Analytical approach to 2d holographic superconductor

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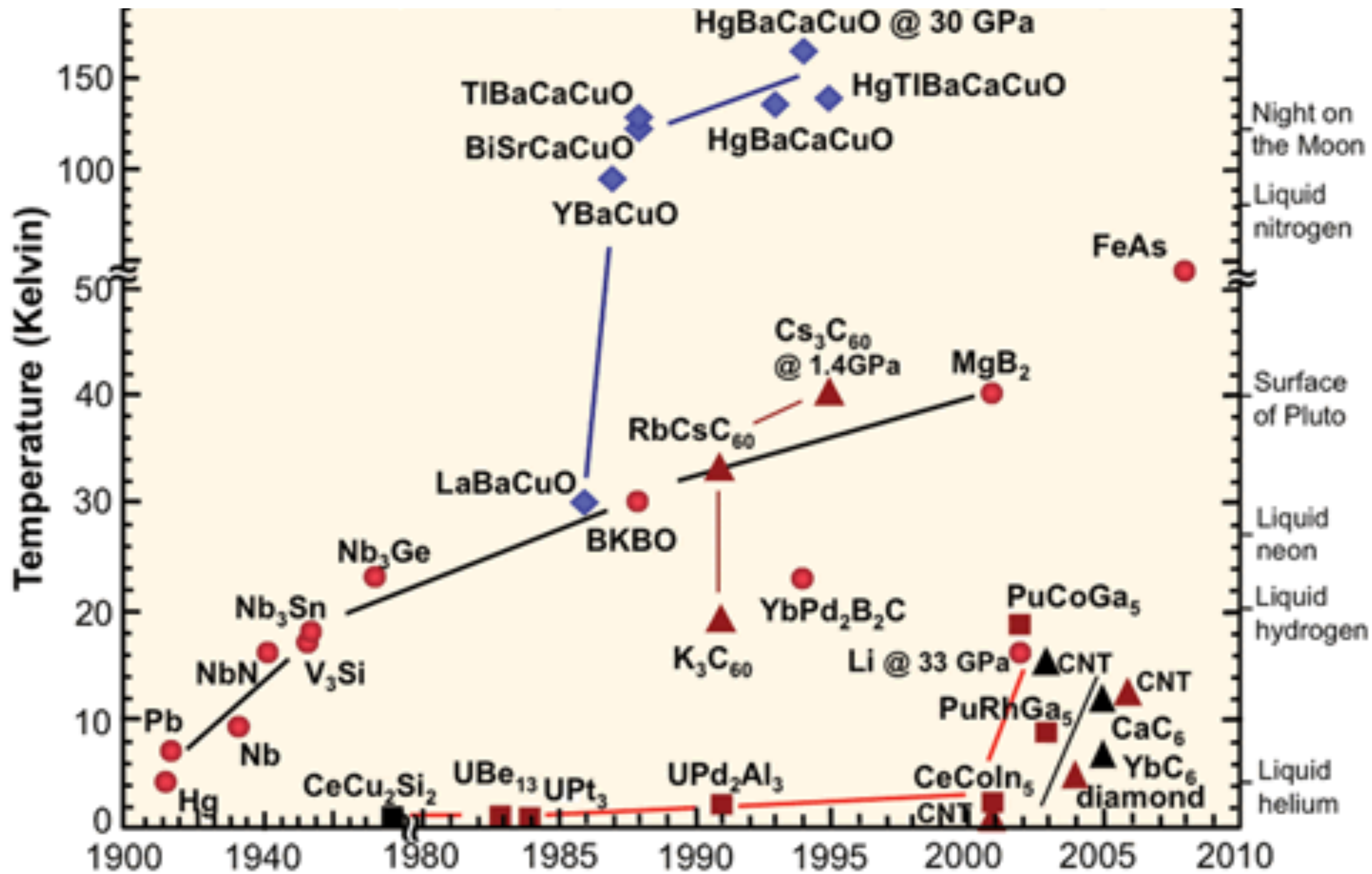
Plan of the talk

- Introducing conventional and non-conventional Superconductors
- Basic ingredients for a holographic Superconductor
- AdS3/CMT2 in the probe limit
- Phase transitions in non-rotating and rotating BTZ Black Hole backgrounds
- Summary and Concluding Remarks

Conventional and non-conventional Superconductors

Brief history of Superconductivity



1911: Conventional Superconductors (e^--ph int.); (B)BCS theory (1957). Magnetism plays a negative role. 1986: HTSC (exotic SCs - cuprates). Superconductivity is provided by magnetic properties of samples. 2008: Fe-based (ferrates LaFeAs(O,F)) SCs. Magnetism is important 1977: ■ Heavy Fermions.

