2d CFT/Gauge/Bethe correspondence

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M. Piątek, *Classical conformal blocks from TBA for the elliptic Calogero-Moser system*, **JHEP 06 (2011) 050**, hep-th/1102.5403

Triple correspondence

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1. Nekrasov-Shatashvili limit:

$$\epsilon_2 \to 0, \quad \epsilon_1 - \text{fixed},$$

2. Bethe/Gauge correspondence.

Basics of the $\mathcal{N}=2$ Gauge Theories

The extended $\mathcal{N}=2$ superspace coordinates: x^μ, $\theta_A^\alpha, \bar\theta_{\dot\alpha}^A$; $A=1,2$.

The $\mathcal{N}=2$ chiral superfield (in terms of $\mathcal{N}=1$ superfields):

$$\begin{aligned}
\Psi &= \Phi(y,\theta_1) + i\sqrt{2}\theta_2^\alpha W_\alpha(y,\theta_1) + \theta_2\theta_2 G(y,\theta_1),\\
W_\alpha(y,\theta) &= -\frac{1}{8}\bar D^2\left(e^{-2V(y,\theta,\bar\theta)} D_\alpha e^{2V(y,\theta,\bar\theta)}\right),\\
G(y,\theta) &= -\frac{1}{2}\int d^2\bar\theta\left[\Phi(y-2i\theta\sigma\bar\theta,\bar\theta)\right]^+ \exp\left[2V(y,\theta,\bar\theta)\right],\\
& \quad y = x^\mu - i\theta_A\sigma^\mu\bar\theta^A.
\end{aligned}$$

The $\mathcal{N}=1$ chiral superfield:

$$\Phi(x,\theta) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x),$$

ϕ — a complex scalar, ψ — a two component spinor, F — an auxiliary field.
The $\mathcal{N}=1$ vector superfield:

$$V(x,\theta,\bar\theta) = \theta\sigma^\mu\bar\theta A_\mu(x) + i\theta^2(\bar\theta\bar\lambda(x)) - i\bar\theta^2(\theta\lambda(x)) + \frac{1}{2}\theta^2\bar\theta^2 D(x),$$

A_μ — a gauge field, $\lambda, \bar\lambda$ — gauginos, D — an auxiliary field.

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$$\frac{1}{4\pi h^{\vee}}\mathfrak{Im}\left[\int d^4xd^4\theta\frac{\tau}{2}\,\mathrm{Tr}\Psi^2\right], \qquad \tau=\frac{\Theta}{2\pi}+\frac{4\pi i}{g^2}.$$

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- More generally (relaxing the renormalizability condition):

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- In the effective theories the only constraint is the holomorphicity. The **effective action** is determined by the prepotential.

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  - If all a_l are different we have the **Coulomb branch** of the theory. In this situation

$$G\to U(1)^r.$$

Seiberg-Witten prepotential

$$\mathcal{F}_{\mathrm{SW}}(\tau, \mathbf{a}, \mathbf{m}) = \mathcal{F}_{\mathrm{class}} + \mathcal{F}_{\mathrm{pert}} + \mathcal{F}_{\mathrm{instanton}}.$$

$$\mathcal{F}_{\mathrm{SW}}^{SU(2)} = \frac{1}{2}\tau a^2 + \frac{i}{2\pi} a^2 \log\frac{a}{\Lambda} + \dots$$

N. Seiberg, E. Witten, Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N}=2$ supersymmetric Yang-Mills theory, Nucl. Phys. **B431** (1994), hep-th/9407087.

N. Seiberg, E. Witten, *Monopoles, duality and chiral symmetry breaking in $N=2$ supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484550, hep-th/9408099.

Nekrasov partition function

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N. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys.7 (2004) 831- 864, hep-th/0206161.

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$$\mathcal{Z}_{\text{Nekrasov}}(\,\cdot\,;\epsilon_1,\epsilon_2) = \mathcal{Z}_{\text{class}}\,\mathcal{Z}_{1-\text{loop}}\,\mathcal{Z}_{\text{inst}}.$$

- For $\epsilon_1, \epsilon_2 \to 0$:

$$\mathcal{Z}_{\text{Nekrasov}}(\,\cdot\,;\epsilon_1,\epsilon_2) \quad\rightarrow\quad \exp\left\{-\frac{1}{\epsilon_1\epsilon_2}\,\mathcal{F}(\,\cdot\,;\epsilon_1,\epsilon_2)\right\},$$

where $\mathcal{F}(\,\cdot\,;\epsilon_1,\epsilon_2)$ becomes the Seiberg-Witten prepotential $\mathcal{F}_{\text{SW}}(\,\cdot\,)$ in the limit $\epsilon_1,\epsilon_2 \to 0$.

