Stringy Differential Geometry, beyond Riemann

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Prologue

- In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.
- String theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ on an equal putting.
- This may suggests the existence of a veiled unifying description of them, beyond Riemann.

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Talk is based on works

in collaboration with Imtak Jeon and Kanghoon Lee

- **Differential geometry with a projection: Application to double field theory**

  JHEP 1104:014 (2011), arXiv:1011.1324

- **Double field formulation of Yang-Mills theory**

  Phys. Lett. B 701:260 (2011), arXiv:1102.0419

- **Stringy differential geometry, beyond Riemann**

  arXiv: 1105.6294

Introduction

Symmetry

- guides the structure of Lagrangians.
- organizes the physical laws into simple forms.
- for example, in Maxwell theory,
  - U(1) gauge symmetry forbids $m^2 A_\mu A^\mu$
  - Lorentz symmetry unifies the original 4 eqs into 2.

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Essence of Riemannian geometry

- Diffeomorphism: $\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$
- $\nabla_\lambda g_{\mu\nu} = 0 \longrightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$
- Curvature: $[\nabla_\mu, \nabla_\nu] \longrightarrow R_g$.

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Closed string

- $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ are on an equal footing completing the massless sector.
- Low energy effective action of them:

$$S_{\rm eff.} = \int {\rm d}x^D \sqrt{-g} e^{-2\phi} \left(R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right)$$

- Diffeomorphism and one-form gauge symmetry are manifest

$$x^\mu \rightarrow x^\mu + \delta x^\mu \,, \qquad B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \,.$$

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T-duality

- Redefine the dilaton,

$$e^{-2d} = \sqrt{-g}e^{-2\phi}$$

- Set a $2D \times 2D$ symmetric matrix,

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}$$

- $A, B, ...$: $2D$-dimensional vector indices.

T-duality

- T-duality is realized by an $\mathbf{O}(D,D)$ rotation,

$$\mathcal{H}_{AB} \longrightarrow L_A{}^C L_B{}^D \mathcal{H}_{CD}\,, \qquad\qquad d \longrightarrow d\,,$$

where

$$L \in \mathbf{O}(D,D)\,.$$

T-duality

- $\mathbf{O}(D,D)$ metric,

$$\mathcal{J}_{AB} := \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$$

freely raises or lowers the $2D$-dimensional vector indices.

Double Field Theory (DFT)

- Hull and Zwiebach , later with Hohm

$$S_{\rm DFT} = \int {\rm d}y^{2D}\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d)\,,$$

where

$$\begin{aligned}\mathcal{R}(\mathcal{H}, d) = \;& \mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\,\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right)\\ & + 4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\,.\end{aligned}$$

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- Yet,

$$\frac{\partial}{\partial\tilde{x}_\mu} \equiv 0\,.$$

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$$\partial_A\partial^A\Phi = 2\frac{\partial^2}{\partial\tilde{x}_\mu\partial x^\mu}\Phi \equiv 0\,, \qquad \partial_A\Phi_1\partial^A\Phi_2 \equiv 0\,.$$

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$$X_L(\sigma^+) = \tfrac{1}{2}(x+\tilde{x}) + \tfrac{1}{2}(p+w)\sigma^+ + \cdots\,,$$
$$X_R(\sigma^-) = \tfrac{1}{2}(x-\tilde{x}) + \tfrac{1}{2}(p-w)\sigma^- + \cdots\,.$$

- Under T-duality,

$$X_L + X_R \longrightarrow X_L - X_R\,,$$

such that

$$(x,\tilde{x},p,w) \longrightarrow (\tilde{x},x,w,p)\,.$$

- Level matching condition for the massless sector,

$$p\cdot w \equiv 0 \quad\Longleftrightarrow\quad \partial_A\partial^A = 2\frac{\partial^2}{\partial\tilde{x}_\mu\partial x^\mu} \equiv 0\,.$$

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- Upon the level matching constraint,

$$S_{\mathrm{DFT}} \Longrightarrow S_{\mathrm{eff.}} = \int \mathrm{d}x^{D}\sqrt{-g}e^{-2\phi}\left(R_{g} + 4(\partial\phi)^{2} - \tfrac{1}{12}H^{2}\right).$$

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- Thus, in the DFT formulation of the effective action by Hull, Zwiebach and Hohm, the $\mathbf{O}(D, D)$ T-duality structure is manifest.