Superconductivity for experimenters

Matthias' rules of looking for new SCs

- High symmetry is good, cubic is best. Sample must have d-electrons (not just s, p nor f). Peaks in DOS are good; certain e^- concentrations are favored.
- Stay away from oxygen, magnetism and insulators!
- Stay away from theorists!

Rules are good for conventional SCs, but aren't good for High-T SCs.

Superconductivity for theorists

Conventional superconductors are described by BCSB (Bardeen-Cooper-Schrieffer-Bogoliubov) theory.

Main ingredients of BCSB are:

- (second order) phase transition (described by the Landau-Ginsburg phenomenological theory)
- the electron Cooper pairs condensate formation
- forming an energy gap

Roughly speaking, the main equation of BCSB theory is the gap equation.

The effective coupling in BCSB is the electron-phonon coupling $\longrightarrow$ a weak coupling constant theory.

HTSCs must be formulated as theories in the strong coupling constant regime.

How to formulate? – Try the AdS/CFT

Basic ingredients for a holographic Superconductor

AdS/CFT on a nutshell

The main principle of the AdS/CFT correspondence
(Maldacena'98; Gubser, Klebanov, Polyakov'98; Witten'98)

Gauge theory on the AdS_{d+1} boundary and at a strong coupling is dual to the bulk gravity with matter at a weak coupling

The Correspondence Dictionary

- Fields in AdS ⟶ local CFT operators
- Spin ⟶ Spin
- Mass ⟶ Scaling dimension Δ
- Gauge fields in AdS ⟶ Boundary currents

$$\Delta(\Delta - d) = m^2 L^2 \quad \text{for Scalar field}$$
$$\Delta(\Delta - d + 2) = m^2 L^2 \quad \text{for Vector field}$$

AdS/CFT on a nutshell

How to define a local CFT operator? Try the asymptotic expansion of a bulk field near the boundary:

$$\Theta(z) = \mathcal{A} z^{\Delta_-} (1 + \dots) + \mathcal{B} z^{\Delta_+} (1 + \dots)$$

Δ is the scaling dimension of the field; Δ(Δ - …)=m^2L^2

$\mathcal{A}$ is the source to the boundary operator $\mathcal{O}$

$\mathcal{B}$ is the expectation value <$\mathcal{O}$>

For a massless vector field with just one temporal component

$$A_t(z \rightsquigarrow 0) = \mathcal{A} z^{(d-2)}(1 + \dots) + \mathcal{B}(1 + \dots)$$

z-dependent part has a fast falloff, hence it's not a background field in the dual theory. Hence $\mathcal{A}$ fixes the electric charge density of the state. The finite part is a chemical potential for the electric charge density. Then *d*=2 one has to use $z^\epsilon = 1 + \epsilon \ln z + \dots, \quad \epsilon = (d-2)$ with rescaling the chemical potential $\mu \rightsquigarrow -\mu/\epsilon$

Basic ingredients for a HolographicSuperConductor

- Take AdS bulk for gravity with matter, and its flat boundary for a gauge CFT
- Put a non-extremal (charged) Black Hole in the bulk
- Take at least Maxwell-scalar interacting fields in the bulk with a large enough charge
- Stay away from experimenters and don’t analyse data!

Then, you’ll get the phase transition on the boundary at some T_c and an energy gap. The charged scalar field will form a BH scalar hair and will condense at the boundary. This condensate simulates the electron Cooper pairs.

