New integrable systems via action-angle variables

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Content

- Action-angle variables
- Generalizations of the Oscillator and Coulomb systems
- Application: Tremblay-Turbiner-Winternitz (TTW) system and its generalizations
- Conformal Mechanics
- Particle moving near the horizon of extreme Reissner-Nordstrom and Kerr black holes

Action-angle variables

Integarbility: $\exists\ F_1 = H, \ldots, F_N$: $\{F_\mu, F_\nu\} = 0$, $\mu, \nu = 1, \ldots N$.
If $M_F = (\mathbf{p}, \mathbf{q} : \mathbf{F} = \mathbf{c} = const)$ is a *compact and connective*, then $\exists\ \mathbf{\Phi} = (\Phi_1, \ldots, \Phi_N) \in S^1 \times \ldots \times S^1$, $\mathbf{I}(\mathbf{F}) = (I_1, \ldots, I_N)$.

$$\frac{d\mathbf{I}}{dt} = 0, \quad \frac{d\mathbf{\Phi}}{dt} = \frac{\partial H(\mathbf{I})}{\partial \mathbf{I}}, \qquad \{I_i, \Phi_j\} = \delta_{ij}.$$

Gauging the integrable system by action-angle variables, we preserve the freedom only in two points:

- in the functional dependence of the Hamiltonian from the action variables, $H = H(\mathbf{I})$;
- in the range of validity of the action variables, $\mathbf{I} \in [\beta^-, \beta^+]$.

Quantization

$$\widehat{\mathbf{I}}\Psi(\Phi) = \mathbf{I}\Psi(\Phi), \qquad \widehat{\mathbf{I}} = -\imath\hbar\frac{\partial}{\partial\boldsymbol{\Phi}}, \quad \Psi = \frac{1}{(2\pi)^{N/2}}\mathrm{e}^{\imath\,\mathbf{I}\boldsymbol{\Phi}}$$

$$I_\mu = \hbar(n_\mu + \frac{1}{2}), \qquad n_\mu \in [\beta_\mu^-, \beta_\mu^+]$$

where n_μ are integer numbers

How to construct the action-angle variables?

$$S(\mathbf{c}, \mathbf{q}) = \int_{\mathbf{F}=\mathbf{c}} \mathbf{p} d\mathbf{q},$$

where $\mathbf{p} = \mathbf{p}(\mathbf{c}, \mathbf{q})$
The Action Variables

$$I_\mu(\mathbf{c}) = \frac{1}{2\pi} \oint_{\gamma_\mu} \mathbf{p} d\mathbf{q},$$

where γ_i is some loop of the level surface $\mathbf{F} = \mathbf{c}$.
The angle variables

$$\mathbf{\Phi} = \frac{\partial S(\mathbf{c}(\mathbf{I}), \mathbf{q})}{\partial \mathbf{I}}$$

General N-dimensional Models

$$\mathrm{I\!R}^N: \; H = \frac{p_r^2}{2} + \frac{\mathcal{H}_{N-1}(I_i)}{r^2} + V(r), \quad V(r) = \left\{ \begin{array}{c} \omega^2 r^2/2 \\ -\gamma/r \end{array} \right.$$

$$S^N: \; H = \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{H}_{N-1}}{r_0^2 \sin^2\chi} + V(\tan\chi), \quad V(r) = \left\{ \begin{array}{c} r_0^2\omega^2 \tan^2\chi/2 \\ -(\gamma\cot\chi)/r_0 \end{array} \right.$$

$$H^N: \quad H = \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{H}_{N-1}}{r_0^2 \sinh^2\chi} + V(\tanh\chi), \quad V(r) = \left\{ \begin{array}{c} r_0^2\omega^2 \tanh^2\chi/2 \\ -(\gamma\coth\chi/r_0 \end{array} \right.$$

$\mathbb{R}^{\mathrm{N}}$ **Action variables**

$$\mathbf{I} = \left(I_r = \frac{1}{\sqrt{2}\pi} \oint dr \sqrt{E - \frac{H_{N-1}(I)}{r^2} - V(r)},\ I_i \right).$$

$\mathbb{R}^{\mathrm{N}}$ **Angle variables**

$$\Phi_r = \frac{1}{\sqrt{2}} \frac{\partial E}{\partial I_r} \int \frac{dr}{\sqrt{E - \frac{\mathcal{H}_{N-1}(I)}{r^2} - V(r)}},$$

$$\Phi_i = \Phi_i^0 + \frac{\partial E/\partial I_i}{\partial E/\partial I_r} \Phi_r - \frac{1}{\sqrt{2}} \frac{\partial H_{N-1}}{\partial I_i} \int \frac{dr}{r^2 \sqrt{E - \frac{\mathcal{H}_{N-1}(I)}{r^2} - V(r)}}.$$