Nekrasov-Shatashvili limit

N. Nekrasov, S. Shatashvili, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, hep-th/0908.4052.

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  - $\mathcal{W}(\,\cdot\,;\epsilon_1) = \mathcal{W}_{\mathsf{pert}}(\,\cdot\,;\epsilon_1) + \mathcal{W}_{\mathsf{inst}}(\,\cdot\,;\epsilon_1)$ is a "deformed" effective **twisted superpotential** of the appropriate $2d$ $\mathcal{N}=2$ gauge theory on $2d$ Ω-background.

Gauge/Bethe correspondence

N. Nekrasov, S. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, In "Cargese 2008, Theory and Particle Physics: the LHC perspective and beyond", 0901.4744.

N. Nekrasov, S. Shatashvili, *Quantum integrability and supersymmetric vacua*, Prog. Theor. Phys. Suppl. 177 (2009) 105-119, 0901.4748.

N. Nekrasov, S. Shatashvili, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, hep-th/0908.4052.

$$\mathcal{W}(\mathbf{a}) = Y(\mathbf{t}).$$

- $Y(\mathbf{t})$ is the Yang-Yang function (Yang's functional) which serves as the potential for Bethe equations;
    - $\mathbf{t} = (t_1, \ldots, t_i)$ — spectral parameters (quasimomenta).

The $\mathcal{N}=2^*$ U(N) twisted superpotential/N-particle elliptic Calogero-Moser Yang's functional

$$\mathcal{W}_{\text{inst}}^{\mathcal{N}=2^*,\text{U(N)}}(q,\mathsf{a},m;\epsilon_1)=\oint_C dz\left[-\frac{1}{2}\,\varphi(z)\log\left(1-q\mathcal{Q}(z)\mathrm{e}^{-\varphi(z)}\right)+\mathrm{Li}_2\left(q\mathcal{Q}(z)\mathrm{e}^{-\varphi(z)}\right)\right].$$

Here $\varphi(x)$ is the solution of the integral equation:

$$\varphi(x)=\oint_C dy\,\mathcal{G}(x-y)\log\left(1-q\mathcal{Q}(y)\mathrm{e}^{-\varphi(y)}\right).$$

$$\mathcal{Q}(x) = \frac{P(x-m)P(x+m+\epsilon_1)}{P(x)P(x+\epsilon_1)}, \qquad P(x)=\prod_{i=1}^{\mathrm{N}}(x-a_i),$$

$$\mathcal{G}(x) = \frac{d}{dx}\log\frac{(x+m+\epsilon_1)(x-m)(x-\epsilon_1)}{(x-m-\epsilon_1)(x+m)(x+\epsilon_1)}.$$

The contour C on the complex plane comes from infinity, goes around the points $a_i+k\epsilon_1$, $i=1,\ldots,\mathrm{N}$, $k=0,1,2,\ldots$ and goes back to infinity. It separates these points and the points $a_i+lm+k\epsilon_1$, $l\in\mathbb{Z}$, $k=-1,-2,\ldots$.

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Chiral part of the AGT conjecture

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Example:

Chiral AGT relation on the **4-punctured Riemann sphere** $\mathcal{C}_{0,4}$

$$q^{\Delta_1+\Delta_2-\Delta}\,\mathcal{F}_{c,\Delta}\left[\begin{matrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{matrix}\right](\,q) = (1-q)^{-\frac{(\mu_1+\mu_2)(\mu_3+\mu_4)}{2\epsilon_1\epsilon_2}}\;\mathcal{Z}^{N_f=4,\mathrm{U}(2)}_{\text{inst}}(q,a,\mu_i;\epsilon_1,\epsilon_2),$$

$$c = \frac{1+6(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2} \equiv 1+6Q^2, \qquad \Delta = \frac{(\epsilon_1+\epsilon_2)^2-4a^2}{4\epsilon_1\epsilon_2},$$