- *What about the diffeomorphism and the one-form gauge symmetry?*

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- *What about the diffeomorphism and the one-form gauge symmetry?*

Diffeomorphism & one-form gauge symmetry

- Introduce a unifying parameter,

$$X^A = (\Lambda_\mu, \delta x^\nu)$$

- Unifying transformation rule, upon the level matching constraint,

$$\delta_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A} X_{C]} \mathcal{H}^C{}_B + 2\partial_{[B} X_{C]} \mathcal{H}_A{}^C \,,$$
$$\delta_X \left(e^{-2d}\right) \equiv \partial_A \left(X^A e^{-2d}\right) \,.$$

- In fact, these coincide with the generalized Lie derivative,

$$\delta_X \mathcal{H}_{AB} = \hat{\mathcal{L}}_X \mathcal{H}_{AB} \,, \qquad \delta_X (e^{-2d}) = \hat{\mathcal{L}}_X (e^{-2d}) = -2(\hat{\mathcal{L}}_X d) e^{-2d} .$$

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Generalized Lie derivative

- Definition, Siegel, Courant, Grana ...

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n} := X^B\partial_B T_{A_1\cdots A_n} + \omega\partial_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}X_B - \partial_B X_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,.$$

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Diffeomorphism & one-form gauge symmetry

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are symmetry of the action by Hull, Zwiebach and Hohm,

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Stringy differential geometry

We propose a novel differential geometry which

- treats the three objects of the massless sector in a unified manner,
- manifests not only diffeomorphism and one-form gauge symmetry but also $\mathbf{O}(D, D)$ T-duality,
- and enables us to rewrite the low energy effective action of them as a single term,

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  is of the most general form to satisfy

$$\mathcal{H}_A{}^C\mathcal{H}_C{}^B=\delta_A{}^B\,,\qquad \mathcal{H}_{AB}=\mathcal{H}_{BA}\,,$$

  and the upper left $D\times D$ block of $\mathcal{H}$ is non-degenerate,

- we focus on a symmetric projection,

$$P_A{}^BP_B{}^C=P_A{}^C\qquad P_{AB}=P_{BA}\,,$$

  which is related to $\mathcal{H}$ by

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Stringy differential geometry

- Three basic objects:

$$\mathcal{J}_{AB}\,, \qquad P_{AB}\,, \qquad d\,.$$

- We postulate a "semi-covariant" derivative, ∇_A,

$$\nabla_C T_{A_1A_2\cdots A_n} = \partial_C T_{A_1A_2\cdots A_n} - \omega\Gamma^B{}_{BC} T_{A_1A_2\cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,.$$

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$$\nabla_C(e^{-2d}) = \partial_C e^{-2d} - \Gamma^B{}_{BC} e^{-2d} = -2(\nabla_C d)e^{-2d}$$

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$$\nabla_A \mathcal{J}_{BC} = 0\,, \qquad \nabla_A P_{BC} = 0\,, \qquad \nabla_A d = 0\,,$$

as for the unifying description of the massless modes
(*cf.* $\nabla_\lambda g_{\mu\nu} = 0$ in Riemannian geometry).

- Further we require,

$$\Gamma_{CAB} + \Gamma_{CBA} = 0\,, \qquad \Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0\,.$$

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- Then, we may replace ∂_A by ∇_A in $\hat{\mathcal{L}}_X$ and also in $[X,Y]^A_{\mathrm{c}}$,

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n} = X^B \nabla_B T_{A_1\cdots A_n} + \omega \nabla_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^n 2\nabla_{[A_i} X_{B]} T_{A_1\cdots A_{i-1}}{}^B{}_{A_{i+1}\cdots A_n}\,,$$

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- *cf.* In Riemannian geometry, torsion free condition implies

$$\mathcal{L}_X T_{\mu_1\cdots\mu_n} = X^\nu \nabla_\nu T_{\mu_1\cdots\mu_n} + \omega \nabla_\nu X^\nu T_{\mu_1\cdots\mu_n} + \sum_{i=1}^n \nabla_{\mu_i} X^\nu T_{\mu_1\cdots\mu_{i-1}\nu\mu_{i+1}\cdots\mu_n}\,,$$