Sphere: $r \to r_0 \sin \chi$; Pseudosphere: $r \to r_0 \sinh \chi$

$\mathbb{R}^N$ **Oscillator**

$$H_{osc} = \omega \left(2I_r + \sqrt{2\mathcal{H}(I_i)}\right).$$

$$\Phi_r = -\arcsin \frac{E - r^2\omega^2}{\sqrt{E^2 - 2\mathcal{H}\omega^2}},$$

$$\Phi_i = \Phi_i^0 + \frac{1}{2\sqrt{2H}} \frac{\partial H}{\partial I_i} \left[\Phi_r - \arcsin \frac{Er^2 - 2\mathcal{H}}{r^2\sqrt{E^2 - 2\omega^2\mathcal{H}}}\right]$$

$\mathbb{R}^N$ **Coulomb**

$$H_{Coulomb} = -\frac{\gamma^2}{2\left(I_r + \sqrt{2\mathcal{H}(I_i)}\right)^2}$$

$$\Phi_r = -\frac{2}{\gamma}\sqrt{E\mathcal{H} - Er(Er + \gamma)} - \arcsin \frac{2Er + \gamma}{\sqrt{4E\mathcal{H} + \gamma^2}}$$

$$\Phi_i = \Phi_i^0 + \sqrt{\frac{2}{\mathcal{H}}}\frac{\partial \mathcal{H}}{\partial I_i} \left[\Phi_r - \frac{1}{2}\arcsin \frac{\gamma r - 2\mathcal{H}}{r\sqrt{4E\mathcal{H} + \gamma^2}}\right]$$

S^N Oscillator

$$H_{Higgs} = \frac{1}{2}\left(2I_\chi + \sqrt{2\mathcal{H}} + \omega\right)^2 - \frac{\omega^2}{2}$$

S^N Coulomb

$$H_{Sch-C} = \frac{1}{2}\left(I_\chi + \sqrt{2\mathcal{H}}\right)^2 - \frac{\gamma^2}{2\left(I_\chi + \sqrt{2\mathcal{H}}\right)^2}$$

H^N **Oscillator**

$$H_{ps} = \frac{\omega^2}{2} - \frac{1}{2}(2I_\chi + \sqrt{2\mathcal{H}} - \omega)^2$$

H^N **Coulomb**

$$H_{Sch-C} = -\frac{1}{2}\left(I_\chi + \sqrt{2\mathcal{H}}\right)^2 - \frac{\gamma^2}{2\left(I_\chi + \sqrt{2\mathcal{H}}\right)^2}.$$

Original Tremblay-Turbiner-Winternitz (TTW) superintegrable system (2009)

$$H_2 = \frac{p_r^2}{2} + \frac{\mathcal{H}_{PT}}{r^2} + \frac{\omega^2 r^2}{2}, \quad \{p_r, r\} = 1,$$

where H_{PT} is the Hamiltonian of Pöshle-Teller(PT) system

$$\mathcal{H}_{PT} = \frac{p_\varphi^2}{2} + \frac{k^2\alpha_1^2}{2\sin^2 k\varphi} + \frac{2k^2\alpha_1^2}{\cos^2 k\varphi}, \qquad k \in \mathcal{N}$$

Action-angle variables for PT (Lechtenfeld,A.N.,Yeghikyan'10)

$$\mathcal{H}_{PT} = \frac{k^2(I + \alpha_1 + \alpha_2)^2}{2}, \quad \Phi_{PT} = \tfrac{1}{2}\arcsin\Big\{\tfrac{1}{a}\big[\cos 2k\varphi + b\big]\Big\},$$

where

$$a = \sqrt{1 - \tfrac{k^2(\alpha_1^2+\alpha_2^2)}{H_{PT}} + \Big(\tfrac{k^2(\alpha_1^2-\alpha_2^2)}{2H_{PT}}\Big)^2},\ b = \tfrac{k^2(\alpha_2^2-\alpha_1^2)}{2H_{PT}}.$$

Generalized TTW systems

$$\mathbb{R}^2: \ \mathrm{H} = \frac{\mathrm{p_r^2}}{2} + \frac{\mathcal{H}_{\mathrm{PT}}}{\mathrm{r}^2} + \mathrm{V(r)}, \quad \mathrm{V(r)} = \begin{cases} \omega^2 r^2/2 \\ -\gamma/r \end{cases}$$

$$S^2: \ H = \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{H}_{PT}}{r_0^2 \sin^2\chi} + V(\tan\chi), \quad V(r) = \begin{cases} r_0^2\omega^2\tan^2\chi/2 \\ -(\gamma\cot\chi)/r_0 \end{cases}$$

$$H^2: \quad H = \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{H}_{PT}}{r_0^2 \sinh^2\chi} + V(\tanh\chi), \quad V(r) = \begin{cases} r_0^2\omega^2\tanh^2\chi/2 \\ -(\gamma\coth\chi)/r_0 \end{cases}$$