$$\Delta_1 = \frac{\frac{1}{4}(\epsilon_1+\epsilon_2)^2-\frac{1}{4}(\mu_1-\mu_2)^2}{\epsilon_1\epsilon_2}, \qquad \Delta_2 = \frac{\frac{1}{2}(\mu_1+\mu_2)(\epsilon_1+\epsilon_2-\frac{1}{2}(\mu_1+\mu_2))}{\epsilon_1\epsilon_2},$$

$$\Delta_3 = \frac{\frac{1}{2}(\mu_3+\mu_4)(\epsilon_1+\epsilon_2-\frac{1}{2}(\mu_3+\mu_4))}{\epsilon_1\epsilon_2}, \qquad \Delta_4 = \frac{\frac{1}{4}(\epsilon_1+\epsilon_2)^2-\frac{1}{4}(\mu_3-\mu_4)^2}{\epsilon_1\epsilon_2}.$$

The NS limit vs. classical limit of conformal blocks

- Recall that:

$$\sqrt{\frac{\epsilon_2}{\epsilon_1}} = b, \qquad c = 1 + 6(b + b^{-1})^2.$$

- If all the conformal weights in the conformal block are "heavy", i.e.:

$$\Delta_i = b^{-2}\,\delta_i$$

then the limit $\epsilon_2 \to 0$ or equivalently $b \to 0$ corresponds to the **classical limit** of conformal blocks in which

$$\Delta_i,\ c\ \longrightarrow \infty.$$

- Classical limit of conformal blocks is not yet completely understood. Up to now it has been explicitly derived only in the case of the conformal block on the **4-punctured Riemann sphere** $\mathcal{C}_{0,4}$:

$$\mathcal{F}_{c=1+6Q^2,\Delta}\left[\begin{matrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{matrix}\right](x) \ \stackrel{b\to 0}{\longrightarrow}\ \exp\left\{\frac{1}{b^2}\,f_\delta\left[\begin{matrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{matrix}\right](x)\right\}.$$

Quantum and classical blocks on the 4-punctured sphere

- Quantum conformal block on the 4-punctured sphere:

$$\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x) \;=\; x^{\Delta-\Delta_2-\Delta_1}\left(1+\sum_{n=1}^{\infty}\mathcal{F}^{\,n}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]x^{\,n}\right),$$

$$\mathcal{F}^{\,n}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] = \sum_{n=|I|=|J|}\langle\nu_{\Delta_4}, V_{\Delta_3}(1)\nu_{\Delta,I}\rangle\;\left[G_{c,\Delta}\right]^{IJ}\;\langle\nu_{\Delta,J}, V_{\Delta_2}(1)\nu_{\Delta_1}\rangle\,.$$

Quantum and classical blocks on the 4-punctured sphere

- $\left[G_{c,\Delta}\right]^{IJ}$ is the inverse of the Gram matrix

$$\left[G_{c,\Delta}\right]_{IJ} = \langle \nu_{\Delta,I}, \nu_{\Delta,J}\rangle$$

of the standard symmetric bilinear form in the Verma module

$$\mathcal{V}_\Delta = \bigoplus_{n=0}^{\infty} \mathcal{V}_\Delta^n,$$

$$\mathcal{V}_\Delta^n = \mathrm{Span}\Big\{\nu_{\Delta,I}^n = L_{-I}\nu_\Delta = L_{-i_k}\ldots L_{-i_2}L_{-i_1}\nu_\Delta \quad : \\ I = (i_k \geq \ldots \geq i_1 \geq 1) \text{ an ordered set of positive integers} \\ \text{of the length } |I| \equiv i_1 + \ldots + i_k = n\Big\}.$$

Quantum and classical blocks on the 4-punctured sphere

- The operator V_Δ in the matrix element is the normalized primary chiral vertex operator acting between the Verma modules:

$$\left\langle \nu_{\Delta_i}, V_{\Delta_j}(z)\nu_{\Delta_k}\right\rangle = z^{\Delta_i-\Delta_j-\Delta_k}.$$

- Covariance property:

$$[L_n, V_\Delta(z)] = z^n\left(z\frac{d}{dz} + (n+1)\Delta\right)V_\Delta(z)\,, \qquad n\in\mathbb{Z}.$$