(Hertog’06; Gubser’08;Hartnoll, Herzog, Horowitz’08)

AdS_3/CMT_2 in the probe limit

AdS_3/CMT_2

Let's focus on AdS_3/CMT_2 correspondence. It is interesting because

- the case includes main ingredients of holographic superconductors
- technically more simple
- less studied
- has a relation to real systems like superconducting nanowires
- has an interesting symmetry structure (pure 2d CFT with infinitely dimensional Conf. symmetry, entropy relation to the algebra central charge etc.)
- though gravity is not dynamical in the bulk, there are AdS Black Holes (Banados, Teitelboim, Zanelli'92; Clement'93)

We are going to study this case analitically, i.e. without using (or rather using in minimal) numerical methods.

AdS_3/CMT_2 setup

In the probe limit, without backreaction on the metric the system is described by the action

$$S = -\int d^3x\,\sqrt{-g}\left(\frac{1}{4}F_{mn}F^{mn} + (\partial_m - iA_m)\Psi(\partial^m + iA^m)\Psi^* + m^2\Psi\Psi^*\right)$$

in the neutral AdS BH background

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + dx^2 + \frac{dz^2}{f(z)}\right), \quad f(z) = 1 - \frac{z^2}{z_H^2}$$

L is a length of AdS, the boundary is located at $z=0$, the BH horizon is at $z=z_H$. This metric solves the Einstein equation

$$R_{mn} - \frac{1}{2}g_{mn}\left(R + \frac{2}{L^2}\right) = 0$$

and the BH temperature is

$$T = \frac{1}{2\pi z_H}$$

AdS_3/CMT_2 setup

The equations of motion of fields in the bulk are

$$\frac{1}{\sqrt{-g}} D_m(\sqrt{-g} D^m \Psi) - m^2 \Psi = 0,$$

$$\partial_n(\sqrt{-g} F^{nm}) + \sqrt{-g} i(\Psi \overline{D^m} \Psi^* - \Psi^* D^m \Psi) = 0$$

$$D\Psi \equiv (\partial_m - iA_m)\Psi, \;\; \overline{D^m}\Psi^* = (D\Psi)^*, \;\; F_{mn} = 2\partial_{[m}A_{n]}$$

We are going to solve EOM with the ansatz

$$\Psi = \psi(z), \qquad A = \phi(z) dt$$

On account of the latter and of the BH metric, EOM becomes

$$\psi'' + \left(\frac{f'}{f} - \frac{1}{z}\right)\psi' + \left(\frac{\phi^2}{f^2} - \frac{m^2 L^2}{z^2 f}\right)\psi = 0,$$

$$\phi'' + \frac{\phi'}{z} - \frac{L^2}{z^2}\frac{2\psi^2}{f}\phi = 0$$

AdS_3/CMT_2 setup

So, we've got the following set of EOM

$$\psi'' + \left(\frac{f'}{f} - \frac{1}{z}\right)\psi' + \left(\frac{\phi^2}{f^2} - \frac{m^2L^2}{z^2 f}\right)\psi = 0,$$

$$\phi'' + \frac{\phi'}{z} - \frac{L^2}{z^2}\frac{2\psi^2}{f}\phi = 0$$

It can be solved numerically

(Ren'10; Liu, Pan, Wang'11),

but we will try it analitically

(like in Gregory, Kanno, Soda'09; Herzog'10; Bellon, Moreno, Schaposnik'10; Chen, Wu'11; Ge'11; Siopsis, Therrien'10)

BoundaryConditions should be specified to this end. At $z=z_H$ we get

$$\phi(z_H) = 0, \quad \psi'(z_H) = \psi(z_H)/2z_H$$

and BCs at the AdS boundary are

$$\phi = \mu \ln z - \rho, \quad \psi = \psi^{(2)} z \rightsquigarrow \langle \mathcal{O} \rangle_\psi = \psi^{(2)}$$

Phase transitions in non-rotating and rotating BTZ BH backgrounds

Phase transitions in BTZ backgrounds

The idea on solving the EOM analytically is quite simple (Gregory, Kanno, Soda'09)

- take the fields series expansions

$$\phi(z) = \phi(Z) + \phi'(Z)(z - Z) + \frac{1}{2}\phi''(Z)(z - Z)^2 + \ldots,$$

$$\psi(z) = \psi(Z) + \psi'(Z)(z - Z) + \frac{1}{2}\psi''(Z)(z - Z)^2 + \ldots$$

  near the horizon ($Z=z_H$) and at the boundary ($Z=0$);
- take the BCs into account;
- evaluate a few first coefficients in the fields series expansions near the boundary and the horizon by use of the BCs and the EOM;
- sew the so obtained fields series expansions, and their first derivatives, at an intermediate point *z*;
- get the result. Fine!