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- Explicitly, the connection is

$$\Gamma_{CAB} = 2\big(P\partial_C P\bar{P}\big)_{[AB]} + 2\big(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\big)\partial_D P_{EC}$$
$$- \tfrac{4}{D-1}\big(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\big)\big(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\big)\,.$$

where $\bar{P}$ is the complementary projection,

$$\bar{P} = \tfrac{1}{2}(1-\mathcal{H})\,.$$

Stringy differential geometry

- Further, we set

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D} \,,$$

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- The connection belongs to the kernel of these rank six-projectors - uniqueness

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- Under $\delta_X \mathcal{H}_{AB} = \hat{\mathcal{L}}_X \mathcal{H}_{AB}$ and $\delta_X d = \hat{\mathcal{L}}_X d$, the diffeomorphism and the one-form gague symmetry, or shortly *double-gauge symmetry*, we obtain

$$(\delta_X - \hat{\mathcal{L}}_X)\Gamma_{CAB} \equiv 2\big[(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE} - \delta_C{}^F \delta_A{}^D \delta_B{}^E\big]\partial_F \partial_{[D} X_{E]}\,,$$

and

$$(\delta_X - \hat{\mathcal{L}}_X)\nabla_C T_{A_1 \cdots A_n} \equiv \sum_i 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BFDE}\partial_F \partial_{[D} X_{E]} T_{\cdots B \cdots}\,.$$

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- However, the characteristic property of our derivative, ∇_A, is that, combined with the projections, it can generate various $\mathbf{O}(D,D)$ and double-gauge covariant quantities:

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Curvature

- The usual curvature,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E}\Gamma_{BED} - \Gamma_{BC}{}^{E}\Gamma_{AED} \,,$$

satisfying

$$[\nabla_A, \nabla_B] T_{C_1 C_2 \cdots C_n} = -\Gamma_{DAB}\nabla^D T_{C_1 C_2 \cdots C_n} + \sum_{i=1}^{n} R_{C_i DAB}\, T_{C_1 \cdots C_{i-1}}{}^{D}{}_{C_{i+1} \cdots C_n} \,,$$

is NOT double-gauge covariant,

$$\delta_X R_{ABCD} \neq \hat{\mathcal{L}}_X R_{ABCD} \,.$$

Generalized curvature

- Instead, we define, as for a key quantity in our formalism,

$$S_{ABCD} := \tfrac{1}{2}\left(\mathcal{R}_{ABCD} + \mathcal{R}_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right) .$$

- This is related to a commutator,

$$P_I{}^A \bar{P}_J{}^B [\nabla_A, \nabla_B] T_C \equiv 2 P_I{}^A \bar{P}_J{}^B S_{CDAB} T^D .$$

- It can be shown, by brute force computation, to meet
  - just like the Riemann curvature,

$$S_{ABCD} = \tfrac{1}{2}(S_{[AB][CD]} + S_{[CD][AB]}) \equiv S_{\{ABCD\}} , \qquad S_{A[BCD]} = 0 ,$$

  - and further

$$P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD} \equiv 0 , \qquad P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} \equiv 0 , \qquad etc.$$

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- Under the double-gauge transformations, we get

$$(\delta_X - \hat{\mathcal{L}}_X)S_{ABCD} \equiv 4\nabla_{\{A}\left[(\mathcal{P}+\bar{\mathcal{P}})_{BCD\}}{}^{EFG}\partial_E\partial_{[F}X_{G]}\,\cdot\right]\,.$$

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Covariant curvature

- Double-gauge covariant rank two-tensor,

$$P_I{}^A \bar{P}_J{}^B S_{AB}\,.$$

- Double-gauge covariant scalar,

$$\mathcal{H}^{AB} S_{AB}\,.$$

In the above, we set

$$S_{AB} = S_{BA} := S^C{}_{ACB}\,,$$

which turns out to be traceless,

$$S^A{}_A \equiv 0\,.$$

Covariant curvature and DFT

- Especially, the covariant scalar constitutes the effective action as

$$\mathcal{H}^{AB}S_{AB} \equiv R_g + 4\Box\phi - 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\,.$$

- It also agrees with Hull, Zwiebach and Hohm,

$$\begin{aligned}\mathcal{H}^{AB}S_{AB} \equiv \quad & \mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right)\\ & +4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\,.\end{aligned}$$