Elliptic quantum block on the 4-punctured sphere

$$
\begin{aligned}
\mathcal{F}_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}(x) \;&=\; x^{\frac{c-1}{24}-\Delta_1-\Delta_2}(1-x)^{\frac{c-1}{24}-\Delta_2-\Delta_3}\\
&\times\theta_3(q)^{\frac{c-1}{2}-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}\,(16q)^{\Delta-\frac{c-1}{24}}\,\mathcal{H}_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}(q),
\end{aligned}
$$

$$
q \equiv q(x) = e^{-\pi\frac{K(1-x)}{K(x)}}, \qquad\qquad K(x) = \int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}},
$$

$$
\mathcal{H}_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}(q) = 1 + \sum_{n=1}^{\infty}(16q)^{\,n}\mathcal{H}^{n}_{c,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix},
$$

$$
\mathcal{H}^{n}_{c,\,\Delta}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix} \;=\; \sum_{\substack{r\geq1\;s\geq1\\ n\geq rs\geq1}} \frac{R^{rs}_{c}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}}{\Delta-\Delta_{rs}(c)}\,\mathcal{H}^{n-rs}_{c,\,\Delta_{rs}(c)+rs}\begin{bmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{bmatrix}, \qquad\qquad n>0.
$$

Classical block on the 4-punctured sphere

$$
\begin{aligned}
f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) &= (\delta-\delta_1-\delta_2)\log x + \sum_{n=1}^{\infty} x^n\, f_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]\\
&= (\delta-\delta_1-\delta_2)\log x + \frac{(\delta+\delta_3-\delta_4)(\delta+\delta_2-\delta_1)}{2\delta}\,x + \ldots\,.
\end{aligned}
$$

Elliptic representation:

$$
\begin{aligned}
f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) &= (\tfrac{1}{4}-\delta_1-\delta_2)\log x + (\tfrac{1}{4}-\delta_2-\delta_3)\log(1-x)\\
&+ (3-4(\delta_1+\delta_2+\delta_3+\delta_4))\log\left(\theta_3(q(x))\right)\\
&+ (\delta-\tfrac{1}{4})\log(16q(x)) + h_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q(x)).
\end{aligned}
$$

Aims and Motivations

- Hance, by combining the **AGT conjecture**, the **NS limit** and the **Gauge/Bethe correspondence** it is possible to link **classical blocks** to **Yang's functionals**.
- The main aim of my work was to exploit this link to shed some light on the **classical block on the sphere** $f_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)$.
- The main motivation for this line of research is the longstanding open problem of the **uniformization** of the 4-punctured Riemann sphere, where the knowledge of the 4-point classical block plays a key role.

Clssical blocks from the elliptic Calogero-Moser Yang's functional

$$
\begin{aligned}
\mathrm{f}_{\frac{1}{4}-\frac{a^2}{\epsilon_1^2}}\left[\begin{matrix}\frac{1}{4}(\frac{3}{4}-\frac{m}{\epsilon_1}(1+\frac{m}{\epsilon_1})) & \frac{1}{4}(\frac{3}{4}-\frac{m}{\epsilon_1}(1+\frac{m}{\epsilon_1})) \\ \frac{3}{16} & \frac{3}{16}\end{matrix}\right](x) \quad &= \quad \left(-\frac{1}{8}+\frac{1}{4}\frac{m}{\epsilon_1}\left(1+\frac{m}{\epsilon_1}\right)\right)\log x \\
&+\left(-\frac{1}{8}+\frac{1}{2}\frac{m}{\epsilon_1}\left(1+\frac{m}{\epsilon_1}\right)\right)\log(1-x) \\
&+2\frac{m}{\epsilon_1}\left(1+\frac{m}{\epsilon_1}\right)\log(\theta_3(q(x))\,\hat{\eta}(q(x))) \\
&-\frac{a^2}{\epsilon_1^2}\log(16q(x))+\frac{1}{\epsilon_1}\,\mathcal{W}_{\rm inst}^{\mathcal{N}=2^*,\mathrm{U}(2)}(q(x),\mathrm{a},m;\epsilon_1).
\end{aligned}
$$

Classical Liouville Theory

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$$g_{ab} = \mathrm{e}^{\varphi}\hat{g}_{ab}$$

on $\mathcal{C}_{g,n}$.