Phase transitions in BTZ backgrounds

Following the described procedure and setting the sew point at $z=1/2z_H$, we get (AJN'11)

$$T_c = \frac{2}{\pi\sqrt{123}} \cdot \mu \approx 0.057 \cdot \mu$$

for the critical temperature of the phase transition, and ($L=1$)

$$\langle \mathcal{O} \rangle_\psi \approx 12.7\,\sqrt{T T_c}\,(1 - T/T_c)^{1/2} \quad \xrightarrow{T \rightsquigarrow T_c} \quad \langle \mathcal{O} \rangle_\psi \approx 12.7\,T_c\,(1 - T/T_c)^{1/2}$$

The critical exponent and the temperature dependence of the scalar EV is typical for the second order phase transitions occurred in superconductors. The numerical coefficient is in a good agreement with that obtained in the numerical studies. There we get

$$\langle \mathcal{O} \rangle_\psi \approx 12.2\,T_c\,(1 - T/T_c)^{1/2} \qquad T_c/\mu \approx 0.136$$

The discrepancy in T_c is typical for this type of analytical calculations, and may be slightly improved by choosing the appropriate sewing point.

Phase transitions in BTZ backgrounds

Let's try to extend the standard setup modifying fields in the bulk, or the background in which they propagate.

A natural modification for AdS_3 Maxwell field consists in adding the Chern-Simons topological term, and to make ED3 massive, i.e.

$$S = S_0(A, \Psi, \partial A, D\Psi) + \frac{\theta}{2}\int A \wedge dA$$

However, such a modification is non-trivial once new magnetic DOF appear in the ansatz for *A*

$$\Psi = \psi(z), \quad A = \phi(z)dt + B(z)dz$$

and it's OK in the probe limit, but it can't be realized in the complete setup with the backreaction on the metric. The reason is all the magnetic BTZ-type solutions in EMS(CS) systems are horizonless

(Clement'95; Hirschmann, Welch'95; Cataldo, Salgado'96; Moussa, Clement'96; Fernando, Mansouri'97;…)

Phase transitions in BTZ backgrounds

But inclusion of external magnetic field can be realized in another way, with taking into account

- the Barnett effect of magnetization of uncharged, but rotated body, and the London magnetic moment, which appears upon rotating a superconductor

  (Barnett'15; London'50)

- the Lense-Thirring dragging force effect which guarantees the AdS boundary rotation in the background of rotating BTZ BH (Lense, Thirring'18)

Hence, we have to put our MS interacting system in the background of a rotating BTZ BH

$$ds^2 = \frac{L^2}{z^2}\left[-f(z)dt^2 + \frac{dz^2}{f(z)} + L^2\left(d\varphi - \frac{Jz^2}{2L^4}dt\right)^2\right]$$

$$f(z) = 1 - \frac{Mz^2}{L^2} + \frac{J^2z^4}{4L^6}$$

Phase transitions in BTZ backgrounds

Now EOM

$$\frac{1}{\sqrt{-g}}D_m(\sqrt{-g}D^m\Psi) - m^2\Psi = 0,$$

$$\partial_n(\sqrt{-g}F^{nm}) + \sqrt{-g}i(\Psi\overline{D^m}\Psi^* - \Psi^* D^m\Psi) = 0$$

have to be solved with the following ansatz

$$\Psi = \Psi(z,\tilde{\varphi}), \quad A = \phi(z)dt + \xi(z)d\tilde{\varphi}, \quad \tilde{\varphi} = L\varphi$$

and in the background

$$ds^2 = \frac{L^2}{z^2}\left[\left(-f(z) + \frac{J^2z^4}{4L^6}\right)dt^2 + \frac{dz^2}{f(z)} + d\tilde{\varphi}^2 - 2\frac{Jz^2}{2L^3}dtd\tilde{\varphi}\right]$$

The system doesn't look simple, so we need a simplification. The small angular momentum approximation, in which

$$\Psi = \Psi(z,\tilde{\varphi}), \quad A = \phi(z)dt + \xi(z)d\tilde{\varphi}, \quad J \ll 1, \quad \xi(z) \ll \phi(z)$$

makes the problem more tractable.

