} M_{ab} + \phi^\alpha Q_\alpha + \bar{\phi}_{\dot\alpha} \bar{Q}^{\dot\alpha},$$

$$R = T_a P_a + R_{0L}{}^{ab} M_{ab} + S^\alpha Q_\alpha + \bar{S}_{\dot\alpha} \bar{Q}^{\dot\alpha}.$$

Zero curvature equations

$$R_L{}^{ab} = d\omega_0^{ab} + \omega_0{}^{ac}\omega_{0c}{}^b = 0,$$

$$T^a = de_0^a + \omega_0{}^{ab} e_{0b} + 2i\phi^\alpha \bar{\phi}^{\dot\alpha} (\sigma^a)_{\alpha\dot\alpha} = 0,$$

$$S^\alpha = d\phi^\alpha + \frac{i}{4}\omega_0{}^{ab}\phi^\beta (\sigma_{ab})_\beta{}^\alpha = 0.$$

Diff. forms: space $\rightarrow$ superspace

$$e_m^a(x)dx^m \rightarrow E_M^a(z)dz^M, \quad \phi_m^\alpha(x)dx^m \rightarrow E_M^\alpha(z)dz^M, \ldots$$

Scalar supermultiplet

Minkowski space

$$R^{a(k)} \stackrel{def}{=} D^L C^{a(k)} + e_b C^{a(k)b} = 0,$$

$$r_\alpha{}^{a(k)} \stackrel{def}{=} D^L \chi_\alpha{}^{a(k)} + e_b \chi_\alpha{}^{a(k)b} = 0.$$

Superspace

$$R^{a(k)} \stackrel{def}{=} D^L C^{a(k)} + E_b C^{a(k)b} - \sqrt{2} E^\alpha \chi_\alpha{}^{a(k)} = 0,$$

$$r_\alpha{}^{a(k)} \stackrel{def}{=} D^L \chi_\alpha{}^{a(k)} + E_b \chi_\alpha{}^{a(k)b} - \sqrt{2} i E^{\dot\alpha} (\sigma_b)_{\alpha\dot\alpha} C^{a(k)b} = 0.$$

Scalar supermultiplet

Minkowski space

$$R \stackrel{def}{=} (D^L + \sigma_{-1})C = 0,$$

$$r \stackrel{def}{=} (D^L + \sigma_{-1})\chi = 0.$$

$\mathrm{H}(\sigma_{-1})$-cohomology analysis.

Superspace

$$R \stackrel{def}{=} (D^L + \sigma_{-1})C + \sigma_{-1/2}\chi = 0,$$

$$r \stackrel{def}{=} (D^L + \sigma_{-1})\chi + \sigma_{-1/2}C = 0.$$

Dynamical fields and equations — ?

σ_--spectral sequences

As for beginning, dynamical fields and equations $\subset \mathrm{H}(\sigma_{-1})$
But these equation can be further analysed!
Compatibility conditions:

$$(\sigma_{-1})^2 = 0, \quad \{\sigma_{-1}, \sigma_{-1/2}\} = 0, \quad (\sigma_{-1/2})^2 + \{D, \sigma_{-1}\} = 0.$$

$$\{\sigma_{-1}, \sigma_{-1/2}\} = 0 \quad \Rightarrow \quad \sigma_{-1/2}(\mathrm{H}_p(\sigma_{-1})) \subset \mathrm{H}_{p+1}(\sigma_{-1}).$$

Introducing $\tilde{\sigma}_{-1/2} = \sigma_{-1/2}|_{\mathrm{H}(\sigma_{-1})}$ we get

$$(\sigma_{-1/2})^2 + \{D, \sigma_{-1}\} = 0 \quad \Rightarrow \quad (\tilde{\sigma}_{-1/2})^2 = 0.$$

We define $\mathrm{H}(\tilde{\sigma}_{-1/2}) \stackrel{def}{=} \mathrm{H}(\sigma_{-1/2}|\sigma_{-1})$ (spectral sequence). They identify dynamical fields and dynamical equations!

Conclusion

- Unfolded equations for $\mathcal{N} = 1$, $d = 4$ chiral supermultiplet are built
- The independence of unfolded equations from base manifold is efficiently used to build superfield formulation from component one
- $\mathrm{H}(\sigma_-)$ technics has been generalized to the case of few negative grade algebraic operators. The dynamical content is defined by spectral sequence.