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since

$$R_{\mathrm{e}^{\sigma}\hat{g}_{ab}} = \mathrm{e}^{-\sigma}\left(R_{\hat{g}_{ab}} - \Delta_{\hat{g}_{ab}}\sigma\right).$$

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Classical Liouville Theory on the n-punctured sphere

- Consider the case of the Riemann surface being $\mathcal{C}_{0,n}$ a punctured sphere and choose complex coordinates on $\mathcal{C}_{0,n}$ in such a way that $z_n = \infty$.
- The solution of the equation:

$$\partial_z \partial_{\bar{z}} \phi(z, \bar{z}) = \frac{\mu}{2}\, \mathrm{e}^{\phi(z,\bar{z})}$$

with the elliptic singularities, namely

$$\phi(z, \bar{z}) = \begin{cases} -2\left(1 - \xi_j\right) \log|z - z_j| + O(1) & \text{as} \quad z \to z_j, \qquad j = 1, \ldots, n-1, \\ -2\left(1 + \xi_n\right) \log|z| + O(1) & \text{as} \quad z \to \infty, \end{cases}$$

$$\text{where} \;\; \xi_i \in \mathbb{R}^+ \setminus \{0\} \quad \text{for all} \;\; i = 1, \ldots, n \quad \text{and} \;\; \sum_{i=1}^{n} \xi_i < n - 2,$$

can be interpreted as the conformal factor of the hyperbolic metric on $\mathcal{C}_{0,n} = \mathbb{C} \setminus \{z_1, \ldots, z_{n-1}\}$ with conical singularities characterized by opening angles $\theta_j = 2\pi\xi_j$ at punctures z_j.

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From this theorem it follows the existence of a meromorphic function

$$\lambda\colon \mathbb{H} \ni \tau \to z = \lambda(\tau) \in \mathbb{H}\diagup G \cong \mathcal{C}$$

called **uniformization**. The map λ is explicitly known only for the 3-punctured sphere and in a few very special, symmetric cases with higher number of punctures. In particular, an explicit construction of this map for the 4-punctured sphere is a longstanding and still open problem.

Riemann-Hilbert problem for the Fuchs equation

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- If λ is the uniformization of $\mathcal{C}_{0,n}$, the inverse

$$\lambda^{-1}\colon \mathcal{C}_{0,n} \ni z \to \tau(z) \in \mathbb{H}$$

is a multi-valued function with branch points z_j and with branches related by the elements T_k of the group G.

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- Define the Schwarzian derivative:

$$\{\psi, z\} = \left[\left(\frac{\psi''}{\psi'}\right)' - \frac{1}{2}\left(\frac{\psi''}{\psi'}\right)^2\right](z).$$

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$$\{\psi, z\} = \left[\left(\frac{\psi''}{\psi'}\right)' - \frac{1}{2}\left(\frac{\psi''}{\psi'}\right)^2\right](z).$$

- One can show that the Schwarzian derivative of λ^{-1} is a holomorphic function on $\mathcal{C}_{0,n}$ of the form:

$$\begin{aligned} \{\lambda^{-1}, z\} &= \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{(z-z_k)^2} + \sum_{k=1}^{n-1}\frac{2c_k}{z-z_k}, \\ \{\lambda^{-1}, z\} &\overset{z\to\infty}{=} \frac{1}{2z^2} + \mathcal{O}(z^{-3}), \end{aligned}$$

where the so-called *accessory parameters* c_j satisfy the relations

$$\sum_{k=1}^{n-1} c_k = 0, \qquad \sum_{k=1}^{n-1}(4c_k z_k + 1) = 1,$$

Riemann-Hilbert problem for the Fuchs equation

- On the other hand it is a well known fact that if $\{\psi_1, \psi_2\}$ is a fundamental system of normalized $(\psi_1\psi_2' - \psi_1'\psi_2 = 1)$ solutions of the Fuchs equation

$$\psi(z)'' + \tfrac{1}{2}\left\{\lambda^{-1}, z\right\}\psi(z) = 0 \tag{1}$$

with $SL(2,\mathbb{R})$ monodromy with respect to all punctures then up to a Möbius transformation the inverse map is

$$\lambda^{-1} = \psi_1(z)/\psi_2(z).$$

- Therefore, it is possible to reformulate the uniformization problem of $\mathcal{C}_{0,n}$ as a kind of the Riemann-Hilbert problem for the Fuchs equation (1).
- !!! Problem: we do not know nothing about the c_k's for $n > 3$.