Phase transitions in BTZ backgrounds

One may wonder on legality of the approximation. But

- BH with large *J* are very unstable due to the super-radiance scattering effect

  (Zeldovich'71; Bardeen, Press, Teukolsky'72; Starobinsky'73)
- the magnetic component of *A* in magnetic rotating BH solutions is related to the temporal component as

$$A_{\tilde{\varphi}} \sim \omega A_t \ \rightsquigarrow \ A_{\tilde{\varphi}} \ll A_t \text{ iff } \omega \ll 1$$

The system of EOM comes in the limit to

$$\psi'' + \left(\frac{f'}{f} - \frac{1}{z}\right)\psi' + \left(\frac{\phi^2}{f^2} - \frac{m^2L^2}{z^2 f} - \frac{z}{Lf}\lambda\right)\psi \approx 0,$$

$$\phi'' + \frac{1}{z}\left(1 + \frac{Jz^3}{L^3}\right)\phi' - \frac{L^2}{z^2}\frac{2\psi^2 e^{2i\alpha\tilde{\varphi}}}{f}\phi \approx 0$$

with the separation constant

$$\lambda \approx \frac{L\alpha^2}{z} + \frac{\alpha J z}{2L^2 f(z)}\phi(z)$$

coming from the solution to the "angular" part of the scalar EOM

Phase transitions in BTZ backgrounds

Doing the same machinery of analytical calculations as in the non-rotating case, we get the following T dependence of the CFT scalar operator EV

$$\langle\mathcal{O}\rangle_\psi \approx 16.68\,T\left(1-\frac{\sqrt{123}}{4\mu}\left[2\pi T+\frac{17\alpha^2}{164\pi T}-\frac{\alpha\mathcal{J}_R}{8\pi^2T^2L^3}\right]\right)^{1/2}$$

Here $\mathcal{J}_R$ is the renormalized angular momentum (see AJN'11 for details), and we are within the approximation

$$\alpha\mathcal{J}_R \ll 1, \quad \alpha \ll 1$$

What we expected to get when the BH becomes rotating? In the background of the rotating BTZ BH the "radial" part of the scalar equation is

$$\partial_m(\sqrt{-g}g^{mn}\partial_n\Psi)+f(A,\partial)\Psi-V(\Psi)=0$$

with some operator $f(A,\partial)$ and an effective potential

$$V(\Psi)=\sqrt{-g}\left[m^2+A_tg^{tt}A_t+A_{\tilde\varphi}g^{\tilde\varphi\tilde\varphi}A_{\tilde\varphi}\right]$$

Phase transitions in BTZ backgrounds

When there are magnetic DOF in *A* the condensation becomes hard in compare to the pure electric case $A_{\tilde{\varphi}} = 0$