Deriving Equations of motion

- Under arbitrary infinitesimal transformations of the dilaton and the projection, we get

$$\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB}\,,$$

where explicitly

$$\begin{aligned}\delta\Gamma_{CAB} = {} & 2P_{[A}{}^{D}\bar{P}_{B]}{}^{E}\nabla_{C}\delta P_{DE} + 2(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E})\nabla_{D}\delta P_{EC} \\ & -\tfrac{4}{D-1}(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D})(\partial_{D}\delta d + P_{E[G}\nabla^{G}\delta P^{E}{}_{D]}) \\ & -\Gamma_{FDE}\,\delta(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}\,.\end{aligned}$$

Deriving Equations of motion

- With $\nabla_A d = 0$, from the manipulation,

$$\delta S_{\text{eff.}} \equiv \int \mathrm{d}y^{2D}\, 2e^{-2d} \left(\delta P^{AB} S_{AB} - \delta d\, \mathcal{H}^{AB} S_{AB} \right) ,$$

and the relation,

$$\delta P = P \delta P \bar{P} + \bar{P} \delta P P ,$$

it is now very easy to derive the equations of motion:

$$P_{(I}{}^{A} \bar{P}_{J)}{}^{B} S_{AB} = 0 , \qquad \mathcal{H}^{AB} S_{AB} = 0 .$$

Double-vielbein

- $\mathcal{J}_{AB}$ and $\mathcal{H}_{AB}$ can be simultaneously diagonalized,

$$\mathcal{J} = \begin{pmatrix} V & \bar{V} \end{pmatrix} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & -\bar{\eta} \end{pmatrix} \begin{pmatrix} V & \bar{V} \end{pmatrix}^t ,$$

$$\mathcal{H} = \begin{pmatrix} V & \bar{V} \end{pmatrix} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \bar{\eta} \end{pmatrix} \begin{pmatrix} V & \bar{V} \end{pmatrix}^t .$$

  Here η and $\bar{\eta}$ are two copies of the D-dimensional Minkowskian metric. Both V and $\bar{V}$ are $2D\times D$ matrices which we name 'double-vielbein '.

- They must satisfy

$$V = PV\,, \qquad V\eta^{-1}V^t = P\,, \qquad V^t\mathcal{J}V = \eta\,, \quad V^t\mathcal{J}\bar{V} = 0\,,$$

$$\bar{V} = \bar{P}\bar{V}\,, \qquad \bar{V}\bar{\eta}\bar{V}^t = -\bar{P}\,, \qquad \bar{V}^t\mathcal{J}\bar{V} = -\bar{\eta}^{-1}\,.$$

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- Double-vielbein is of the following general form, Siegel, Hassan

$$V_{Am} = \frac{1}{\sqrt{2}}\begin{pmatrix} (e^{-1})_m{}^{\mu} \\ (B+e)_{\nu m} \end{pmatrix}, \quad \bar{V}_A{}^{\bar{n}} = \frac{1}{\sqrt{2}}\begin{pmatrix} (\bar{e}^{-1})^{\bar{n}\mu} \\ (\bar{B}-\bar{e})_{\nu}{}^{\bar{n}} \end{pmatrix}.$$

- Here, $e_\mu{}^m$ and $\bar{e}_\nu{}^{\bar{n}}$ are two copies of the D-dimensional vielbein corresponding to the same spacetime metric,

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- Double-vielbein is of the following general form, Siegel, Hassan

$$V_{Am} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_m{}^{\mu} \\ (B+e)_{\nu m} \end{pmatrix}, \quad \bar{V}_A{}^{\bar{n}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})^{\bar{n}\mu} \\ (\bar{B}-\bar{e})_{\nu}{}^{\bar{n}} \end{pmatrix}.$$

  - Here, $e_\mu{}^m$ and $\bar{e}_\nu{}^{\bar{n}}$ are two copies of the D-dimensional vielbein corresponding to the same spacetime metric,

$$e_\mu{}^m e_{\nu m} = \bar{e}_\mu{}^{\bar{n}} \bar{e}_{\nu\bar{n}} = g_{\mu\nu}\,.$$

  - We set $B_{\mu m} = B_{\mu\nu}(e^{-1})_m{}^{\nu}$, $\bar{B}_{\mu\bar{n}} = B_{\mu\nu}(\bar{e}^{-1})_{\bar{n}}{}^{\nu}$, *etc.*