Polyakov conjecture

- The properly defined and normalized Liouville action functional evaluated on the classical solution $\phi(z, \bar{z})$ is the generating functional for the accessory parameters:

$$c_j = -\frac{\partial S_{\rm L}^{\rm cl}[\phi]}{\partial z_j}.$$

Zamolodchikovs conjecture

The 4-point function of the Liouville DOZZ theory with the operators insertions at $z_4 = \infty$, $z_3 = 1$, $z_2 = x$, $z_1 = 0$, can be expressed as an integral of s-channel conformal blocks and DOZZ couplings over the continuous apectrum of the theory. In the semiclassical limit the integrand can be written in terms of 3-point **classical Liouville actions** and the **classical block**:

$$\left\langle \mathsf{V}_{\alpha_4}(\infty,\infty)\mathsf{V}_{\alpha_3}(1,1)\mathsf{V}_{\alpha_2}(x,\bar{x})\mathsf{V}_{\alpha_1}(0,0)\right\rangle \sim \int\limits_0^\infty dp\, e^{-\frac{1}{b^2}S_{\rm L}^{\rm cl}(\delta_i,x;\delta)},$$

where $\delta = 1/4 + p^2$ and

$$S_{\rm L}^{\rm cl}(\delta_i,x;\delta) = S_{\rm L}^{\rm cl}(\delta_4,\delta_3,\delta)+S_{\rm L}^{\rm cl}(\delta,\delta_2,\delta_1)-f_\delta\begin{bmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{bmatrix}(x)-\bar{f}_\delta\begin{bmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{bmatrix}(\bar{x}).$$

Zamolodchikovs conjecture

In the classical limit $b \to 0$ the integral is dominated by its saddle point value. One thus gets

$$
\begin{aligned}
S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i; x, \delta_s) &= S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4, \delta_3, \delta_s) + S_{\mathrm{L}}^{\mathrm{cl}}(\delta_s, \delta_2, \delta_1) \\
&- \mathrm{f}_{\delta_s}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) - \bar{\mathrm{f}}_{\delta_s}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](\bar{x})
\end{aligned}
$$

where $\delta_s = \frac{1}{4} + p_s(x)^2$ and the **saddle point Liouville momentum** $p_s(x)$ is determined by

$$
\frac{\partial}{\partial p} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, x; \tfrac{1}{4} + p^2)_{|p=p_s} = 0\ . \tag{2}
$$

Accessory parameter vs. eCM Yang's functional

In the case of $\mathcal{C}_{0,4}$ the Polyakov conjecture gives

$$
\begin{aligned}
c_2(x) &= -\frac{\partial}{\partial x} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, x) \\
&= -\frac{\partial}{\partial p} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, x, \tfrac{1}{4} + p^2)\Big|_{p=p_s(x)} \frac{\partial p_s(x)}{\partial x} - \frac{\partial}{\partial x} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, x, \tfrac{1}{4} + p^2)\Big|_{p=p_s(x)} \\
&= -\frac{\partial}{\partial x} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, x, \tfrac{1}{4} + p^2)\Big|_{p=p_s(x)} = \frac{\partial}{\partial x} \mathrm{f}_{\frac{1}{4}+p^2}\left[\begin{smallmatrix}\delta_3 & \delta_2 \\ \delta_4 & \delta_1\end{smallmatrix}\right](x)\Big|_{p=p_s(x)}.
\end{aligned}
$$

Accessory parameter vs. eCM Yang's functional

In particular, one gets

$$\begin{aligned} c_2(x) &= -\frac{\pi E(x)K(1-x)p_s(x)^2}{2(x-1)xK(x)^2} - \frac{4x-1-4\xi^2}{16(x-1)x} \\ &+ \frac{2m(m+\epsilon_1)}{\epsilon_1^2}\frac{\partial}{\partial x}\log\hat{\eta}(q(x)) \\ &+ \frac{1}{\epsilon_1}\frac{\partial}{\partial x}\mathcal{W}_{\rm inst}^{\mathcal{N}=2^*,\mathrm{U}(2)}(q(x),a,m;\epsilon_1)\Big|_{\frac{ia}{\epsilon_1}=p_s(x)}, \end{aligned}$$

$$\xi = \sqrt{\tfrac{1}{4}+\tfrac{m}{\epsilon_1}(1+\tfrac{m}{\epsilon_1})}.$$

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- A relation with generalized matrix models
- A relation with quantum integrable systems
- An extension to the 4d $\mathcal{N} = 2$ SU(N) gauge theories (on the $2d$ CFT side appears conformal Toda theory)