Proof of superintegrability. Oscillator case

$$\mathcal{H}_{osc} = \mathcal{H}_{osc}(2I_r + k\tilde{I}_{PT}) = \begin{cases} \omega(2I_r + k\tilde{I}_{PT}) & \text{for} \quad \mathbb{R}^2 \\ \frac{1}{2}(2I_\chi + k\tilde{I}_{PT} + \omega)^2 - \frac{\omega^2}{2} & \text{for} \quad S^2 \\ -\frac{1}{2}(2I_\chi + k\tilde{I}_{PT} - \omega)^2 + \frac{\omega^2}{2} & \text{for} \quad H^2 \end{cases}$$

$$\frac{d\Phi_r}{dt} = 2\omega_{osc}(2I_r + k\tilde{I}_{PT}), \quad \frac{d\Phi_\varphi}{dt} = k\omega_{osc}(2I_r + k\tilde{I}_{PT}), \quad \omega_{osc}(x) = \frac{d\mathcal{H}_{osc}(x}{dx}$$

Hence, for $k = m/n$, where m, n are integer number the trajectories are closed.

Additional constant of motion

$$I_{add} = \cos(n\Phi_r - 2m\Phi_\varphi).$$

Superintegrability. Coulomb case

$$\mathcal{H}_C = \mathcal{H}_C(I_r + k\tilde{I}_{PT})$$

Equations of motion

$$\frac{d\Phi_r}{dt} = 2\omega_C(I_r + k\tilde{I}_{PT}), \ \frac{d\Phi_\varphi}{dt} = k\omega_C(I_r + k\tilde{I}_{PT}), \quad \omega_C(x) = \frac{d\mathcal{H}_C(x)}{dx}.$$

For $k = m/n$ the trajectories are closed.

Additional constant of motion

$$I_{add} = \cos(n\Phi_r - m\Phi_\varphi).$$

Conformal mechanics

$$\{H, D\} = 2H, \qquad \{K, D\} = -2K, \qquad \{H, K\} = D$$

"Radial" Coordinates

$$r \equiv \sqrt{2K}, \quad p_r \equiv \frac{D}{\sqrt{2K}} : \ \{p_r, r\} = 1,$$

$$H = \frac{p_r^2}{2} + \frac{\mathcal{H}_{N-1}}{r^2}, \qquad D = rp_r \qquad K = \frac{r^2}{2}.$$

where

$$\mathcal{H}_{N-1} \equiv \mathcal{I} = 4KH - D^2 \quad : \{p_r, \mathcal{I}\} = \{r, \mathcal{I}\} = \{H, \mathcal{I}\} = 0$$

Particle near extreme black hole horizon

$$H = r\left(\sqrt{(rp_r)^2 + L(\theta, p_\theta, p_\varphi)} - q(p_\varphi)\right),$$

$$\mathcal{I} = \frac{1}{2}\left(L(p_\theta, \theta, p_\varphi) - q(p_\varphi)\right) = \mathcal{I}(I_a)$$

"Canonical" formulation

$$H = \frac{P_X^2}{2} + \frac{\mathcal{I}}{X^2}, \quad \Omega = dP_X \wedge dX + dI_a \wedge d\widetilde{\Phi}^a,$$

$$X = \sqrt{2K}, \quad P_X = \frac{D}{\sqrt{2K}},$$

$$\widetilde{\Phi}^a = \Phi^a + \frac{1}{2}\int d(XP_X)\frac{\partial \log\left(\sqrt{(XP_X)^2 + L(I)} + q(I)\right)}{\partial I_a}$$

Reissner-Nordström case

$$2\mathcal{I} = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} + (mM)^2 - (eq)^2, \quad q(p_\varphi) = eq$$

Action-Angle formulation

$$2\mathcal{I} = (I_1 + I_2)^2 + (mM)^2 - (eq)^2$$

We have got conventional $3d$ conformal mechanics with coupling constant $2g = (mM)^2 - (eq)^2$

Kerr case

$$2\mathcal{I} = p_\theta^2 + \left[\left(\frac{1+\cos^2\theta}{2\sin\theta} \right)^2 - 1 \right] p_\varphi^2 + \frac{m^2}{2}(1+\cos^2\theta), \;\; q(p_\varphi) = p_\varphi$$

Action-Angle formulation

$$\mathcal{I} = \frac{1}{2}\left(\sqrt{(m^2+l_2^2)^2 + 4\pi(m^2-l_2^2)l_1^2} - m^2 - l_2^2 \right)$$

Thank you for your attention!