Double-vielbein

- We may identify $(B+e)_{\mu}{}^{m}$ and $(\bar{B}-\bar{e})_{\nu}{}^{\bar{n}}$ as two copies of the vielbein for the winding mode coordinate, $\tilde{x}_{\mu}$, since

$$(B+e)_{\mu}{}^{m}(B+e)_{\nu m}=(\bar{B}-\bar{e})_{\mu}{}^{\bar{n}}(\bar{B}-\bar{e})_{\nu\bar{n}}=(g-Bg^{-1}B)_{\mu\nu}\,.$$

Double vielbein

- Internal symmetry group is

$$\mathbf{SO}(1,D{-}1)\times\overline{\mathbf{SO}}(1,D{-}1)\,,$$

  of which the former and the latter rotates each unbarred and barred small Roman alphabet index.

- $\mathbf{O}(D,D)$ acts only on the capital indices.
- Hence, both V_{Am} and $\bar{V}_{A}{}^{\bar{n}}$ are $\mathbf{O}(D,D)$ vectors.

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Double vielbein

- Further, V_{Am} and $\bar{V}_{A}{}^{\bar{n}}$ are double-gauge covariant vectors,

$$\delta_X V_{Am} \equiv X^B \partial_B V_{Am} + 2\partial_{[A} X_{B]} V^B{}_m = \hat{\mathcal{L}}_X V_{Am} \,,$$

$$\delta_X \bar{V}_{A}{}^{\bar{n}} \equiv X^B \partial_B \bar{V}_{A}{}^{\bar{n}} + 2\partial_{[A} X_{B]} \bar{V}^{B\bar{n}} = \hat{\mathcal{L}}_X \bar{V}_{A}{}^{\bar{n}} \,.$$

Ability of the double-vielbein

- Double-vielbein can pull back the chiral and the anti-chiral $2D$ indices to the more familiar D-dimensional ones without losing any information, since it is an invertible process.
- We pull back the double-gauge covariant rank two-tensor to obtain,

$$S_{AB}V^{A}{}_{m}\bar{V}^{B}{}_{\bar{n}} = R_{m\bar{n}} + 2D_{m}D_{\bar{n}}\phi - \tfrac{1}{4}H_{m\mu\nu}H_{\bar{n}}{}^{\mu\nu} + (\partial^{\lambda}\phi)H_{\lambda m\bar{n}} - \tfrac{1}{2}\nabla^{\lambda}H_{\lambda m\bar{n}}\,.$$

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Another ability of the double-vielbein

- We may construct a rank four tensor:

$$R_{mnpq} + D_{(p}H_{q)mn} - \tfrac{1}{4}H_{mn}{}^{r}H_{pqr} - \tfrac{3}{4}H_{m[n}{}^{r}H_{pq]r}\ ,$$

which may provide a powerful tool to organize the higher derivative corrections to the effective action $\longrightarrow$ Vanhove's talk

Application to Yang-Mills

- We postulate a vector potential, $\mathcal{V}_A$, which is
  - $\mathbf{O}(D, D)$ and double-gauge covariant,
  - and transforms under non-Abelian gauge symmetry, $\mathbf{g} \in \mathbf{G}$,

$$\mathcal{V}_A \quad \longrightarrow \quad \mathbf{g}\mathcal{V}_A\mathbf{g}^{-1} - i(\partial_A\mathbf{g})\mathbf{g}^{-1} \,.$$

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Application to Yang-Mills

- The usual field strength,

$$F_{AB} = \partial_A \mathcal{V}_B - \partial_B \mathcal{V}_A - i\,[\mathcal{V}_A, \mathcal{V}_B]\ ,$$

is YM gauge covariant, but it is NOT double-gauge covariant,

$$\delta_X F_{AB} \neq \hat{\mathcal{L}}_X F_{AB}\,.$$

Application to Yang-Mills

- Instead, we consider with the semi-covariant derivative,

$$\mathcal{F}_{AB} := \nabla_A \mathcal{V}_B - \nabla_B \mathcal{V}_A - i\,[\mathcal{V}_A, \mathcal{V}_B] = F_{AB} - \Gamma^C{}_{AB} \mathcal{V}_C \,.$$