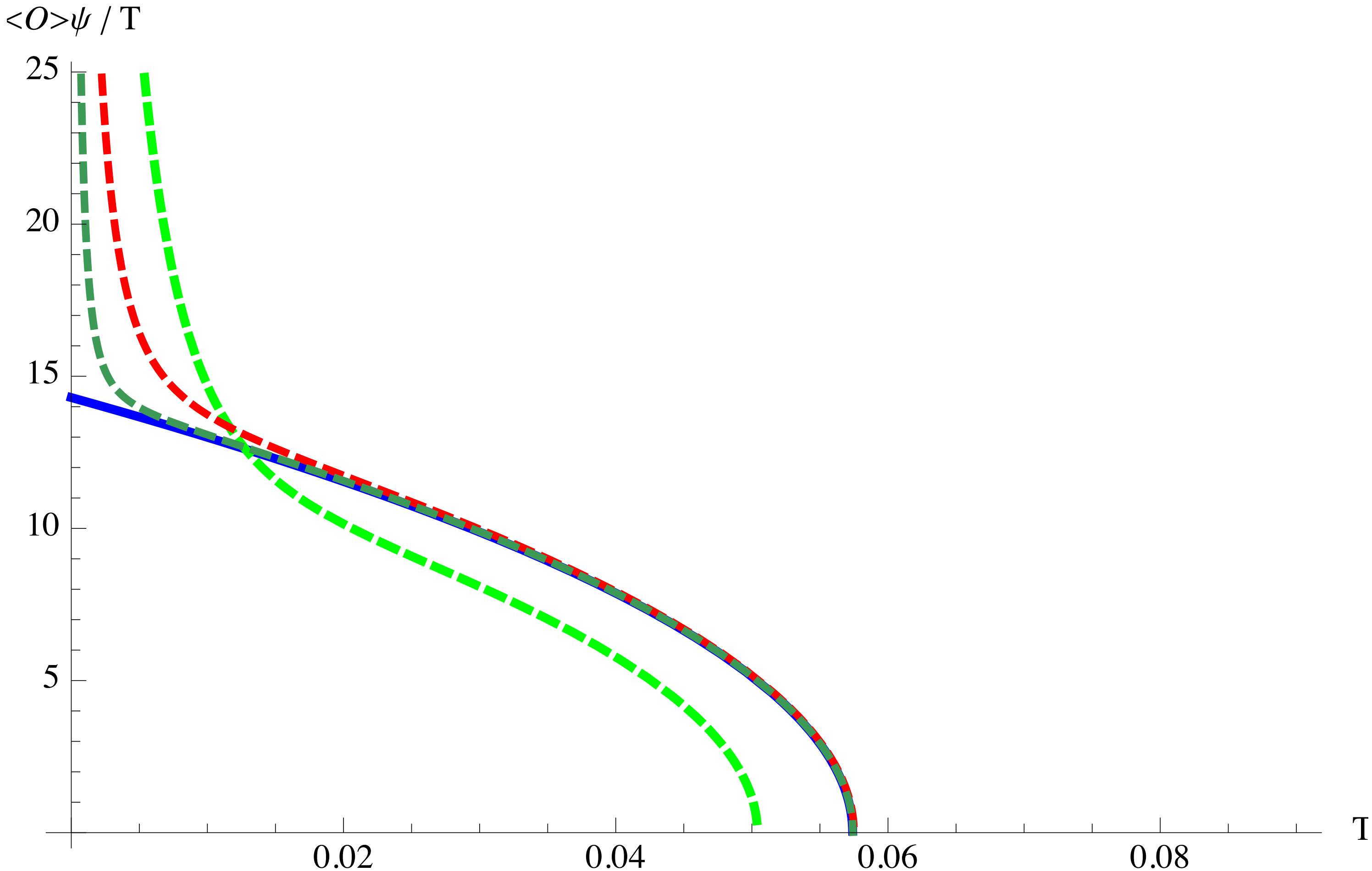
$$V(\Psi) = \sqrt{-g}\left[m^2 + A_t g^{tt} A_t + A_{\tilde{\varphi}} g^{\tilde{\varphi}\tilde{\varphi}} A_{\tilde{\varphi}}\right]$$

when $g^{tt} < 0$, and the effective mass decreases. In the electro-magnetic case, due to $g^{\tilde{\varphi}\tilde{\varphi}} > 0$, the effective mass gets decreased smaller, making the condensation hard.

Though we have not magnetic DOF in the ansatz, external magnetic field is modeled by the rotation. Hence we expect the critical temperature of the phase transition will become lower in compare to the non-rotating case.