- While this is neither YM gauge nor double-gauge covariant,

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Application to Yang-Mills

That is to say, ${P_A}^C{\bar{P}_B}^D\mathcal{F}_{CD}$ is fully covariant with respect to

- $\mathbf{O}(D,D)$ T-duality
- Gauge symmetry
  - Double gauge = Diffeomorphism + one form gauge symmetry
  - Yang-Mills gauge

Yang-Mills action

- Our double field formulation of Yang-Mills action is

$$S_{\mathrm{YM}} = g_{\mathrm{YM}}^{-2} \int \mathrm{d}y^{2D}\ e^{-2d}\,\mathrm{Tr}\Big(P^{AB}\bar{P}^{CD}\mathcal{F}_{AC}\mathcal{F}_{BD}\Big)\ ,$$

- Manifestly, $O(D,D)$ T-duality, double-gauge and Yang-Mills gauge covariant.

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Yang-Mills in components

- Decompose the vector potential into chiral and anti-chiral ones,

$$\mathcal{V}_A = V_A^+ + V_A^- \,,$$

$$V_A^+ = P_A{}^B \mathcal{V}_B \,, \qquad V_A^- = \bar{P}_A{}^B \mathcal{V}_B \,.$$

- Their general forms are

$$V_A^+ = \tfrac{1}{2}\left(\begin{array}{c} A^{+\lambda} \\ (g+B)_{\mu\nu} A^{+\nu} \end{array}\right) , \qquad V_A^- = \tfrac{1}{2}\left(\begin{array}{c} -A^{-\lambda} \\ (g-B)_{\mu\nu} A^{-\nu} \end{array}\right) .$$

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- With the field redefinition,

$$A_\mu := \frac{1}{2}(A^+_\mu + A^-_\mu)\,, \qquad \phi_\mu := \frac{1}{2}(A^+_\mu - A^-_\mu)\,,$$

 we get

$$\mathcal{V}_A = \begin{pmatrix} \phi^\lambda \\ A_\mu + B_{\mu\nu}\phi^\nu \end{pmatrix}.$$

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- Turning off the $\tilde{x}$-dependence reduces the action to

$$S_{\mathrm{YM}} \equiv g_{\mathrm{YM}}^{-2} \int \mathrm{d}x^D \; \sqrt{-g} e^{-2\phi} \, \mathrm{Tr}\Big(-\tfrac{1}{4}\hat{f}^{\mu\nu}\hat{f}_{\mu\nu}\Big) \; ,$$

where

$$\hat{f}_{\mu\nu} := f_{\mu\nu} - D_\mu\phi_\nu - D_\nu\phi_\mu + i\,[\phi_\mu, \phi_\nu] + H_{\mu\nu\lambda}\phi^\lambda \, ,$$

and

$$\mathrm{Tr}\Big(\hat{f}_{\mu\nu}\hat{f}^{\mu\nu}\Big) = \mathrm{Tr}\Big(f_{\mu\nu}f^{\mu\nu} + 2D_\mu\phi_\nu D^\mu\phi^\nu + 2D_\mu\phi_\nu D^\nu\phi^\mu - [\phi_\mu, \phi_\nu][\phi^\mu, \phi^\nu]$$
$$+2i\,f_{\mu\nu}[\phi^\mu, \phi^\nu] + 2\,(f^{\mu\nu} + i[\phi^\mu, \phi^\nu])\,H_{\mu\nu\sigma}\phi^\sigma + H_{\mu\nu\sigma}H^{\mu\nu}{}_\tau\phi^\sigma\phi^\tau\Big) \, .$$

- Similar to topologically twisted Yang-Mills, but differs in detail.
- Curved D-branes are known to convert adjoint scalars into one-form,

$$\phi^a \longrightarrow \phi_\mu , \qquad \text{Bershadsky}$$

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Concluding remarks

- **O**(D, D) T-duality, diffeomorphism, one-form gauge symmetry fixes the low energy effective action,

$$S_{\text{eff.}} = \int \mathrm{d}x^D \; e^{-2d}\,\mathcal{H}^{AB} S_{AB}\,.$$

- Supersymmetrization, Higher derivative corrections – *in progress*.
- Application to 'doubled sigma model' and generalization to $\mathcal{M}$-theory are of interest Ivanov, Hull, Berman, Perry, Bergshoeff
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Thank you.