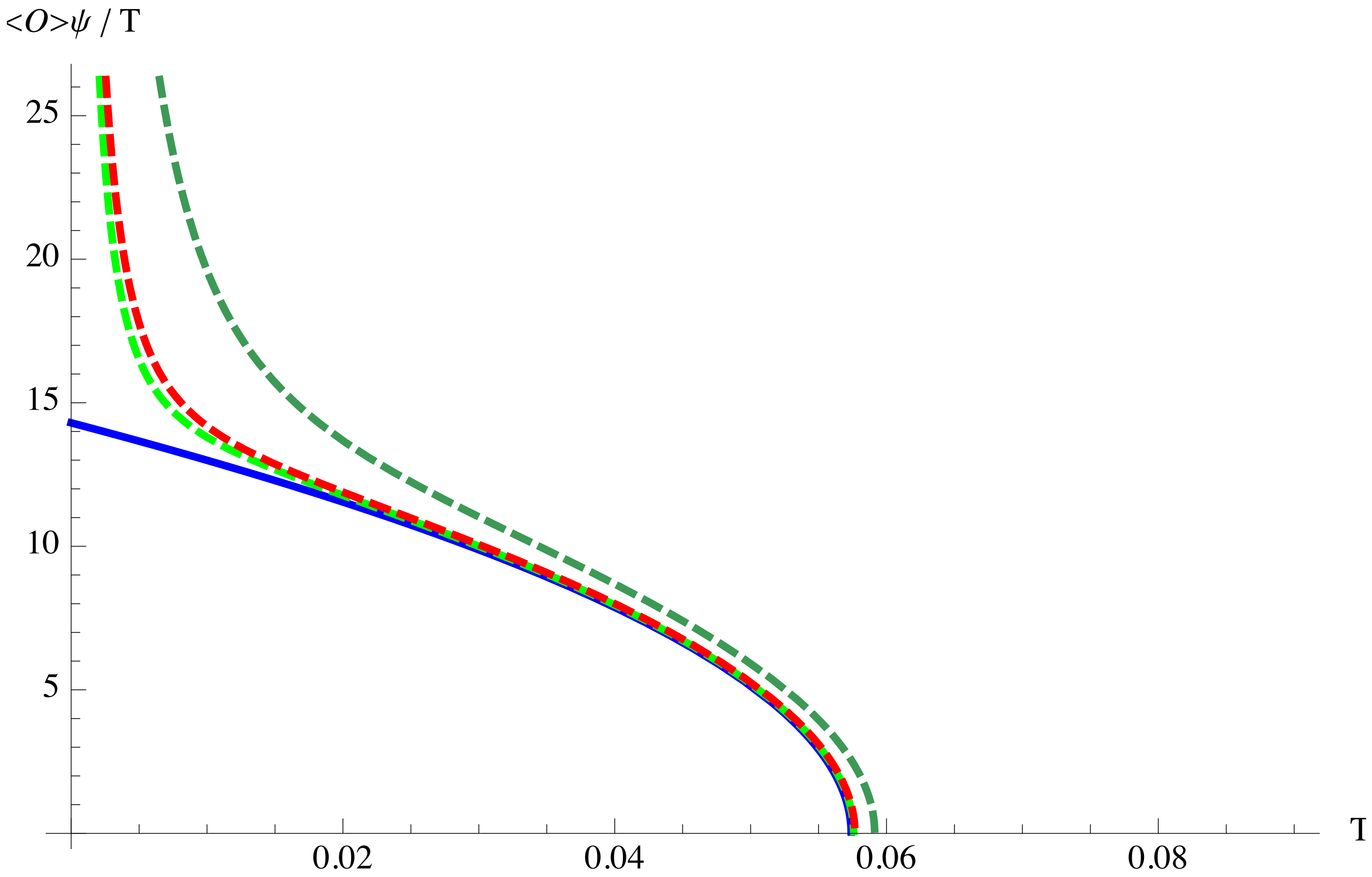
To check it, let's put the scalar operator EV on the plot.

Phase transitions in BTZ backgrounds



Plot for different (α , J_R): (0.0,0.0) blue curve, (0.003,0.01) deep green, (0.03,0.01) red, (0.3,0.01) green

Phase transitions in BTZ backgrounds



Plot for different (α, J_R): (0.0,0.0) blue curve, (0.003,0.01) green, (0.003,0.15) red, (0.003,1.0) deep green

Summary and Concluding Remarks

Summary and Concluding Remarks

- We have filled the gap in applying the analytical methods to a 2D holographic superconductor and in the probe limit.
- We have found a good agreement between the boundary scalar operator EV in analytical and numerical approaches, but the value of the critical temperature, estimated analytically, is about twice less than that reproduced in the numerical calculations. This discrepancy is typical within the approach we followed, as it comes from Table 1 in Chen&Wu arXiv:1103.5130[hep-th]. It can be slightly improved by the appropriate choice of the sewing point, however the coefficient in the scalar CFT operator will be changed.
- We have observed that in dependence on the choice of free parameters of the model (α, J_R) one may encounters as “normal” lowering of the critical temperature due to the BH rotating, as well as “abnormal” T_c increasing. It would be interesting to reproduce this effect in numerical simulations of the complete problem with backreaction, and out of the small angular momentum approximation.